EXERCISE SET 2,
MS-A0503, FIRST COURSE IN PROBABILITY AND STATISTICS

Explorative exercises

I will expect that you study the explorative problems BEFORE the first lecture of the week. In a manner of speaking, the lectures will contain the "solutions" to the explorative problems, meaning that by solving the problems you have taken important steps towards developing the theory on your own, before I present the theory on the lectures. For the explorative problems, it is very strongly recommended that you work on them in groups.

Problem 1. Prove that the *binomial coefficients* $\binom{n}{k}$ satisfy the identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

(1) Using the interpretation of $\binom{n}{k}$ as the number of combinations of k out of n elements.
(2) Using the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Problem 2. A school class consists of 26 unrelated children, Alice, Bob, Camilla, David etc., each of which have the same probability of being born on any of the 365 days in a year.

(1) Without counting, guesstimate the probability that two of the children share a birthday.
(2) What is the probability that Alice and Bob have the same birthday?
(3) What is the probability that some other kid has the same birthday as Alice? (hint: consider the complementary event)
(4) What is the probability that two of the children share a birthday. (hint: consider the complementary event)

Problem 3. Twins can either be identical or fraternal. Identical wins share the exact same DNA, and in particular are necessarily of the same gender, whereas fraternal twins share only as much DNA as siblings do in general, and so are equally likely to have the same or different gender. A study shows that among all pairs of twins, 64 percent share the same gender. Estimate the fraction of all twin pairs that are identical twins.

Problem 4. The deadly disease D is carried by 0.1% of the population in a country. A blood test can determine whether you have D. However, regardless of whether or not you have the disease, the test shows the wrong result 0.5% of the time. If the test tells that you carry D, what is the probability that you actually have the disease?

Problem 5. An unfair six-sided die comes up i with probability p_i, for $i = 1, \dots, 6$, where $p_1 + \cdots + p_6 = 1$. Recall the definition of the *expected value* $E(X)$ of the outcome X:

$$E(X) = 1p_1 + 2p_2 + 3p_3 + 4p_4 + 5p_5 + 6p_6.$$

Now let $(y_1, y_2, y_3, y_4, y_5)$ be the outcomes of five independent rolls of the die. Which of the following is true?

(1) The expected value of mean(y) equals $E(X)$.
(2) The expected value of median(y) equals $E(X)$.

Difficult bonus problem: For the statement(s) that are not necessarily true, describe a probability distribution for which they are not true.

1. Homework problems

The homework problems are reported during the second exercise session of the week. You are allowed and encouraged to work in groups, but every student should be prepared to present the solutions individually. During the last exercise session of the week, the teacher will ask you to mark what problems you have solved, and you get points according to how many problems you marked as solved. If you mark a problem as solved, however, you should also be prepared to present your solution in front of the class.

Homework 1. A deck of cards consists of 52 cards, labelled with one of four suits and one of thirteen ranks. A poker hand consists of five cards drawn uniformly at random from the deck.

(1) A poker hand is a "pair" if two of the cards have the same rank, and all others have different ranks. What is the probability of a pair?
(2) A poker hand is a "full house" if two of the cards have the same rank, and three other cards have the same ranks. What is the probability of a full house?
(3) A poker hand is a "flush" if all of the cards have the same suit. What is the probability of a flush?

Homework 2. A red die, a blue die, and a yellow die (all fair and six-sided) are rolled, and show the numbers R, B, Y respectively.

(1) What is the probability that no two of the dice land on the same number?
(2) Given that no two of the dice land on the same number, what is the conditional probability that $R < B < Y$?
(3) What is $P(R < B < Y)$?
(4) How many triples of integers (b, y, r) exist with $1 \leq b < y < r \leq 6$?

Homework 3. One urn has 4 red balls and 1 white ball; a second urn has 2 red balls and 3 white balls. We roll a fair die to determine which urn to take a ball from: If the die comes up 6 then we draw from the first urn, and otherwise we draw a ball from the second urn. If the drawn ball is red, what is the probability that it came out of the second urn?

Week 2, Problem 1

1) # Hands with a pair =

$$\underset{\substack{\uparrow\\\text{rank of}\\\text{pair}}}{13} \cdot \underset{\substack{\uparrow\\\text{suits}\\\text{of pair}}}{\binom{4}{2}} \cdot \underset{\substack{\uparrow\\\text{ranks of}\\\text{other cards}}}{\binom{12}{3}} \cdot \underset{\substack{\uparrow\\\text{suits of}\\\text{other cards}}}{4^3}$$

poker hands $= \binom{52}{5}$.

$$\mathbb{P}[\text{pair}] = \frac{13 \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot 4^3}{\binom{52}{5}}$$

$$= \ldots = \frac{2^5 \cdot 11}{7^2 \cdot 17} \approx 0.43$$

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2) # Full house $=$ $\underset{\substack{\uparrow\\\text{rank}\\\text{of}\\\text{triple}}}{13} \cdot \underset{\substack{\uparrow\\\text{rank}\\\text{of}\\\text{pair}}}{12} \cdot \overset{\substack{\text{suits of triple}\\\downarrow}}{\binom{4}{3}} \cdot \overset{\substack{\text{suits of}\\\text{pair}\\\downarrow}}{\binom{4}{2}}$

$$\mathbb{P}[\text{pair}] = \frac{13 \cdot 12 \cdot \binom{4}{3} \cdot \binom{4}{2}}{\binom{52}{5}} = \frac{2 \cdot 3}{5 \cdot 7^2 \cdot 17} \approx 0.0014$$

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3) # Flush $= \overset{\substack{\text{suit}\\\downarrow}}{4} \cdot \binom{13}{5} \leftarrow$ ranks

$$\mathbb{P}[\text{Flush}] = \frac{4 \cdot \binom{13}{5}}{\binom{52}{5}} = \frac{3 \cdot 11}{2^2 \cdot 5 \cdot 7^2 \cdot 17} \approx 0.0020$$

Week 2, Problem 2

1) $\mathbb{P}[\text{all dice distinct}] = \dfrac{6\cdot 5\cdot 4}{6^3} = \dfrac{5}{9}$

2) Three different possible orders, all equally likely,

so $\mathbb{P}[R<B<Y \mid R,B,Y \text{ distinct}] = \dfrac{1}{6}$.

3) $\mathbb{P}[R<B<Y] = \mathbb{P}[\text{distinct}]\cdot\mathbb{P}[R<B<Y \mid \text{distinct}]$

$$= \frac{5}{9}\cdot\frac{1}{6} = \frac{5}{2\cdot 3^3}$$

4) $\#\{(b,y,r) : 1\le b<y<r\le 6\} = \#\left\{(b,y,r) : \begin{matrix}1\le b\le 6\\ 1\le y\le 6\\ 1\le r\le 6\end{matrix}\right\}\cdot\mathbb{P}[b<y<r]$

when b,y,r uniform die rolls

$$= 6^3\cdot\frac{5}{2\cdot 3^3} = 2^2\cdot 5 = 20$$

This problem can also be solved by enumerating all 20 possibilities, or

as $\binom{6}{3} = 20$

sets of 3 distinct elements from $\{1,\dots,6\}$.

Week 2, Problem 3

Let D be the event that the die shows "6", let R be the event that we draw a red ball. We want to compute $\mathbb{P}[\bar{D} \mid R]$.

$$\mathbb{P}[D] = \frac{1}{6}$$
$$\mathbb{P}[R \mid D] = \frac{4}{5}$$
$$\mathbb{P}[R \mid \bar{D}] = \frac{2}{5}$$

So

$$\mathbb{P}[\bar{D} \mid R] = \frac{\mathbb{P}[\bar{D} \,\&\, R]}{\mathbb{P}[R]} = \frac{\mathbb{P}[\bar{D}]\,\mathbb{P}[R \mid \bar{D}]}{\mathbb{P}[D]\,\mathbb{P}[R \mid D] + \mathbb{P}[\bar{D}]\,\mathbb{P}[R \mid \bar{D}]}$$

$$= \frac{\frac{5}{6} \cdot \frac{2}{5}}{\frac{1}{6} \cdot \frac{4}{5} + \frac{5}{6} \cdot \frac{2}{5}} = \frac{10}{14} = \frac{5}{7}$$

Answer: With probability $\frac{5}{7}$ the ball came from the second urn