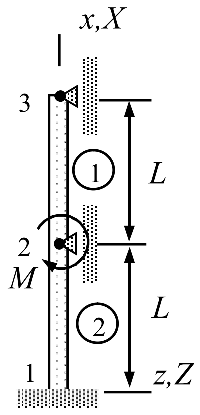
Name________________________ Student number________________



Home assignment 2

Beam structure of the figure is loaded by a point moment acting on node 2. Determine the rotations θ_{Y2} and θ_{Y3} by using two beam elements. Displacements are confined to the XZ-plane. The cross-section properties of the beam A, I and Young's modulus of the material E are constants.

Solution template

Virtual work expression for the displacement analysis consists of parts coming from internal and external forces δW^{int} and δW^{ext}. For the beam bending mode in xz-plane and cubic interpolation of the nodal values, the element contributions are

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}, \quad \delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{f_z h}{12} \begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix},$$

The element contribution of the point force/moment follows from the definition or work and is given by

$$\delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \\ \delta u_{Z1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix} + \begin{Bmatrix} \delta\theta_{X1} \\ \delta\theta_{Y1} \\ \delta\theta_{Z1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} M_X \\ M_Y \\ M_Z \end{Bmatrix}.$$

For beam 1, $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}}$ is given by

$$\delta W^1 = -\begin{Bmatrix} 0 \\ \delta\theta_{Y2} \\ 0 \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ \theta_{Y2} \\ 0 \\ \theta_{Y3} \end{Bmatrix} = -\begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 4\dfrac{EI}{L} & 2\dfrac{EI}{L} \\ 2\dfrac{EI}{L} & 4\dfrac{EI}{L} \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix},$$

For beam 2, $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}}$ is given by (when written in a form which is compatible with the first element contribution)

$$\delta W^2 = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_{Y2} \end{Bmatrix} = -\begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 4\dfrac{EI}{L} & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix}.$$

Virtual work expression of the point moment (written in a form which is compatible with the other element contributions)

$$\delta W^3 = \begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} -M \\ 0 \end{Bmatrix}.$$

Virtual work expression of structure is sum of the element contributions

$$\delta W = \delta W^1 + \delta W^2 + \delta W^3 = -\begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} 8\dfrac{EI}{L} & 2\dfrac{EI}{L} \\ 2\dfrac{EI}{L} & 4\dfrac{EI}{L} \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix} - \begin{Bmatrix} -M \\ 0 \end{Bmatrix} \right).$$

Principle of virtual work and the fundamental lemma of variation calculus imply that

$$\begin{bmatrix} 8\dfrac{EI}{L} & 2\dfrac{EI}{L} \\ 2\dfrac{EI}{L} & 4\dfrac{EI}{L} \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix} - \begin{Bmatrix} -M \\ 0 \end{Bmatrix} = 0.$$

Solution to the linear equations system is given by

$$\theta_{Y2} = -\frac{1}{7}\frac{ML}{EI} \quad \text{and} \quad \theta_{Y3} = \frac{1}{14}\frac{ML}{EI}.$$ ←