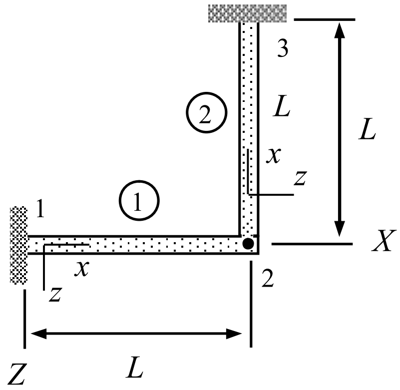
Name________________________ Student number________________

Home assignment 2

The XZ-plane structure shown consists of two beams of equal properties. Assuming that the beams are inextensible in the axial directions, derive the angular speed of free vibrations. Young's modulus E and density ρ of the beam material and the second moment of area I are constants. Assume that the inertia term due to rotation is negligible.



Solution template

As the beams are inextensible in the axial directions, the active degree of the structure is θ_{Y2}. Let us start with the element contributions. Only the virtual work expressions of the internal and inertia forces (available in the formulae collection) are needed.

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\text{T}} \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix},$$

$$\delta W^{\text{ine}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\text{T}} \left(\frac{\rho I_{yy}}{30h} \begin{bmatrix} 36 & -3h & -36 & -3h \\ -3h & 4h^2 & 3h & -h^2 \\ -36 & 3h & 36 & 3h \\ -3h & -h^2 & 3h & 4h^2 \end{bmatrix} + \frac{\rho A h}{420} \begin{bmatrix} 156 & -22h & 54 & 13h \\ -22h & 4h^2 & -13h & -3h^2 \\ 54 & -13h & 156 & 22h \\ 13h & -3h^2 & 22h & 4h^2 \end{bmatrix} \right) \begin{Bmatrix} \ddot{u}_{z1} \\ \ddot{\theta}_{y1} \\ \ddot{u}_{z2} \\ \ddot{\theta}_{y2} \end{Bmatrix},$$

Virtual work expressions of internal and inertia forces for beam 1 simplify to

$$\delta W^{\text{int}} = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Y2} \end{Bmatrix}^{\text{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_{Y2} \end{Bmatrix} = -\delta\theta_{Y2} 4\frac{EI}{L}\theta_{Y2},$$

$$\delta W^{\text{ine}} = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Y2} \end{Bmatrix}^{\text{T}} \frac{\rho A L}{420} \begin{bmatrix} 156 & -22L & 54 & 13L \\ -22L & 4L^2 & -13L & -3L^2 \\ 54 & -13L & 156 & 22L \\ 13L & -3L^2 & 22L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \ddot{\theta}_{Y2} \end{Bmatrix} = -\delta\theta_{Y2} \frac{\rho A L^3}{105} \ddot{\theta}_{Y2},$$

giving

$$\delta W^1 = -\delta\theta_{Y2}(4\frac{EI}{L}\theta_{Y2} + \frac{\rho AL^3}{105}\ddot{\theta}_{Y2}) .$$

Virtual work expressions of internal and inertia forces for beam 2 simplify to

$$\delta W^{\text{int}} = -\begin{Bmatrix} 0 \\ \delta\theta_{Y2} \\ 0 \\ 0 \end{Bmatrix}^{\text{T}} \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ \theta_{Y2} \\ 0 \\ 0 \end{Bmatrix} = -\delta\theta_{Y2} 4\frac{EI}{L}\theta_{Y2} ,$$

$$\delta W^{\text{ine}} = -\begin{Bmatrix} 0 \\ \delta\theta_{Y2} \\ 0 \\ 0 \end{Bmatrix}^{\text{T}} \frac{\rho AL}{420}\begin{bmatrix} 156 & -22L & 54 & 13L \\ -22L & 4L^2 & -13L & -3L^2 \\ 54 & -13L & 156 & 22L \\ 13L & -3L^2 & 22L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ \ddot{\theta}_{Y2} \\ 0 \\ 0 \end{Bmatrix} = -\delta\theta_{Y2}\frac{\rho AL^3}{105}\ddot{\theta}_{Y2} ,$$

giving

$$\delta W^2 = -\delta\theta_{Y2}(4\frac{EI}{L}\theta_{Y2} + \frac{\rho AL^3}{105}\ddot{\theta}_{Y2}) .$$

Virtual work expression of structure is the sum of element contributions.

$$\delta W = \delta W^1 + \delta W^2 = -\delta\theta_{Y2}(8\frac{EI}{L}\theta_{Y2} + 2\frac{\rho AL^3}{105}\ddot{\theta}_{Y2}) \quad .$$

Finally, principle of virtual work and the fundamental lemma of variation calculus implies the ordinary differential equation of the form $\ddot{\theta}_{Y2} + \omega^2\theta_{Y2} = 0$

$$\ddot{\theta}_{Y2} + 420\frac{EI}{\rho AL^4}\theta_{Y2} = 0 .$$

Therefore, the angular speed of the free vibrations ω is given by

$$\omega = \frac{2}{L^2}\sqrt{105\frac{EI}{\rho A}} . \quad \leftarrow$$