Name________________________________ Student number_____________________

# Home assignment 1

Vibration of a torsion bar is described by the second order ordinary differential equations

$$\frac{GI_{rr}}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} \theta_{X2} \\ \theta_{X3} \end{Bmatrix} + \frac{\rho I_{rr} L}{6}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_{X2} \\ \ddot{\theta}_{X3} \end{Bmatrix} = 0$$

in which $I_{rr}$ is the second moment of area with respect to the axis of the bar, $G$ is the shear modulus, and $\rho$ is the density of material. Derive the angular speeds and the corresponding modes of the free vibrations.

**Solution template**

The set of ordinary differential equations as given by the principle of virtual work $\mathbf{M}\ddot{\mathbf{a}} + \mathbf{K}\mathbf{a} = 0$ consists of the inertia and stiffness parts. The symmetric mass matrix $\mathbf{M}$ and the stiffness matrix $\mathbf{K}$ depend on the structure. Angular speeds of the free vibrations are the eigenvalues of $\mathbf{\Omega} = \sqrt{\mathbf{M}^{-1}\mathbf{K}}$. In practice, it is easier to calculate first the eigenvalues $\mathbf{\Omega}^2 = \mathbf{M}^{-1}\mathbf{K}$ as the eigenvalues of $\mathbf{\Omega}$ are the square roots of those for $\mathbf{\Omega}^2$ and the eigenvectors coincide.

In the present case, the matrices are

$$\mathbf{K} = \frac{GI_{rr}}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \text{ and } \mathbf{M} = \frac{\rho I_{rr} L}{6}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \quad \Leftrightarrow \quad \mathbf{M}^{-1} = \frac{1}{\rho I_{rr} L}\begin{bmatrix} 8/5 & -2/5 \\ -2/5 & 8/5 \end{bmatrix}.$$

Therefore

$$\mathbf{\Omega}^2 = \mathbf{M}^{-1}\mathbf{K} = \frac{G}{\rho L^2}\begin{bmatrix} 18/5 & -12/5 \\ -12/5 & 18/5 \end{bmatrix}.$$

In the eigenvalue problem of matrix $\mathbf{A}$, the goal is to find all pairs $(\lambda, \mathbf{x})$ such that $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$. The linear homogeneous equation system can have a non-zero solution only if $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. The eigenvalues are obtained as solutions to this characteristic equation. The characteristic equation for the eigenvalues of $\mathbf{\Omega}^2$ is

$$\det(\mathbf{\Omega}^2 - \lambda\mathbf{I}) = (\frac{18}{5}\frac{G}{\rho h^2} - \lambda)^2 - (\frac{12}{5}\frac{G}{\rho h^2})^2 = 0.$$

The two solutions for the eigenvalues are ($(a-\lambda)^2 - b^2 = 0 \;\Leftrightarrow\; \lambda = a \pm b$)

$$\lambda_1 = \frac{6}{5}\frac{G}{\rho L^2} \quad \text{and} \quad \lambda_2 = 6\frac{G}{\rho L^2}.$$

The corresponding eigenvectors are obtained as solutions to $(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$. The eigenvectors are not unique and it is enough to find some of them. However, the eigenvectors should be linearly independent so that, e.g., the zero vector is not a valid choice.

$$\lambda_1: \qquad \frac{G}{\rho L^2}\begin{bmatrix} 12/5 & -12/5 \\ -12/5 & 12/5 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 \quad \Rightarrow \quad \mathbf{x}_1 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix},$$

$$\lambda_2: \qquad \frac{G}{\rho L^2}\begin{bmatrix} -12/5 & -12/5 \\ -12/5 & -12/5 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 \quad \Rightarrow \quad \mathbf{x}_2 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}.$$

The representation of the matrix in terms of its eigenvalues and eigenvectors $\mathbf{\Omega}^2 = \mathbf{X}\boldsymbol{\lambda}\mathbf{X}^{-1}$ implies that $\mathbf{\Omega} = \mathbf{X}\sqrt{\boldsymbol{\lambda}}\mathbf{X}^{-1}$. As taking a square root of the diagonal matrix means just taking the square roots of the diagonal terms, the angular speeds of the free vibrations

$$\omega_1 = \sqrt{\lambda_1} = \sqrt{\frac{6}{5}\frac{G}{\rho L^2}} \quad \text{and} \quad \omega_2 = \sqrt{\lambda_2} = \sqrt{6\frac{G}{\rho L^2}}.$$ ←