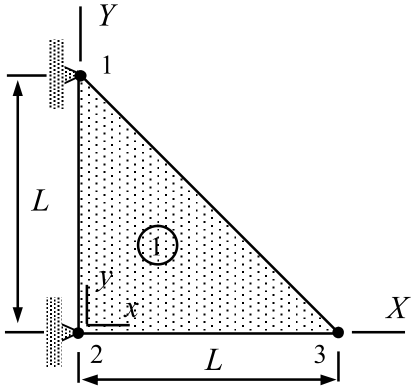
Name________________________ Student number________________

Home assignment 3



A thin triangular slab of thickness h is fixed on edge 1-2. Determine the angular speeds of free vibrations. Assume plane stress conditions and that the material properties E, ν, and ρ are constants. Use the approximations $u=(x/L)u_{X3}$ and $v=(x/L)u_{Y3}$ in which the nodal values u_{X3} and u_{Y3} are functions of time.

Solution

The virtual work densities of the internal and inertia forces for the thin slab model (plane stress conditions assumed) are given by

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\sigma} \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} \text{ and } \delta w_{\Omega}^{\text{ine}} = -\begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} t\rho \begin{Bmatrix} \ddot{u} \\ \ddot{v} \end{Bmatrix}$$

where the elasticity matrix of the plane stress

$$[E]_{\sigma} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

Expressions of linear shape functions in the material $xy-$coordinates can be deduced from the figure. Only the shape function $N_3 = x/L$ of node 3 is actually needed. Hence

$$u = \frac{x}{L}u_{X3} \quad \Rightarrow \quad \frac{\partial u}{\partial x} = \frac{1}{L}u_{X3}, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial \delta u}{\partial x} = \frac{1}{L}\delta u_{X3}, \quad \frac{\partial \delta u}{\partial y} = 0, \quad \delta u = \frac{x}{L}\delta u_{X3}, \quad \ddot{u} = \frac{x}{L}\ddot{u}_{X3}$$

$$v = \frac{x}{L}u_{Y3} \quad \Rightarrow \quad \frac{\partial v}{\partial x} = \frac{1}{L}u_{Y3}, \quad \frac{\partial v}{\partial y} = 0, \quad \frac{\partial \delta v}{\partial x} = \frac{1}{L}\delta u_{Y3}, \quad \frac{\partial \delta v}{\partial y} = 0, \quad \delta v = \frac{x}{L}\delta u_{Y3}, \quad \ddot{v} = \frac{x}{L}\ddot{u}_{Y3}$$

When the approximation is substituted there, virtual work density of internal forces simplifies to

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta u_{X3} \\ 0 \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{1}{L}\frac{hE}{2(1-\nu^2)}\begin{bmatrix} 2 & 2\nu & 0 \\ 2\nu & 2 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix}\frac{1}{L}\begin{Bmatrix} u_{X3} \\ 0 \\ u_{Y3} \end{Bmatrix} \quad \Leftrightarrow$$

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{hE}{2L^2(1-\nu^2)} \begin{bmatrix} 2 & 0 \\ 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix}.$$

As the integrand is constant, integration over the triangular domain gives

$$\delta W^{\text{int}} = \int_A \delta w_{\Omega}^{\text{int}} \mathrm{d}A = \delta w_{\Omega}^{\text{int}} \frac{L^2}{2} = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{hE}{4(1-\nu^2)} \begin{bmatrix} 2 & 0 \\ 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix}.$$

Virtual work density of the inertia forces simplifies to

$$\delta w_{\Omega}^{\text{ine}} = -\begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} h\rho \begin{Bmatrix} \ddot{u} \\ \ddot{v} \end{Bmatrix} = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} h\rho\left(\frac{x}{L}\right)^2 \begin{Bmatrix} \ddot{u}_{X3} \\ \ddot{u}_{Y3} \end{Bmatrix}.$$

Integration over the domain occupied by the element gives

$$\delta W^{\text{ext}} = \int_A \delta w_{\Omega}^{\text{ine}} \mathrm{d}A = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \int_0^L \left(\int_0^{L-x} h\rho\left(\frac{x}{L}\right)^2 dy\right) dx \begin{Bmatrix} \ddot{u}_{X3} \\ \ddot{u}_{Y3} \end{Bmatrix} = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{\rho h L^2}{12} \begin{Bmatrix} \ddot{u}_{X3} \\ \ddot{u}_{Y3} \end{Bmatrix}.$$

Virtual work expression of the structure takes the form

$$\delta W = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \left(\frac{hE}{4(1-\nu^2)} \begin{bmatrix} 2 & 0 \\ 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix} + \frac{\rho h L^2}{12} \begin{Bmatrix} \ddot{u}_{X3} \\ \ddot{u}_{Y3} \end{Bmatrix} \right).$$

Principle of virtual work $\delta W = 0 \; \forall \delta \mathrm{a}$ and the fundamental lemma of variation calculus give

$$\frac{hE}{4(1-\nu^2)} \begin{bmatrix} 2 & 0 \\ 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix} + \frac{\rho h L^2}{12} \begin{Bmatrix} \ddot{u}_{X3} \\ \ddot{u}_{Y3} \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} \ddot{u}_{X3} \\ \ddot{u}_{Y3} \end{Bmatrix} + \frac{3E}{\rho L^2(1-\nu^2)} \begin{bmatrix} 2 & 0 \\ 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix} = 0.$$

As the two ordinary differential equations are not connected and of the form $\ddot{u} + \omega^2 u = 0$, the angular speeds of free vibrations are

$$\omega_1 = \sqrt{\frac{6E}{\rho L^2(1-\nu^2)}} \quad \text{and} \quad \omega_2 = \sqrt{\frac{3E}{\rho L^2(1+\nu)}}.$$