2 VIBRATION OF STRUCTURES

LEARNING OUTCOMES

Students are able to solve the weekly lecture problems, home problems, and exercise problems on the topics of week 3:

- Vibration problem, natural frequencies and modes of vibration, solution to a free vibration problem in terms of the modes.

- Time dependent linear elasticity problem, principle of virtual work in a time-dependent case and vibration analysis by FEM.

- Inertia term element contributions for the solid-, beam-, plate-, and rigid body elements.

LINEAR ELASTICITY

Balance of mass (def. of a body or a material volume) Mass of a body is constant

Balance of linear momentum (Newton 2) The rate of change of linear momentum within a material volume equals the external force resultant acting on the material volume. ←

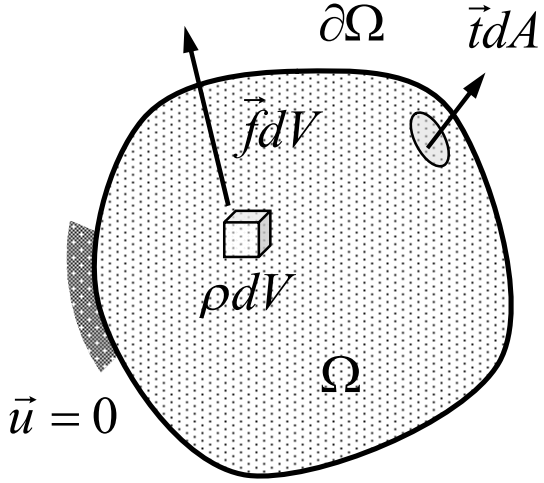
Balance of angular momentum (Cor. of Newton 2) The rate of change of angular momentum within a material volume equals the external moment resultant acting on the material volume. ←

Balance of energy (Thermodynamics 1)

Entropy growth (Thermodynamics 2)

BOUNDARY-INITIAL VALUE PROBLEM

Assuming equilibrium of a solid body (a set of particles) inside domain Ω, the aim is to find displacement $\vec{u}$ of the particles as functions of time, when external forces or boundary conditions are changed in some manner:



Equilibrium equations $\nabla \cdot \vec{\vec{\sigma}} + \vec{f} = \rho \ddot{\vec{u}}$ in Ω, $t > 0$

Hooke's law $\vec{\vec{\sigma}} = \frac{E}{1+\nu}\left(\frac{\nu}{1-2\nu}\vec{\vec{I}}\nabla \cdot \vec{u} + \vec{\vec{\varepsilon}}\right)$ in Ω, $t > 0$

Boundary conditions $\vec{n} \cdot \vec{\vec{\sigma}} = \vec{t}$ or $\vec{u} = \vec{g}$ on $\partial\Omega$, $t > 0$

Initial conditions $\vec{u} = \vec{u}_0$ and $\dot{\vec{u}} = \dot{\vec{u}}_0$ in Ω at $t = 0$

The balance law of angular momentum is satisfied 'a priori' by the form of Hooke's law.

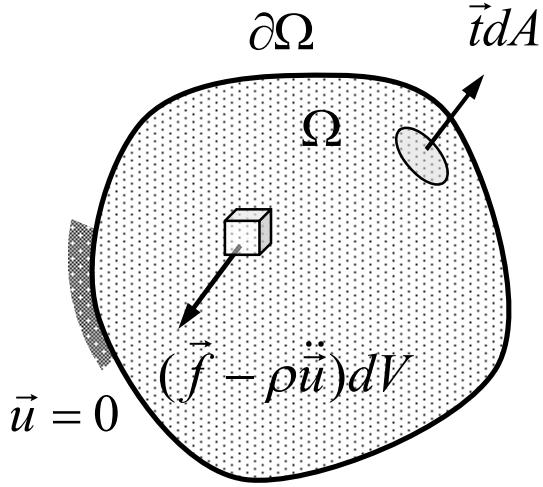
PRINCIPLE OF VIRTUAL WORK

Principle of virtual work $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} + \delta W^{\text{ine}} = 0 \quad \forall \delta \vec{u}$ is just one form of the equations of motion, where

$$\delta W^{\text{int}} = -\int_{\Omega} \left(\begin{Bmatrix} \delta\varepsilon_{xx} \\ \delta\varepsilon_{yy} \\ \delta\varepsilon_{zz} \end{Bmatrix}^{\text{T}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} + \begin{Bmatrix} \delta\gamma_{xy} \\ \delta\gamma_{yz} \\ \delta\gamma_{zx} \end{Bmatrix}^{\text{T}} \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} \right) dV ,$$

$$\delta W^{\text{ext}} = \int_{\Omega} \left(\begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\text{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} \right) dV + \int_{\partial\Omega} \left(\begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\text{T}} \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix} \right) dA ,$$

$$\delta W^{\text{ine}} = -\int_{\Omega} \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\text{T}} \rho \begin{Bmatrix} \ddot{u}_x \\ \ddot{u}_y \\ \ddot{u}_z \end{Bmatrix} dV .$$



In connection with the principle, time is considered as a parameter!

VIRTUAL WORK DENSITIES

Virtual work densities of the internal forces, inertia forces, external volume forces, and external surface forces are

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta\varepsilon_{xx} \\ \delta\varepsilon_{yy} \\ \delta\varepsilon_{zz} \end{Bmatrix}^{\text{T}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} - \begin{Bmatrix} \delta\gamma_{xy} \\ \delta\gamma_{yz} \\ \delta\gamma_{zx} \end{Bmatrix}^{\text{T}} \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} \text{ and } \delta w_{\Omega}^{\text{ine}} = -\begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\text{T}} \rho \begin{Bmatrix} \ddot{u}_x \\ \ddot{u}_y \\ \ddot{u}_z \end{Bmatrix},$$

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\text{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} \text{ and } \delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\text{T}} \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}.$$

Virtual work densities consist of terms containing kinematic quantities and their “work conjugates” !

2.1 PERIODIC MOTION AND VIBRATION

Constrained motion: there exists c and C such that $c < x(t) < C$

Periodic motion: there exists T such that $x(t+T) = x(t)$ for any t

Vibration: "periodic motion near static equilibrium"

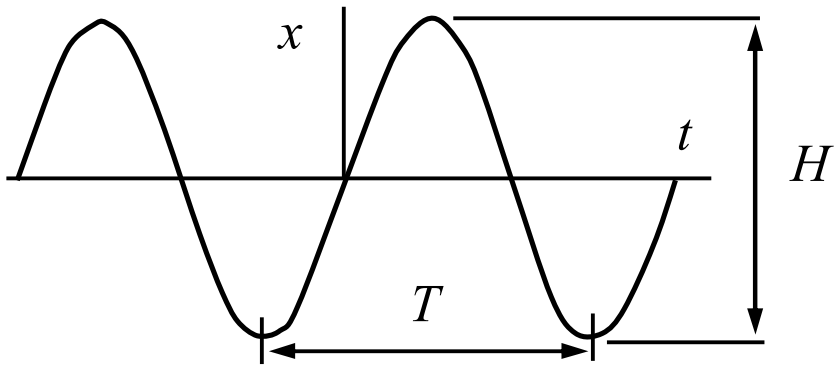
Harmonic vibration: $x(t) = X \sin \omega t$

Period T

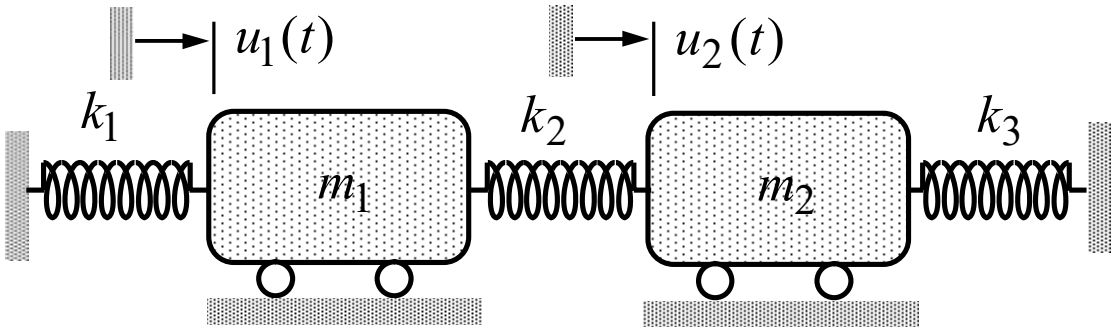
Frequency $f = 1/T$

Angular speed $\omega = 2\pi f$

Amplitude $X = H/2$



FREE UNDAMPED VIBRATION



Initial value problem $\mathbf{M}\ddot{\mathbf{a}} + \mathbf{K}\mathbf{a} = 0 \quad t > 0\,, \quad \dot{\mathbf{a}} = \dot{\mathbf{a}}_0 \quad t = 0,$ and $\mathbf{a} = \mathbf{a}_0 \quad t = 0$

Solution $\mathbf{a}(t) = \cos(\boldsymbol{\Omega} t)\mathbf{a}_0 + \sin(\boldsymbol{\Omega} t)\boldsymbol{\Omega}^{-1}\dot{\mathbf{a}}_0$

Problem parameter $\boldsymbol{\Omega} = (\mathbf{M}^{-1}\mathbf{K})^{1/2} = \mathbf{X}\boldsymbol{\omega}\mathbf{X}^{-1}$

In practice, the main task is to find the eigenvalue decomposition $\boldsymbol{\Omega} = \mathbf{X}\boldsymbol{\omega}\mathbf{X}^{-1}$ or it's form $\boldsymbol{\Omega}^2 = \mathbf{X}\boldsymbol{\omega}^2\mathbf{X}^{-1}$ (see: the definition of matrix function)!

MATRIX ALGEBRA

Transpose	$\mathbf{A}^{\mathrm{T}}$	$A_{ij}^{\mathrm{T}} = A_{ji}$
Inverse	$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$	$\sum_{k\in\{1\ldots n\}} A_{ik}A_{kj}^{-1} = \delta_{ij}$
Derivative	$\dot{\mathbf{x}}$	$\dot{x}_i = dx_i / dt$

Linear equation system: Find $\mathbf{x}$ such that $\mathbf{A}\mathbf{x} = \mathbf{b}$

Eigenvalue problem: Find all $(\lambda, \mathbf{x})$ such that $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$

Eigenvalue decomposition: $\mathbf{A} = \mathbf{X}\boldsymbol{\lambda}\mathbf{X}^{-1}$, where $\mathbf{X} = [\mathbf{x}_1 \quad \ldots \quad \mathbf{x}_n]$ and $\boldsymbol{\lambda} = diag[\lambda_1 \ \ldots \ \lambda_n]$

Matrix function: If $\mathbf{A} = \mathbf{X}\boldsymbol{\lambda}\mathbf{X}^{-1}$, then $f(\mathbf{A}) = \mathbf{X}f(\boldsymbol{\lambda})\mathbf{X}^{-1}$

EXAMPLE 2.1 Determine the angular speeds and modes of free vibrations, when the differential equations in their standard form are given by

$$\begin{Bmatrix} \ddot{a}_1 \\ \ddot{a}_2 \end{Bmatrix} + \boldsymbol{\Omega}^2 \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = 0 \quad \text{in which} \quad \boldsymbol{\Omega}^2 = \begin{bmatrix} 3 & -1/3 \\ -3 & 3 \end{bmatrix}.$$

Answer $\omega_1 = \sqrt{2}$, $\mathbf{x}_1 = \begin{Bmatrix} 1 \\ 3 \end{Bmatrix}$ and $\omega_2 = 2$, $\mathbf{x}_2 = \begin{Bmatrix} 1 \\ -3 \end{Bmatrix}$

- Characteristic equation $\det(\boldsymbol{\Omega}^2 - \lambda\mathbf{I}) = \det\begin{bmatrix} 3-\lambda & -1/3 \\ -3 & 3-\lambda \end{bmatrix} = (3-\lambda)^2 - 1 = 0$

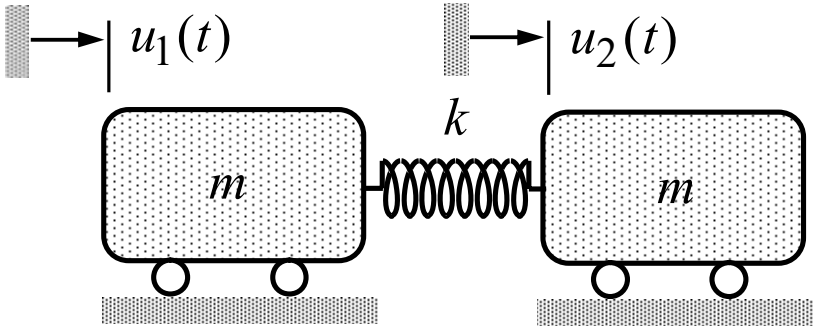
  Mode $\lambda_1 = 2$: $\begin{bmatrix} 3-2 & -1/3 \\ -3 & 3-2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 \quad \Rightarrow \quad \mathbf{x}_1 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 3 \end{Bmatrix}$

  Mode $\lambda_2 = 4$: $\begin{bmatrix} 3-4 & -1/3 \\ -3 & 3-4 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 \quad \Rightarrow \quad \mathbf{x}_2 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -3 \end{Bmatrix}$

- Eigenvalue decomposition $\boldsymbol{\Omega}^2 = \mathbf{X}\boldsymbol{\omega}^2\mathbf{X}^{-1} = \begin{bmatrix} 1 & 1 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3 & -3 \end{bmatrix}^{-1}$

- Positive square root $\boldsymbol{\Omega} = \mathbf{X}\boldsymbol{\omega}\mathbf{X}^{-1} = \begin{bmatrix} 1 & 1 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3 & -3 \end{bmatrix}^{-1}$ ←

EXAMPLE 2.2 Write down the equations of motion for the system shown consisting of two particles and a spring. After that, determine the angular speeds and modes of free vibrations.



Answer $\omega_1 = 0$ and $\mathbf{x}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$ (translation mode),

$$\omega_2 = \sqrt{2\frac{k}{m}} \quad \text{and} \quad \mathbf{x}_2 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \quad \text{(vibration mode).}$$

- Matrices $\mathbf{M}=\begin{bmatrix} 0 & m \\ m & 0 \end{bmatrix}$ and $\mathbf{K}=\begin{bmatrix} -k & k \\ k & -k \end{bmatrix}$ give

$$\mathbf{\Omega}^2=\mathbf{M}^{-1}\mathbf{K}=\begin{bmatrix} 0 & m \\ m & 0 \end{bmatrix}^{-1}\begin{bmatrix} -k & k \\ k & -k \end{bmatrix}=\frac{k}{m}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

- Angular speeds of the free vibration modes are the eigenvalues of $\mathbf{\Omega}$. Let us calculate first the eigenvalues of $\mathbf{\Omega}^2$ and the corresponding modes

$$\det(\mathbf{\Omega}^2-\lambda\mathbf{I})=\det\left(\begin{bmatrix} \frac{k}{m}-\lambda & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m}-\lambda \end{bmatrix}\right)=(\frac{k}{m}-\lambda)^2-(\frac{k}{m})^2=0 \quad \Rightarrow \quad \lambda\in\{0,2\frac{k}{m}\}.$$

$$\lambda_1 = 0: \quad \begin{bmatrix} \frac{k}{m} - 0 & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m} - 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 \quad \Rightarrow \quad \mathbf{x}_1 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix},$$

$$\lambda_2 = 2\frac{k}{m}: \quad \begin{bmatrix} \frac{k}{m} - 2\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m} - 2\frac{k}{m} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 \quad \Rightarrow \quad \mathbf{x}_2 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}.$$

- Angular speeds of the free vibrations and the corresponding modes are

$$\omega_1 = \sqrt{\lambda_1} = 0, \ \mathbf{x}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad \text{and} \quad \omega_2 = \sqrt{\lambda_2} = \sqrt{2\frac{k}{m}}, \ \mathbf{x}_2 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}.$$

TIME-INTEGRATION

In one-step methods for second order initial value problems, temporal domain is divided into sub-domains $t \in [t^{(i-1)}, t^{(i)}] \quad i \in \{1 \ldots n\}$. Differential equations are replaced by difference equations:

initial conditions

$$\left\{ \begin{array}{c} \mathbf{a}_0 \\ \hline \mathbf{a}_1 \Delta t \end{array} \right\}^{(i+1)} = \left[\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{A}_{21} & \mathbf{A}_{22} \end{array} \right] \left\{ \begin{array}{c} \mathbf{a}_0 \\ \hline \mathbf{a}_1 \Delta t \end{array} \right\}^{(i)} + \left\{ \begin{array}{c} \mathbf{B}_1 \\ \hline \mathbf{B}_2 \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{c} \mathbf{a}_0 \\ \hline \mathbf{a}_1 \Delta t \end{array} \right\}^{(0)} = \left\{ \begin{array}{c} \underline{\mathbf{a}}_0 \\ \hline \underline{\mathbf{a}}_1 \Delta t \end{array} \right\}$$

Iteration on the difference equations gives values of the unknowns $\mathbf{a}_0^{(i)}$ and their first time-derivatives $\mathbf{a}_1^{(i)}$ at $t^{(i)} \quad i \in \{0 \ldots n\}$. Iteration matrix $\mathbf{A}$ depends on the mass matrix $\mathbf{M}$, stiffness matrix $\mathbf{K}$, and the step size Δt. Vector $\mathbf{B}$ depends also on the external forces.

ONE-STEP INTEGRATION METHODS

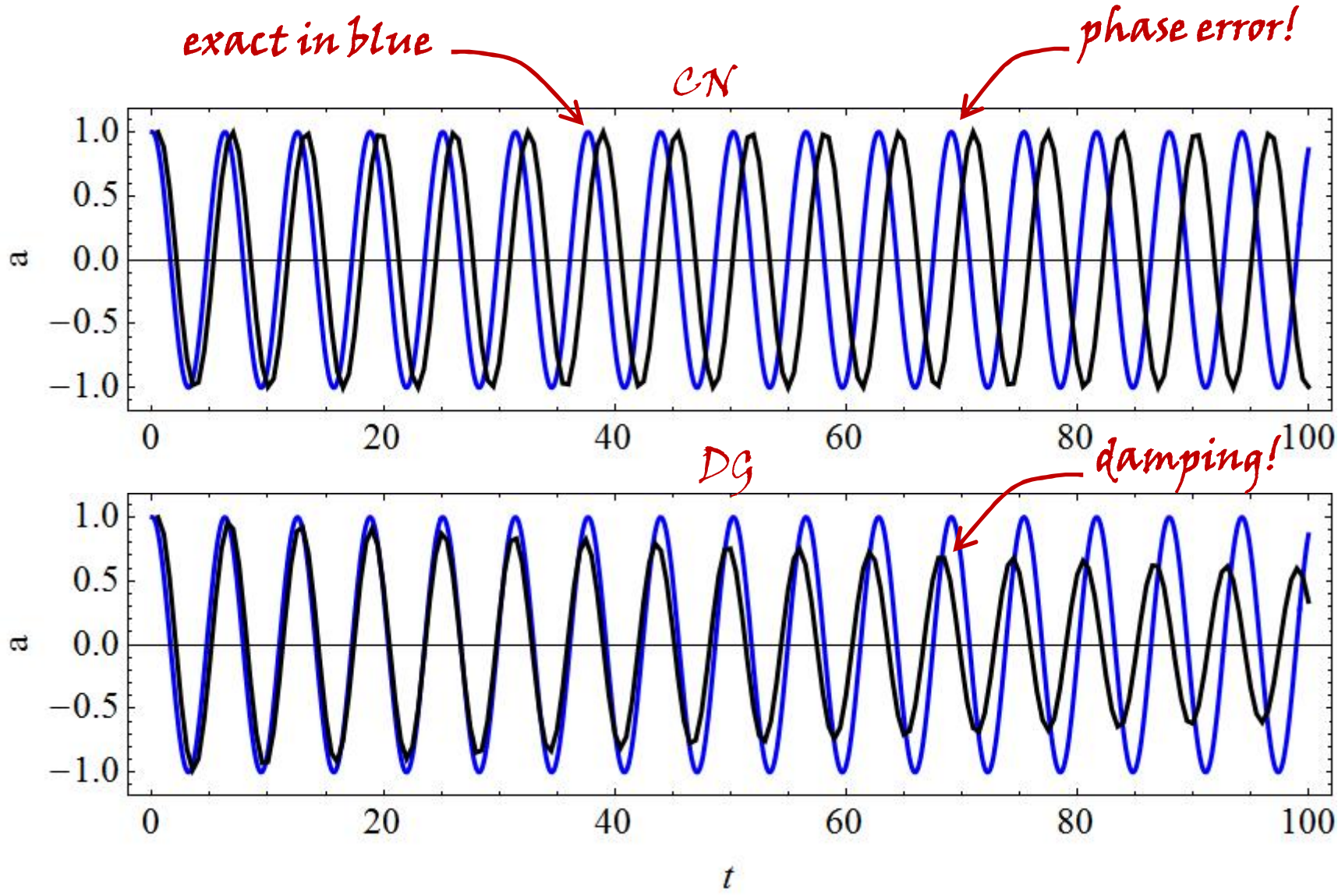
The recipes for a single equation and an equation systems are the same. For problem $\mathbf{M}\ddot{\mathbf{a}} + \mathbf{K}\mathbf{a} = 0$

Crank-Nicholson:
$$\begin{Bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \Delta t \end{Bmatrix}^{(i+1)} = \begin{bmatrix} \mathbf{I} & -\mathbf{I}/2 \\ \alpha/2 & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{I} & \mathbf{I}/2 \\ -\alpha/2 & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \Delta t \end{Bmatrix}^{(i)}$$

Disc. Galerkin:
$$\begin{Bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \Delta t \end{Bmatrix}^{(i+1)} = \begin{bmatrix} \alpha & \mathbf{I} - \alpha/2 \\ -\mathbf{I} - \alpha/2 & \alpha/3 \end{bmatrix}^{-1} \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & -\mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \Delta t \end{Bmatrix}^{(i)}$$

The proper step-size of depends on the largest eigenvalue of parameter $\alpha = \mathbf{M}^{-1}\mathbf{K}\Delta t^2$. A small amount of numerical damping is advantageous, if the step-size, according to the largest eigenvalue, becomes impractically small.

ACCURACY AND STABILITY $\Delta t = 1/2$ & $\alpha = 1/4$



2.2 VIBRATION ANALYSIS

- Model the structure as a collection of beam, plate, etc. elements. Derive the element contributions δW^e and express the nodal displacement and rotation components of the material coordinate system in terms of those in the structural coordinate system.

- Sum the element contributions to end up with the virtual work expression of the structure $\delta W = \sum_{e \in E} \delta W^e$. Re-arrange to get $\delta W = -\delta \mathbf{a}^{\mathrm{T}} (\mathbf{M}\ddot{\mathbf{a}} + \mathbf{K}\mathbf{a} - \mathbf{F})$.

- Use the principle of virtual work $\delta W = 0 \ \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus for $\delta \mathbf{a} \in \mathbb{R}^n$ to deduce the system equations $\mathbf{M}\ddot{\mathbf{a}} + \mathbf{K}\mathbf{a} - \mathbf{F} = 0$.

- Solve the equations for the natural angular speeds of vibrations and the corresponding modes or solve for the displacements and rotations as functions of time.

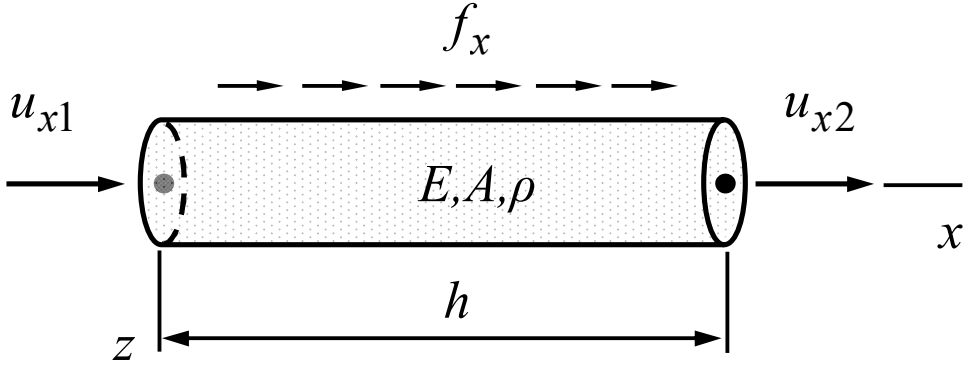
BAR ELEMENT

Assuming a linear approximation to the axial displacement $u(x,t)$, virtual work expressions of the internal, external, and inertia forces take the forms

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\text{T}} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix},$$

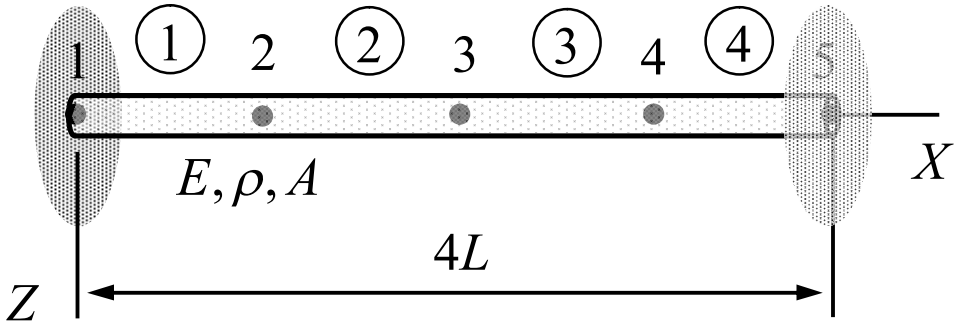
$$\delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\text{T}} \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix},$$

$$\delta W^{\text{ine}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\text{T}} \frac{\rho A h}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{x1} \\ \ddot{u}_{x2} \end{Bmatrix}.$$



Above, f_x and E, A, ρ are taken as constants.

EXAMPLE 2.3. Consider the free vibrations of the bar shown, when material properties E, ρ and cross-sectional area A are constants. Determine the set of ordinary differential equations giving as their solution the nodal displacements (assuming that initial displacement and velocity are known). Use four elements of equal size.



Answer

$$\frac{EA}{L}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{X3} \\ u_{X4} \end{Bmatrix} + \frac{\rho AL}{6}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}\begin{Bmatrix} \ddot{u}_{X2} \\ \ddot{u}_{X3} \\ \ddot{u}_{X4} \end{Bmatrix} = 0$$

- Let us assume that the structural and material coordinate systems coincide (for simplicity). Virtual work expressions of the elements taking into account the internal and inertia parts are

$$\delta W^1 = -\begin{Bmatrix} 0 \\ \delta u_{X2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} + \frac{\rho AL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} 0 \\ \ddot{u}_{X2} \end{Bmatrix} \right),$$

$$\delta W^2 = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} + \frac{\rho AL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{X2} \\ \ddot{u}_{X3} \end{Bmatrix} \right),$$

$$\delta W^3 = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{X4} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{X4} \end{Bmatrix} + \frac{\rho AL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{X3} \\ \ddot{u}_{X4} \end{Bmatrix} \right),$$

$$\delta W^4 = -\begin{Bmatrix} \delta u_{X4} \\ 0 \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{X4} \\ 0 \end{Bmatrix} + \frac{\rho AL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{X4} \\ 0 \end{Bmatrix} \right).$$

- Virtual work expression of the structure is the sum of the element contributions $\delta W = \sum \delta W^e$. When expressed in the "standard form"

$$\delta W = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \\ \delta u_{X4} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{X3} \\ u_{X4} \end{Bmatrix} + \frac{\rho AL}{6} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{X2} \\ \ddot{u}_{X3} \\ \ddot{u}_{X4} \end{Bmatrix} \right).$$

- Principle of virtual work $\delta W = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus for $\delta \mathbf{a}$ give

$$\frac{EA}{L} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{X3} \\ u_{X4} \end{Bmatrix} + \frac{\rho AL}{6} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{X2} \\ \ddot{u}_{X3} \\ \ddot{u}_{X4} \end{Bmatrix} = 0.$$

- Mathematica code of the course can be used to build the set of ordinary differential equations and check the outcome of hand calculations:

	type	properties	geometry
1	BEAM	{{E, G, ρ}, {A, I, I}, {0, 0, 0}}	Line[{1, 2}]
2	BEAM	{{E, G, ρ}, {A, I, I}, {0, 0, 0}}	Line[{2, 3}]
3	BEAM	{{E, G, ρ}, {A, I, I}, {0, 0, 0}}	Line[{3, 4}]
4	BEAM	{{E, G, ρ}, {A, I, I}, {0, 0, 0}}	Line[{4, 5}]

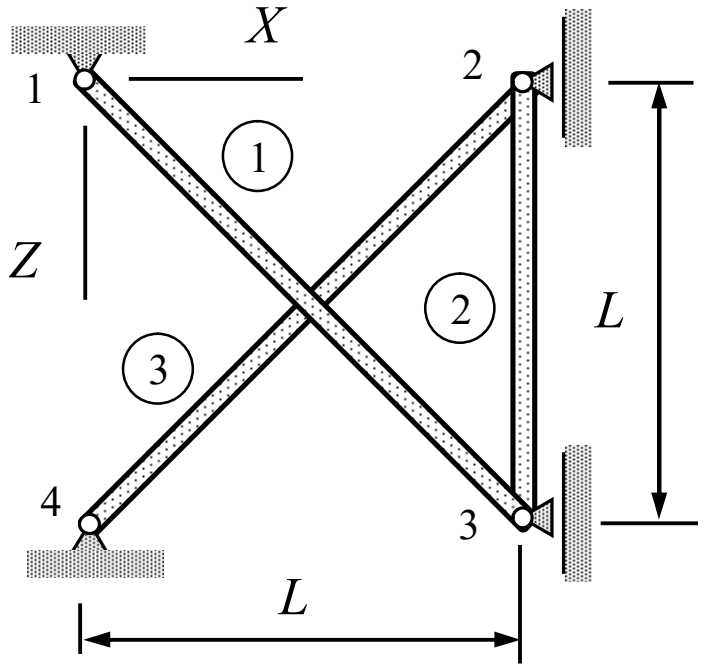
	{X,Y,Z}	$\{u_X,u_Y,u_Z\}$	$\{\Theta_X,\Theta_Y,\Theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{uX[1], 0, 0}	{0, 0, 0}
3	{2 L, 0, 0}	{uX[2], 0, 0}	{0, 0, 0}
4	{3 L, 0, 0}	{uX[3], 0, 0}	{0, 0, 0}
5	{4 L, 0, 0}	{0, 0, 0}	{0, 0, 0}

In Mathematica representation, derivatives with respect to time are indicated by indices. Therefore, e.g., $u_{Xn} \sim uX[n,0]$, $\ddot{u}_{Xn} \sim uX[n,2]$ (zero order derivative means function itself):

$$\delta W = -\begin{pmatrix} \delta uX[1,0] \\ \delta uX[2,0] \\ \delta uX[3,0] \end{pmatrix}^T \left(\begin{pmatrix} \frac{2AE}{L} & -\frac{AE}{L} & 0 \\ -\frac{AE}{L} & \frac{2AE}{L} & -\frac{AE}{L} \\ 0 & -\frac{AE}{L} & \frac{2AE}{L} \end{pmatrix} \right.$$

$$\left. \begin{pmatrix} uX[1,0] \\ uX[2,0] \\ uX[3,0] \end{pmatrix} + \begin{pmatrix} \frac{2AL\rho}{3} & \frac{AL\rho}{6} & 0 \\ \frac{AL\rho}{6} & \frac{2AL\rho}{3} & \frac{AL\rho}{6} \\ 0 & \frac{AL\rho}{6} & \frac{2AL\rho}{3} \end{pmatrix} \begin{pmatrix} uX[1,2] \\ uX[2,2] \\ uX[3,2] \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right)$$

EXAMPLE 2.4. Consider free vibrations of a truss of three bar elements of which bar 2 is inextensible and bars 1 and 3 massless. Determine the displacement of node 2 as function of time. Initially, displacements are zeros and velocity of nodes 2 and 3 are $\dot{U}$ downwards. Use linear bar elements. Cross-sectional areas of bars 1 and 3 are A and that of bar 2 $\sqrt{8}A$.



Answer $u_{Z2}(t) = 2\dot{U}\sqrt{\frac{L^2\rho}{E}}\sin(\frac{1}{2}t\sqrt{\frac{E}{L^2\rho}})$

- Only the displacements of nodes 2 and 3 in the $Z-$direction matter. As bar 2 is known to be rigid, vertical displacements of nodes 2 and 3 coincide i.e. $u_{Z2}=u_{Z3}$. From the figure, the nodal displacement and length of bar 1 are $u_{x1}=0$, $u_{x2}=u_{Z2}/\sqrt{2}$ and $h=\sqrt{2}L$. As the bar is assumed to be massless, inertia term vanishes and

$$\delta W^1=-\begin{Bmatrix}0\\ \delta u_{Z2}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{\sqrt{8}L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ u_{Z2}\end{Bmatrix}=-\delta u_{Z2}\frac{EA}{\sqrt{8}L}u_{Z2}.$$

- The relationships for bar 2 are $u_{x1}=u_{Z2}$, $u_{x2}=u_{Z2}$ and $h=L$. The cross-sectional area is $\sqrt{8}A$. As the axial displacements coincide, internal part vanishes and

$$\delta W^2=-\begin{Bmatrix}\delta u_{Z2}\\ \delta u_{Z2}\end{Bmatrix}^{\mathrm{T}}\frac{\rho\sqrt{8}AL}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}\ddot{u}_{Z2}\\ \ddot{u}_{Z2}\end{Bmatrix}=-\rho\sqrt{8}AL\delta u_{Z2}\ddot{u}_{Z2}.$$

- The relationships for bar 3 are $u_{x1}=0,\ u_{x2}=-u_{Z2}/\sqrt{2}$ and $h=\sqrt{2}L$. As the bar is assumed to be massless

$$\delta W^3=-\begin{Bmatrix}0\\-\delta u_{Z2}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{\sqrt{8}L}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\begin{Bmatrix}0\\-u_{Z2}\end{Bmatrix}=-\delta u_{Z2}\frac{EA}{\sqrt{8}L}u_{Z2}.$$

- Virtual work expression of the structure is the sum of element contributions

$$\delta W=\delta W^1+\delta W^2+\delta W^3=-\delta u_{Z2}\left(\frac{EA}{\sqrt{2}L}u_{Z2}+\rho\sqrt{8}AL\ddot{u}_{Z2}\right).$$

- Principle of virtual work $\delta W=0\quad\forall\delta\mathbf{a}$ and the fundamental lemma of variation calculus $\delta\mathbf{a}^{\mathrm{T}}\mathbf{F}=0\ \forall\delta\mathbf{a}\quad\Leftrightarrow\quad\mathbf{F}=0$ imply that

$$\frac{EA}{\sqrt{2}L}u_{Z2}+\rho\sqrt{8}AL\ddot{u}_{Z2}=0.$$

- What remains, is solving for the displacement as function of time with the additional information of the problem description. The initial value problem consists of the differential equation and two initial conditions:

$$\ddot{u}_{Z2} + \frac{1}{4}\frac{E}{\rho L^2} u_{Z2} = 0 \quad t > 0, \quad u_{Z2}(0) = 0 \quad \text{and} \quad \dot{u}_{Z2}(0) = \dot{U}.$$

- Solution to the equations is given by

$$u_{Z2}(t) = 2\dot{U}\sqrt{\frac{L^2\rho}{E}}\sin\left(\frac{1}{2}t\sqrt{\frac{E}{L^2\rho}}\right) \quad t > 0.$$

- Mathematica code of the course can be used to solve the set of ordinary differential equations for the nodal displacements and rotations in simple cases and check the outcome of the hand calculations:

	type	properties	geometry
1	BAR	{{E, 0}, {A}, {0}}	Line[{1, 3}]
2	BAR	$\{\{E, \rho\}, \{2\sqrt{2}\ A\}, \{0\}\}$	Line[{2, 3}]
3	BAR	{{E, 0}, {A}, {0}}	Line[{4, 2}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{0, 0, uZ[2]}	{0, 0, 0}
3	{L, 0, L}	{0, 0, uZ[2]}	{0, 0, 0}
4	{0, 0, L}	{0, 0, 0}	{0, 0, 0}

$$\left\{uZ[2] \to \frac{2\,\dot{U}\,\text{Sin}\left[\frac{1}{2}\,t\,\sqrt{\frac{E}{L^2\,\rho}}\right]}{\sqrt{\frac{E}{L^2\,\rho}}}\right\}$$

2.3 ELEMENT CONTRIBUTIONS

Virtual work expressions for solid, beam, plate elements combine virtual work densities representing the model and the element shape and type dependent approximation. To derive the expression for an element:

- Start with the virtual work densities $\delta w_{\Omega}^{\text{int}}$, $\delta w_{\Omega}^{\text{ine}}$, and $\delta w_{\Omega}^{\text{ext}}$ of the formulae collection.

- Represent the unknown functions by interpolation of the nodal displacement and rotations (see formulae collection). Substitute the approximations into the density expressions.

- Integrate the virtual work density over the domain occupied by the element to get δW.

ELEMENT APPROXIMATION

In MEC-E8001 element approximation is a polynomial interpolant of the nodal displacement and rotations in terms of shape functions. In vibration analysis, shape functions depend on *x, y, z* and the nodal values on time *t*.

Approximation $\mathbf{u} = \mathbf{N}^{\mathrm{T}}\mathbf{a}$ *always of the same form!*

Shape functions $\mathbf{N} = \{N_1(x,y,z) \;\; N_2(x,y,z) \;\; \ldots \;\; N_n(x,y,z)\}^{\mathrm{T}}$

Parameters $\mathbf{a} = \{\mathrm{a}_1(t) \;\; \mathrm{a}_2(t) \;\; \ldots \;\; \mathrm{a}_n(t)\}^{\mathrm{T}}$

Nodal parameters $\mathrm{a} \in \{u_x, u_y, u_z, \theta_x, \theta_y, \theta_z\}$ may be only displacement or rotation components or a mixture of them (as with the beam model).

SOLID MODEL

The model does not contain any assumptions in addition to those of the linear elasticity theory.

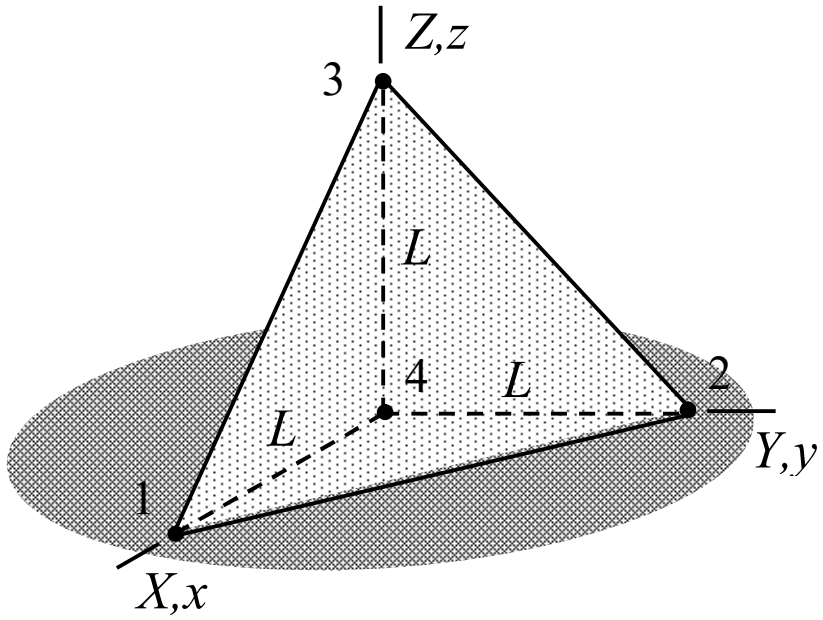
$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta w/\partial z \end{Bmatrix}^{\text{T}} [E] \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial w/\partial z \end{Bmatrix} - \begin{Bmatrix} \partial\delta u/\partial y + \partial\delta v/\partial x \\ \partial\delta v/\partial z + \partial\delta w/\partial y \\ \partial\delta w/\partial x + \partial\delta u/\partial z \end{Bmatrix}^{\text{T}} G \begin{Bmatrix} \partial u/\partial y + \partial v/\partial x \\ \partial v/\partial z + \partial w/\partial y \\ \partial w/\partial x + \partial u/\partial z \end{Bmatrix},$$

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\text{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix}, \quad \delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\text{T}} \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}, \text{ and } \delta w_{\Omega}^{\text{ine}} = -\begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\text{T}} \rho \begin{Bmatrix} \ddot{u} \\ \ddot{v} \\ \ddot{w} \end{Bmatrix}.$$

The solution domain can be represented, e.g., by tetrahedron elements with linear interpolation of $u(x,y,z,t)$, $v(x,y,z,t)$, and $w(x,y,z,t)$ in spatial coordinates.

EXAMPLE 2.5 Consider a tetrahedron of edge length L, density ρ, and elastic properties E and $\nu = 0$ on a horizontal floor. Calculate the displacement $u_{Z3}(t)$ of node 3 with one tetrahedron element and linear approximation. Assume that $u_{X3} = u_{Y3} = 0$, the bottom surface is fixed, and $u_{Z3} = U$ and $\dot{u}_{Z3} = 0$ at $t = 0$. Stress vanishes at the initial geometry when $u_{Z3} = 0$.

Answer: $u_{Z3}(t) = U\cos(t\sqrt{10\frac{E}{\rho L^2}})$



- Linear shape functions can be deduced directly from the figure $N_1 = x/L$, $N_2 = y/L$, $N_3 = z/L$, and $N_4 = 1 - x/L - y/L - z/L$. However, only the shape function of node 3 is needed as the other nodes are fixed. Approximations to the displacement components are

$$u = 0,\ v = 0, \text{ and } w = \frac{z}{L} u_{Z3}, \text{ giving } \frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = 0,\ \frac{\partial w}{\partial z} = \frac{1}{L} u_{Z3}, \text{ and } \ddot{w} = \frac{z}{L} \ddot{u}_{Z3}.$$

- When the approximation is substituted there, the virtual work densities of the internal external, and inertia forces simplify to (here $\nu = 0$)

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} 0 \\ 0 \\ \partial \delta w / \partial z \end{Bmatrix}^{\text{T}} \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \partial w / \partial z \end{Bmatrix} = -\frac{E}{L^2} u_{Z3} \delta u_{Z3},$$

$$\delta w_\Omega^{\text{ine}} = -\begin{Bmatrix}\delta u\\ \delta v\\ \delta w\end{Bmatrix}^{\mathrm{T}} \rho \begin{Bmatrix}\ddot{u}\\ \ddot{v}\\ \ddot{w}\end{Bmatrix} = -\begin{Bmatrix}0\\ 0\\ z/L\,\delta u_{Z3}\end{Bmatrix}^{\mathrm{T}} \rho \begin{Bmatrix}0\\ 0\\ z/L\,\ddot{u}_{Z3}\end{Bmatrix} = -\rho\left(\frac{z}{L}\right)^2 \delta u_{Z3}\ddot{u}_{Z3}.$$

- Virtual work expressions are obtained as integrals of densities over the volume:

$$\delta W^{\text{int}} = \int_\Omega \delta w_\Omega^{\text{int}} dV = \int_0^L \int_0^{L-z} \int_0^{L-z-y} \delta w_\Omega^{\text{int}} dxdydz = -\frac{1}{6} EL\delta u_{Z3} u_{Z3},$$

$$\delta W^{\text{ine}} = \int_\Omega \delta w_\Omega^{\text{ine}} dV = \int_0^L \int_0^{L-z} \int_0^{L-z-y} \delta w_\Omega^{\text{ine}} dxdydz = -\frac{L^3\rho}{60} \delta u_{Z3}\ddot{u}_{Z3}.$$

- Finally, principle of virtual work $\delta W = 0$ with $\delta W = \delta W^{\text{int}} + \delta W^{\text{ine}}$ implies that

$$\frac{1}{6} EL u_{Z3} + \frac{L^3\rho}{60}\ddot{u}_{Z3} = 0 \quad \Leftrightarrow \quad \ddot{u}_{Z3} + 10\frac{E}{\rho L^2} u_{Z3} = 0.$$

The standard form!

- Solution to the ordinary differential equations with the initial conditions $u_{Z3} = U$ and $\dot{u}_{Z3} = 0$ at $t = 0$ is given by (

$$u_{Z3}(t) = U\cos(\sqrt{10\frac{E}{\rho L^2}}t).$$

BEAM MODEL

Virtual work density of the inertia term is of the same form as that of the external force, if the distributed external force is replaced by the "inertia force" (not a true force actually). Virtual work density of the inertia forces of the beam is given by

$$\delta w_{\Omega}^{\text{ine}} = -\begin{Bmatrix} \delta u \\ \delta\psi \\ -\delta\theta \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} A & -S_z & -S_y \\ -S_z & I_{zz} & I_{yz} \\ -S_y & I_{zy} & I_{yy} \end{bmatrix} \rho \begin{Bmatrix} \ddot{u} \\ \ddot{\psi} \\ -\ddot{\theta} \end{Bmatrix} - \begin{Bmatrix} \delta v \\ \delta w \\ \delta\phi \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} A & 0 & -S_y \\ 0 & A & S_z \\ -S_y & S_z & I_{rr} \end{bmatrix} \rho \begin{Bmatrix} \ddot{v} \\ \ddot{w} \\ \ddot{\phi} \end{Bmatrix},$$

in which (Bernoulli constraints) $\psi = dv/dx$, $\theta = -dw/dx$, $\ddot{\psi} = d\ddot{v}/dx$, and $\ddot{\theta} = -d\ddot{w}/dx$. The terms for the bar, torsion bar, and the two ending modes follow from the generic expression above. Often, the rotation terms in bending are omitted as negligible.

- Let us consider the inertia term per unit length under the kinematic assumptions of the Timoshenko beam model $u_x = u + z\theta - y\psi$, $u_y = v - z\phi$, and $u_z = w + y\phi$

$$\delta w_\Omega^{\text{ine}} = -\int_A (\rho \ddot{\vec{u}} \cdot \delta \vec{u}) dA = (\delta w_\Omega^{\text{ine}})_x + (\delta w_\Omega^{\text{ine}})_y + (\delta w_\Omega^{\text{ine}})_z, \quad \text{where}$$

$$(\delta w_\Omega^{\text{ine}})_x = -\int_A \delta u_x \rho \ddot{u}_x dA = -\begin{Bmatrix} \delta u \\ \delta\psi \\ -\delta\theta \end{Bmatrix}^{\text{T}} \int_A \begin{bmatrix} 1 & -y & -z \\ -y & y^2 & yz \\ -z & zy & z^2 \end{bmatrix} \rho dA \begin{Bmatrix} \ddot{u} \\ \ddot{\psi} \\ -\ddot{\theta} \end{Bmatrix},$$

$$(\delta w_\Omega^{\text{ine}})_y = -\int_A \delta u_y \rho \ddot{u}_y dA = -\begin{Bmatrix} \delta v \\ \delta\phi \end{Bmatrix}^{\text{T}} \int_A \begin{bmatrix} 1 & -z \\ -z & z^2 \end{bmatrix} \rho dA \begin{Bmatrix} \ddot{v} \\ \ddot{\phi} \end{Bmatrix},$$

$$(\delta w_\Omega^{\text{ine}})_z = -\int_A \delta u_z \rho \ddot{u}_z dA = -\begin{Bmatrix} \delta w \\ \delta\phi \end{Bmatrix}^{\text{T}} \int_A \begin{bmatrix} 1 & y \\ y & y^2 \end{bmatrix} \rho dA \begin{Bmatrix} \ddot{w} \\ \ddot{\phi} \end{Bmatrix}.$$

- Assuming that cross-section geometry and density are constants, integration over the area gives with the assumptions $S_y = S_z = 0$ and $I_{yz} = I_{zy} = 0$

$$(\delta w_{\Omega}^{\text{ine}})_x = -\begin{Bmatrix} \delta u \\ \delta\psi \\ -\delta\theta \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} A & -S_z & -S_y \\ -S_z & I_{zz} & I_{yz} \\ -S_y & I_{zy} & I_{yy} \end{bmatrix} \rho \begin{Bmatrix} \ddot{u} \\ \ddot{\psi} \\ -\ddot{\theta} \end{Bmatrix},$$

$$(\delta w_{\Omega}^{\text{ine}})_y + (\delta w_{\Omega}^{\text{ine}})_z = -\begin{Bmatrix} \delta v \\ \delta v \\ \delta\phi \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} A & 0 & -S_y \\ 0 & A & S_z \\ -S_y & S_z & I_{rr} \end{bmatrix} \rho \begin{Bmatrix} \ddot{v} \\ \ddot{w} \\ \ddot{\phi} \end{Bmatrix},$$

in which $I_{rr} = I_{yy} + I_{zz}$, $\psi = dv/dx$ and $\theta = -dw/dx$ (Bernoulli constraints).

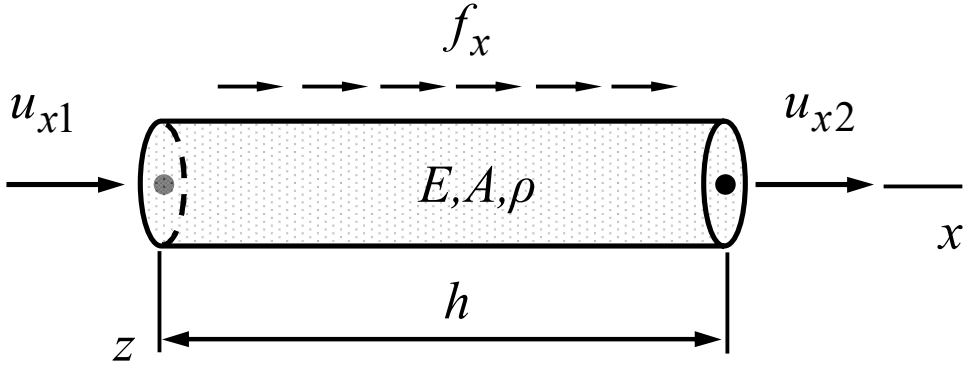
BAR MODE

Bar mode element contribution follows with the assumptions $v=0,\ w=0,\ \phi=0$, and a linear approximation to $u(x)$

$$\delta W^{\text{int}} = -\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\text{T}} \frac{EA}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix},$$

$$\delta W^{\text{ext}} = \begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\text{T}} \frac{f_x h}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix},$$

$$\delta W^{\text{ine}} = -\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\text{T}} \frac{\rho A h}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}\ddot{u}_{x1}\\ \ddot{u}_{x2}\end{Bmatrix}.$$



Above, f_x and E, A, ρ are taken as constants.

- Virtual work density of the inertia term is of the same form as the terms coming from the external distributed forces with $f_x = -\rho A\ddot{u}$ (inertia force per unit length). Hence virtual work densities are

$$\delta w_\Omega^{\text{int}} = -\frac{d\delta u}{dx} EA \frac{du}{dx}, \quad \delta w_\Omega^{\text{ext}} = \delta u f_x, \quad \text{and } \delta w_\Omega^{\text{ine}} = -\delta u \rho A \ddot{u}.$$

  Cross-sectional area A, Young's modulus E, density ρ, and external force per unit length f_x may depend on x and time t.

- Element approximation of the bar model with semi-discretization $u(x,t) = \mathbf{N}(x)^{\mathrm{T}} \mathbf{a}(t)$

$$u = \frac{1}{h}\{h-x \;\; x\}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} \quad \Rightarrow \frac{du}{dx} = \frac{1}{h}\{-1 \;\; 1\}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} \quad \text{and} \quad \ddot{u} = \frac{1}{h}\{h-x \;\; x\}\begin{Bmatrix} \ddot{u}_{x1} \\ \ddot{u}_{x2} \end{Bmatrix}$$

- Virtual work density of the inertia force (expressions for the internal and external forces have been discussed in MEC-E1050) is given by

$$\delta w_{\Omega}^{\text{ine}} = -\delta u \rho A \ddot{u} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{\rho A}{h^2} \begin{bmatrix} (h-x)^2 & x(h-x) \\ x(h-x) & x^2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{x1} \\ \ddot{u}_{x2} \end{Bmatrix}$$

- Assuming that A, ρ are constants, integration over the length gives

$$\delta W^{\text{ine}} = \int_0^h \delta w_{\Omega}^{\text{ine}} dx = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{\rho A h}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{x1} \\ \ddot{u}_{x2} \end{Bmatrix}. \leftarrow$$

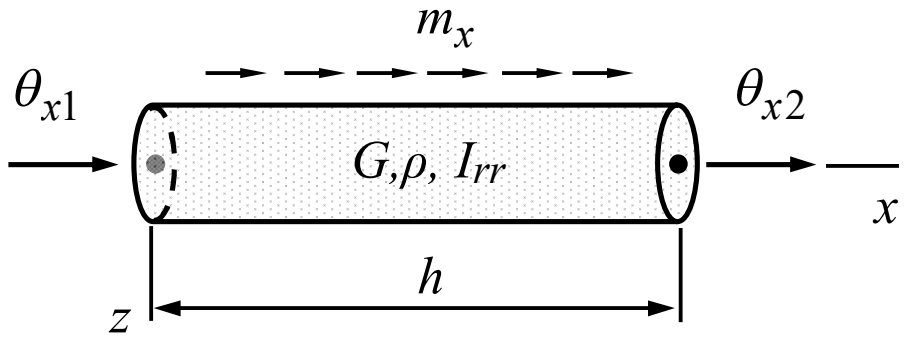
TORSION MODE

Torsion mode element contribution follows with the assumptions $u=0,\ v=0,\ w=0$, and a linear approximation to $\phi(x)$

$$\delta W^{\text{int}} = -\begin{Bmatrix}\delta\theta_{x1}\\ \delta\theta_{x2}\end{Bmatrix}^{\text{T}} \frac{GI_{rr}}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}\theta_{x1}\\ \theta_{x2}\end{Bmatrix},$$

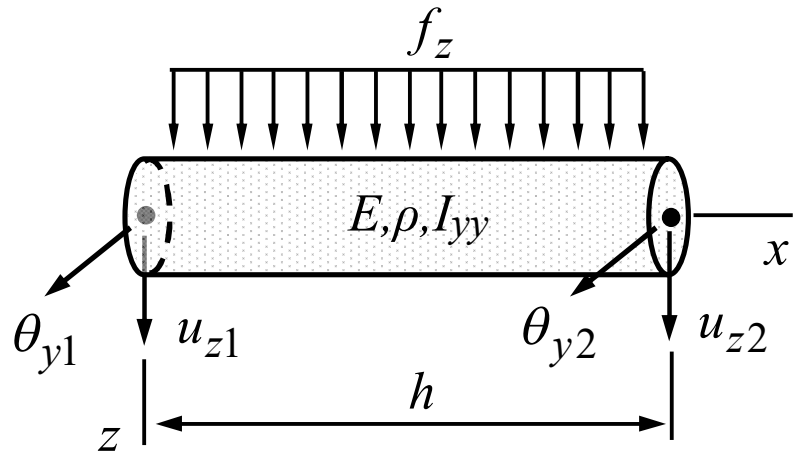
$$\delta W^{\text{ext}} = \begin{Bmatrix}\delta\theta_{x1}\\ \delta\theta_{x2}\end{Bmatrix}^{\text{T}} \frac{m_x h}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix},$$

$$\delta W^{\text{ine}} = -\begin{Bmatrix}\delta\theta_{x1}\\ \delta\theta_{x2}\end{Bmatrix}^{\text{T}} \frac{\rho I_{rr} h}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}\ddot{\theta}_{x1}\\ \ddot{\theta}_{x2}\end{Bmatrix}.$$



Above, m_x, G, ρ and I_{rr} are assumed to be constants.

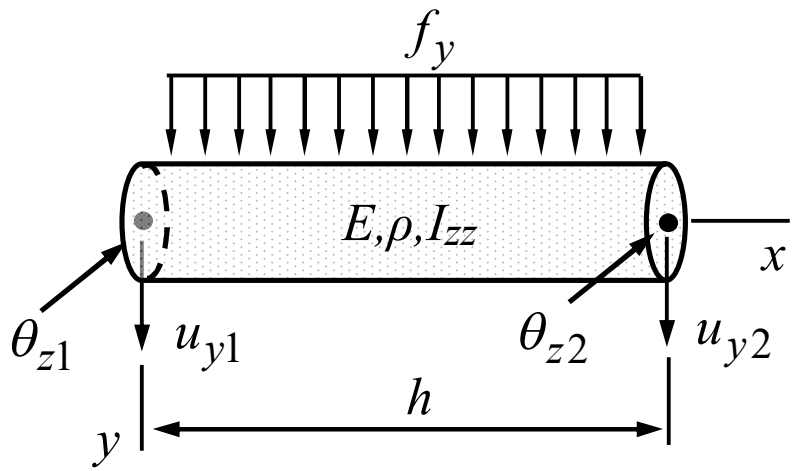
BENDING MODE (*xz*-plane)



$$\delta W^{\text{ine}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \hline \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{\rho I_{yy}}{30h} \left[\begin{array}{cc|cc} 36 & -3h & -36 & -3h \\ -3h & 4h^2 & 3h & -h^2 \\ \hline -36 & 3h & 36 & 3h \\ -3h & -h^2 & 3h & 4h^2 \end{array}\right] + \frac{\rho Ah}{420} \left[\begin{array}{cc|cc} 156 & -22h & 54 & 13h \\ -22h & 4h^2 & -13h & -3h^2 \\ \hline 54 & -13h & 156 & 22h \\ 13h & -3h^2 & 22h & 4h^2 \end{array}\right] \right) \begin{Bmatrix} \ddot{u}_{z1} \\ \ddot{\theta}_{y1} \\ \hline \ddot{u}_{z2} \\ \ddot{\theta}_{y2} \end{Bmatrix}$$

The first term is negligible whenever a beam element is thin in the sense $\alpha = t / h << 1$!

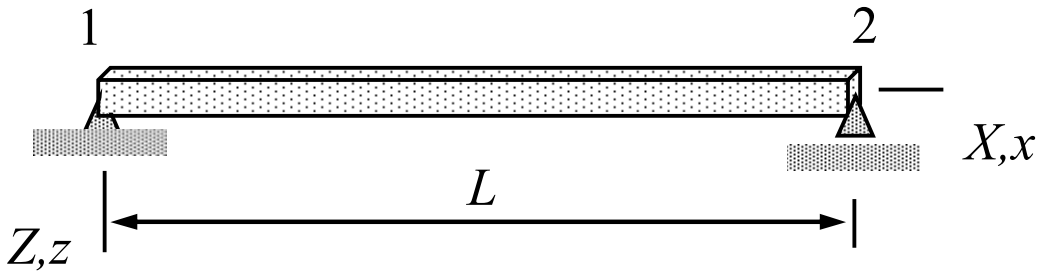
BENDING MODE (*xy*-plane)



$$\delta W^{\text{ine}} = -\begin{Bmatrix} \delta u_{y1} \\ \delta\theta_{z1} \\ \delta u_{y2} \\ \delta\theta_{z2} \end{Bmatrix}^{\text{T}} \left(\frac{\rho I_{zz}}{30h} \left[\begin{array}{cc|cc} 36 & 3h & -36 & 3h \\ 3h & 4h^2 & -3h & -h^2 \\ \hline -36 & -3h & 36 & -3h \\ 3h & -h^2 & -3h & 4h^2 \end{array}\right] + \frac{\rho Ah}{420} \left[\begin{array}{cc|cc} 156 & 22h & 54 & -13h \\ 22h & 4h^2 & 13h & -3h^2 \\ \hline 54 & 13h & 156 & -22h \\ -13h & -3h^2 & -22h & 4h^2 \end{array}\right] \right) \begin{Bmatrix} \ddot{u}_{y1} \\ \ddot{\theta}_{z1} \\ \ddot{u}_{y2} \\ \ddot{\theta}_{z2} \end{Bmatrix}$$

The first term is negligible whenever a beam element is thin in the sense $\alpha = t / h << 1$!

EXAMPLE 2.6 Consider bending of a simply supported beam of length L in $XZ-$plane. Determine the ordinary differential equations giving as their solution the rotation components of the end nodes as functions of time. Determine also the natural angular speeds of free vibrations and the corresponding modes. Cross-section properties A, I and material properties E, ρ are constants.



Answer $\omega_1 = \sqrt{2520\dfrac{EI}{\rho AL^4}}$, $\begin{Bmatrix}\theta_{Y1}\\ \theta_{Y2}\end{Bmatrix}_1 = \begin{Bmatrix}1\\ 1\end{Bmatrix}$ and $\omega_2 = \sqrt{120\dfrac{EI}{\rho AL^4}}$, $\begin{Bmatrix}\theta_{Y1}\\ \theta_{Y2}\end{Bmatrix}_2 = \begin{Bmatrix}-1\\ 1\end{Bmatrix}$

- As the material and structural coordinate systems coincide here, virtual work expression taking into account the internal and inertia forces simplifies to (the second bending term is omitted in the inertia part)

$$\delta W = -\begin{Bmatrix} \delta\theta_{Y1} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L^3} \begin{bmatrix} 4L^2 & 2L^2 \\ 2L^2 & 4L^2 \end{bmatrix} \begin{Bmatrix} \theta_{Y1} \\ \theta_{Y2} \end{Bmatrix} + \frac{\rho AL}{420} \begin{bmatrix} 4L^2 & -3L^2 \\ -3L^2 & 4L^2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_{Y1} \\ \ddot{\theta}_{Y2} \end{Bmatrix} \right) = 0.$$

- Principle of virtual work and the fundamental lemma of variation calculus give the ordinary differential equations

$$\frac{EI}{L} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{Bmatrix} \theta_{Y1} \\ \theta_{Y2} \end{Bmatrix} + \frac{\rho AL^3}{420} \begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_{Y1} \\ \ddot{\theta}_{Y2} \end{Bmatrix} = 0.$$

- Angular speeds of free vibrations $\boldsymbol{\omega}$ are the eigenvalues of $\boldsymbol{\Omega}$ which is related with the matrices of the differential equations by $\boldsymbol{\Omega}^2 = \mathbf{M}^{-1}\mathbf{K} = \mathbf{X}\boldsymbol{\omega}^2\mathbf{X}^{-1}$

$$\boldsymbol{\Omega}^2 = 120\frac{EI}{\rho AL^4}\begin{bmatrix}11 & 10\\ 10 & 11\end{bmatrix} = \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\frac{EI}{\rho AL^4}\begin{bmatrix}2520 & 0\\ 0 & 120\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}^{-1}.$$

- The latter form of $\boldsymbol{\Omega}^2$ (eigenvalue decomposition) gives

$$\omega_1 = \sqrt{2520\frac{EI}{\rho AL^4}},\ \mathbf{x}_1 = \begin{Bmatrix}1\\ 1\end{Bmatrix} \text{ and } \omega_2 = \sqrt{120\frac{EI}{\rho AL^4}},\ \mathbf{x}_2 = \begin{Bmatrix}-1\\ 1\end{Bmatrix}.$$

- Mathematica code takes into account both inertia terms

	type	properties	geometry
1	BEAM	{{E, G, ρ}, {A, I, I}, {0, 0, 0}}	Line[{1, 2}]

	$\{X,Y,Z\}$	$\{u_X,u_Y,u_Z\}$	$\{\theta_X,\theta_Y,\theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, θY[1], 0}
2	{L, 0, 0}	{0, 0, 0}	{0, θY[2], 0}

$$\left\{\left\{\omega[1] \to 6\sqrt{70}\sqrt{\frac{E\,I\left(A\,L^2+10\,I\right)}{\left(A^2\,L^6+52\,A\,L^4\,I+420\,L^2\,I^2\right)\rho}},\ \{\theta Y[1]\to 1,\ \theta Y[2]\to 1\}\right\},\right.$$

$$\left.\left\{\omega[2] \to 2\sqrt{30}\sqrt{\frac{E\,I\left(A\,L^2+42\,I\right)}{\left(A^2\,L^6+52\,A\,L^4\,I+420\,L^2\,I^2\right)\rho}},\ \{\theta Y[1]\to -1,\ \theta Y[2]\to 1\}\right\}\right\}$$

PLATE MODEL

The generic element contribution of plate is obtained by combining the virtual work expressions of thin slab and plate bending modes. Assuming that the origin of the material coordinate system is placed at the mid-plane and material properties are constants through the thickness, virtual work density is given by

$$\delta w_{\Omega}^{\text{ine}} = -\begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} t\rho \begin{Bmatrix} \ddot{u} \\ \ddot{v} \\ \ddot{w} \end{Bmatrix} - \begin{Bmatrix} \partial \delta w / \partial x \\ \partial \delta w / \partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3 \rho}{12} \begin{Bmatrix} \partial \ddot{w} / \partial x \\ \partial \ddot{w} / \partial y \end{Bmatrix}.$$

The planar solution domain (reference-plane) can be represented by triangular or rectangular elements. Interpolation of displacement components should be continuous and $w(x, y)$ should have also continuous derivatives at the element interfaces.

- Let us consider (first) the virtual work density of the inertia forces under the kinematic assumptions of the Reissner-Mindlin model $u_x = u + \theta z$, $u_y = v - \phi z$, and $u_z = w$.

$$\delta w_\Omega^{\text{ine}} = -\int_t (\rho \ddot{\vec{u}} \cdot \delta \vec{u}) dz = (\delta w_\Omega^{\text{ine}})_x + (\delta w_\Omega^{\text{ine}})_y + (\delta w_\Omega^{\text{ine}})_z, \quad \text{where}$$

$$(\delta w_\Omega^{\text{ine}})_x = -\int \delta u_x \rho \ddot{u}_x dz = -\begin{Bmatrix} \delta u \\ -\delta\theta \end{Bmatrix}^{\mathrm{T}} \int \begin{bmatrix} 1 & -z \\ -z & z^2 \end{bmatrix} \rho dz \begin{Bmatrix} \ddot{u} \\ -\ddot{\theta} \end{Bmatrix},$$

$$(\delta w_\Omega^{\text{ine}})_y = -\int \delta u_y \rho \ddot{u}_y dz = -\begin{Bmatrix} \delta v \\ \delta\phi \end{Bmatrix}^{\mathrm{T}} \int \begin{bmatrix} 1 & -z \\ -z & z^2 \end{bmatrix} \rho dz \begin{Bmatrix} \ddot{v} \\ \ddot{\phi} \end{Bmatrix},$$

$$(\delta w_\Omega^{\text{ine}})_z = -\int \delta u_z \rho \ddot{u}_z dz = -\delta w \int \rho dz \ddot{w}.$$

- Assuming that thickness and density are constants, and the origin of the $z-$axis is placed at the geometric centroid, integration over the thickness $z \in [-t/2, t/2]$ gives

$$(\delta w_\Omega^{\text{ine}})_x = -\begin{Bmatrix} \delta u \\ -\delta\theta \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} t & 0 \\ 0 & t^3/12 \end{bmatrix} \rho \begin{Bmatrix} \ddot{u} \\ -\ddot{\theta} \end{Bmatrix},$$

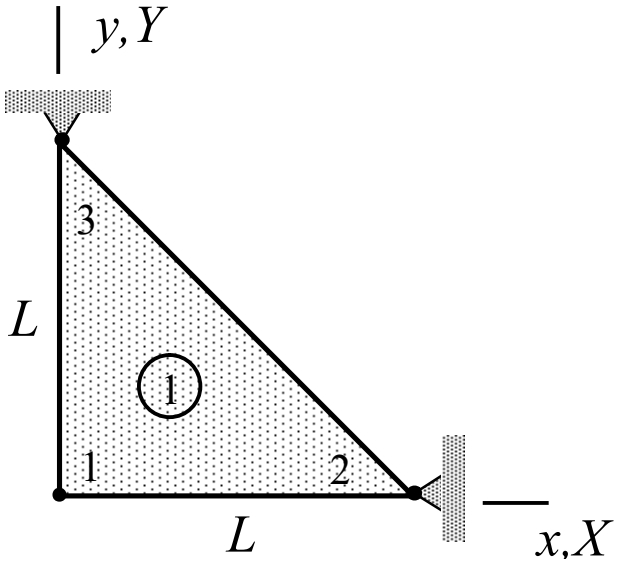
$$(\delta w_\Omega^{\text{ine}})_y = -\begin{Bmatrix} \delta v \\ \delta\phi \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} t & 0 \\ 0 & t^3/12 \end{bmatrix} \rho \begin{Bmatrix} \ddot{v} \\ \ddot{\phi} \end{Bmatrix},$$

$$(\delta w_\Omega^{\text{ine}})_z = -\delta w t \rho \ddot{w}.$$

- Summing up the terms with the Kirchhoff constraints $\phi = \partial w / \partial y$ and $\theta = -\partial w / \partial x$ (to end up with the Kirchhoff model expressions) gives the final form:

$$\delta w_\Omega^{\text{ine}} = -\begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} t\rho \begin{Bmatrix} \ddot{u} \\ \ddot{v} \\ \ddot{w} \end{Bmatrix} - \begin{Bmatrix} \partial\delta w/\partial x \\ \partial\delta w/\partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3\rho}{12} \begin{Bmatrix} \partial\ddot{w}/\partial x \\ \partial\ddot{w}/\partial y \end{Bmatrix}.$$

EXAMPLE 2.7 Consider the thin triangular structure shown. Young's modulus E, Poisson's ratio ν, and thickness h are constants. Assume plane-stress conditions and derive the ordinary differential equations giving as their solutions the free vibrations of the structure.



Answer: $$\frac{1}{4}\frac{hE}{1-\nu^2}\begin{bmatrix}3-\nu & 1+\nu\\ 1+\nu & 3-\nu\end{bmatrix}\begin{Bmatrix}u_{X1}\\ u_{Y1}\end{Bmatrix}+\frac{L^2}{12}h\rho\begin{Bmatrix}\ddot{u}_{X1}\\ \ddot{u}_{Y1}\end{Bmatrix}=0$$

- Nodes 2 are 3 are fixed and the non-zero displacements/rotations are u_{X1} and u_{Y1}. Linear shape functions $N_1 = (L-x-y)/L$, $N_2 = x/L$ and $N_3 = y/L$ are easy to deduce from the figure. Therefore

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \frac{L-x-y}{L} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} \quad \Rightarrow$$

$$\begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial x \end{Bmatrix} = -\frac{1}{L} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}, \quad \begin{Bmatrix} \partial u/\partial y \\ \partial v/\partial y \end{Bmatrix} = -\frac{1}{L} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}, \text{ and } \begin{Bmatrix} \ddot{u} \\ \ddot{v} \end{Bmatrix} = \frac{L-x-y}{L} \begin{Bmatrix} \ddot{u}_{X1} \\ \ddot{u}_{Y1} \end{Bmatrix}.$$

- Virtual work densities of internal and inertia forces are given by

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \\ \delta u_{X1} + \delta u_{Y1} \end{Bmatrix}^{\text{T}} \frac{1}{L^2} \frac{hE}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \\ u_{X1} + u_{Y1} \end{Bmatrix},$$

$$\delta w_{\Omega}^{\text{ine}} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} h\rho \left(\frac{L-x-y}{L}\right)^2 \begin{Bmatrix} \ddot{u}_{X1} \\ \ddot{u}_{Y1} \end{Bmatrix}.$$

- Integration over the triangular domain gives (integrand of the internal part is constant)

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \\ \delta u_{X1} + \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{1}{2}\frac{hE}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \\ u_{X1} + u_{Y1} \end{Bmatrix} \Leftrightarrow$$

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{1}{4}\frac{hE}{1-\nu^2}\begin{bmatrix} 3-\nu & 1+\nu \\ 1+\nu & 3-\nu \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix},$$

$$\delta W^{\text{ine}} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} h\rho \begin{Bmatrix} \ddot{u}_{X1} \\ \ddot{u}_{Y1} \end{Bmatrix} \int_0^L \int_0^{L-x} \left(\frac{L-x-y}{L}\right)^2 dydx \Rightarrow$$

$$\delta W^{\text{ine}} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{L^2}{12} h\rho \begin{Bmatrix} \ddot{u}_{X1} \\ \ddot{u}_{Y1} \end{Bmatrix}.$$

- Principle of virtual work in the form $\delta W = \delta W^{\text{int}} + \delta W^{\text{ine}} = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

$$\delta W = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \left(\frac{1}{4} \frac{hE}{1-\nu^2} \begin{bmatrix} 3-\nu & 1+\nu \\ 1+\nu & 3-\nu \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} + \frac{L^2}{12} h\rho \begin{Bmatrix} \ddot{u}_{X1} \\ \ddot{u}_{Y1} \end{Bmatrix} \right) = 0 \quad \forall \begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix} \Rightarrow$$

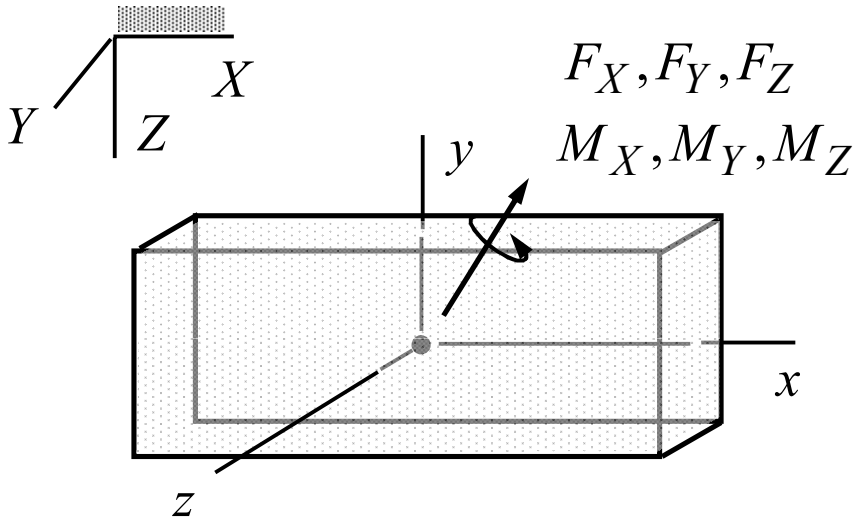
$$\frac{1}{4} \frac{hE}{1-\nu^2} \begin{bmatrix} 3-\nu & 1+\nu \\ 1+\nu & 3-\nu \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} + \frac{L^2}{12} h\rho \begin{Bmatrix} \ddot{u}_{X1} \\ \ddot{u}_{Y1} \end{Bmatrix} = 0 \,.$$

RIGID BODY

Inertia term takes into account translation and rotation parts which depend on the mass m and the 3×3 inertia matrix $\mathbf{J}$. For a sphere $\mathbf{J} = \frac{2}{5} mR^2 \mathbf{I}$ ($\mathbf{I}$ is the 3×3 unit matrix).

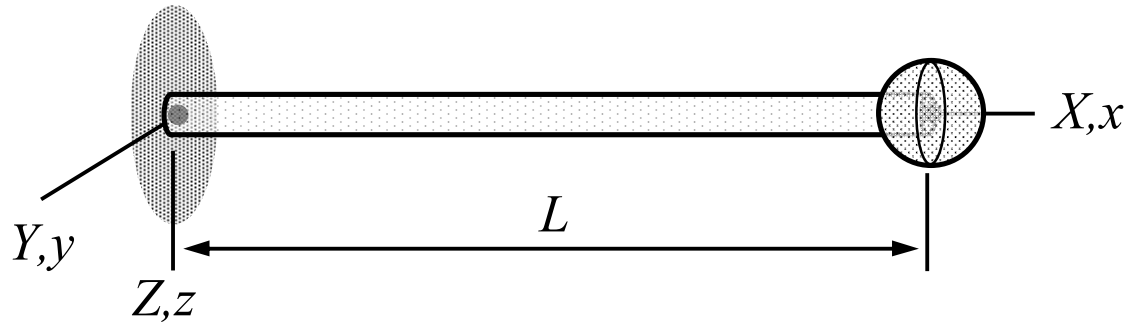
$$\delta W^{\text{ext}} = \begin{Bmatrix} \delta u_X \\ \delta u_Y \\ \delta u_Z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix} + \begin{Bmatrix} \delta\theta_X \\ \delta\theta_Y \\ \delta\theta_Z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} M_X \\ M_Y \\ M_Z \end{Bmatrix},$$

$$\delta W^{\text{ine}} = -\begin{Bmatrix} \delta u_X \\ \delta u_Y \\ \delta u_Z \end{Bmatrix}^{\mathrm{T}} m \begin{Bmatrix} \ddot{u}_X \\ \ddot{u}_Y \\ \ddot{u}_Z \end{Bmatrix} - \begin{Bmatrix} \delta\theta_x \\ \delta\theta_y \\ \delta\theta_z \end{Bmatrix}^{\mathrm{T}} \mathbf{J} \begin{Bmatrix} \ddot{\theta}_x \\ \ddot{\theta}_y \\ \ddot{\theta}_z \end{Bmatrix}.$$



The form above assumes that the first moments of mass and the off-diagonal terms of the second moments of mass vanish (origin of the material coordinate system at the center of mass etc.). Expressions for large rotations are more complex.

EXAMPLE 2.8. The mass of a cantilever (circular cross section) is negligible compared to the mass of a rigid spherical body welded to the free end. Determine the angular speeds and modes of the free vibrations. The mass of the sphere is m and the moment of inertia $\mathbf{J} = \frac{1}{5} mL^2 \mathbf{I}$.



Answer $\omega_3 = \omega_4 = \sqrt{2\frac{EI}{mL^3}}$, $\omega_5 = \omega_6 = \sqrt{30\frac{EI}{mL^3}}$, $\omega_2 = \sqrt{10\frac{GI}{mL^3}}$, and $\omega_1 = \sqrt{\frac{EA}{mL}}$

- Frequency analysis by the Mathematica code gives

$$\omega[1] \to \sqrt{\frac{A\,E}{L\,m}} \qquad \{uX[2] \to 1,\ uY[2] \to 0,\ uZ[2] \to 0,\ \theta X[2] \to 0,\ \theta Y[2] \to 0,\ \theta Z[2] \to 0\}$$

$$\omega[2] \to \sqrt{10}\ \sqrt{\frac{G\,I}{L^3\,m}} \qquad \{uX[2] \to 0,\ uY[2] \to 0,\ uZ[2] \to 0,\ \theta X[2] \to 1,\ \theta Y[2] \to 0,\ \theta Z[2] \to 0\}$$

$$\omega[3] \to \sqrt{2}\ \sqrt{\frac{E\,I}{L^3\,m}} \qquad \left\{uX[2] \to 0,\ uY[2] \to \frac{3\,L}{5},\ uZ[2] \to 0,\ \theta X[2] \to 0,\ \theta Y[2] \to 0,\ \theta Z[2] \to 1\right\}$$

$$\omega[4] \to \sqrt{2}\ \sqrt{\frac{E\,I}{L^3\,m}} \qquad \left\{uX[2] \to 0,\ uY[2] \to 0,\ uZ[2] \to -\frac{3\,L}{5},\ \theta X[2] \to 0,\ \theta Y[2] \to 1,\ \theta Z[2] \to 0\right\}$$

$$\omega[5] \to \sqrt{30}\ \sqrt{\frac{E\,I}{L^3\,m}} \qquad \left\{uX[2] \to 0,\ uY[2] \to -\frac{L}{3},\ uZ[2] \to 0,\ \theta X[2] \to 0,\ \theta Y[2] \to 0,\ \theta Z[2] \to 1\right\}$$

$$\omega[6] \to \sqrt{30}\ \sqrt{\frac{E\,I}{L^3\,m}} \qquad \left\{uX[2] \to 0,\ uY[2] \to 0,\ uZ[2] \to \frac{L}{3},\ \theta X[2] \to 0,\ \theta Y[2] \to 1,\ \theta Z[2] \to 0\right\}$$