MS-E1991 Calculus Of Variations
Lebesgue Differentiation Theorem

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Abstract

In this project work, we study the set of non-Lebesgue points of Newtonian functions when $1 < p < \infty$.

1 Lebesgue's Differentiation Theorem

In this section we give some background.

Theorem 1.1. *(Lebesgue's differentiation theorem) Let $f : X \to Y$, Y is a Banach space, be a locally integrable function in a doubling metric measure space (X, d, μ). Then*

$$\lim_{r\to 0} \fint_{B(x,r)} f(y)d\mu(y) = f(x) \tag{1.1}$$

for μ-almost every $x \in X$.

This theorem is known as the classic Lebesgue differentiation theorem which states that the derivative of the integral exists and is equal to $f(x)$ at almost every point $x \in X$. The theorem asserts that almost every point is a Lebesgue point for a locally integrable function. By Lebesgue points, we mean

Definition 1.2. *(Lebesgue point) A point $x \in X$ is a Lebesgue point of a locally integrable function $f : X \to Y$, if*

$$\lim_{r\to 0} \fint_{B(x,r)} |f(y) - f(x)|\, d\mu(y) = 0 \tag{1.2}$$

Clearly, (1.2) implies (1.1). In general, (1.2) claims that the average $|f - f(x)|$ are small on a small balls centered at x. In other words, function f does not oscillate too much at its Lebesgue points in an average sense.

The concept of Lebesgue points is a weaker property of continuity, i.e., Eq. (1.2) is a form of continuity in integral average sense. A continuous function have Lebesgue point everywhere, however, the converse is not true. The example below will illustrate the claim.

Example 1.3. *Let* $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\sum_{i=1}^{\infty}u_n(x)$, *where*

$$u_n(x)=\begin{cases}2n^3x-2n^2, & \textit{if}\ \ \frac{1}{n}\le x\le \frac{1}{n}+\frac{1}{2n^3},\\ -2n^3x+2n^2+2, & \textit{if}\ \ \frac{1}{n}+\frac{1}{2n^3}\le x\le \frac{1}{n}+\frac{1}{n^3},\\ 0, & \textit{otherwise.}\end{cases}$$

The above function is discontinuous at the origin, $x=0$ since $f(0)=0$ and arbitrarily close to 0 there are point where $f(x)=1$, but still have Lebesgue points everywhere even at $x=0$.

2 Lebesgue points and Sobolev Spaces

In the preceding section we have seen that functions in L^1_{loc} have Lebesgue points almost everywhere, that is the set of non-Lebesgue points has μ-measure zeros. However, even a set of measure zero can be relatively large. Naturally a question arises that if the function is more regular, is the exceptional set smaller? To answer such question, we study Lebesgue points for Sobolev spaces on a metric measure space. Sobolev functions are defined only up to a set of measure zero, but they can be defined pointwise up to a set of capacity zero. The existence of Lebesgue points is proven to be outside a set of capacity zero for such functions. Thus the concept of capacity plays a key role in understanding the pointwise behaviour of Sobolev functions.

Throughout this work, we denote $X=(X,d,\mu)$ a metric space endowed with metric d and be a nontrivial locally finite outer Borel regular measure μ on X. Recall that a metric measure space equipped with a doubling measure μ implies σ-finite and thus X can be written as a countable union of balls of finite measure. The locally finite property means that for every point $x\in X$ there is an $r>0$ such that $\mu(B(x,r))<\infty$. The outer measure μ is Borel regular if it is a Borel measure and for every $E\subset X$ there is a Borel set $B\subset X$ such that $E\subset B$ and $\mu(E)=\mu(B)$. The measure μ is said to be doubling if there exists a constant $c_\mu\ge 1$, called doubling constant of μ, such that

$$\mu(B(x,2r))\le c_\mu\mu(B(x,r)) \tag{2.1}$$

for every ball in X. An iteration of the doubling property implies that if $B(y,R)$ is a ball in X, $x\in B(y,R)$ and $0<r<R<\infty$, then

$$\frac{\mu(B(x,r))}{\mu(B(y,R))}\le c\Big(\frac{r}{R}\Big)^Q \tag{2.2}$$

for some $c=c(c_\mu)$ and $Q=\log c_\mu/\log 2$. The exponent Q plays as a counterpart of dimension related to the measure.

An ε-*separated set*, $\varepsilon>0$, in a metric space is a set such that every two distinct points in the set have distance at least ε. A metric space X is called *doubling with constant* N, where $N\ge 1$ is an integer, if, for every ball $B(x,r)$, every $r/2$-separated subset of $B(x,r)$ has at most N points. It is clear that every subset of a doubling space is doubling with the same constant. The motivation behind this is that we want to show that if a doubling metric space X is equipped with a nontrivial locally finite doubling measure then X is separable.

Lemma 2.1. *Assume that X is a doubling space with a constant N , then every ball in X can be covered by at most C^k balls of radius $2^{-k}r$, where $k > 1$ is an integer.*

The above lemma asserts that if X is a doubling metric space, for each $K \geq 1$ there is a constant $C_K > 1$ such that for every $r > 0$ we can find a countable cover of X of form $\{B(x_i, r)\}_i$ such that

$$\sum_{i=1} \chi_{B(x_i,Kr)} \leq C_K. \tag{2.3}$$

This means every point $x \in X$ is contained in at most C_K balls of Kr radius.

Lipschitz partition of unity We can find a partition of unity subordinate to the above cover: for every i there is a C/r-Lipschitz function $\varphi_{r,i} : X \to [0, 1]$ such that the support of $\varphi_{r,i}$ lies in $B(x_i, 2r)$ and $\sum_{i=1} \varphi_{r,i} \equiv 1$. The construction of this partition of unity can be found section 4.1.

Discrete convolution We define a discrete convolution of a measurable function $u : X \to Y$

$$u_r(x) := \sum_{i=1} \varphi_{r,i}(x) u_{B(x_i,r)}. \tag{2.4}$$

Discrete maximal function Let $r_j, j = 1, 2, \dots,$ be enumeration of the positive rationals. Observe that for each of such radius we can choose a covering $\{B(x_i, r_j)\}$ of X as above. Define the discrete maximal function

$$M^*u(x) := \sup_j |u|_{r_j}(x) \tag{2.5}$$

for every $x \in X$.

Recall the definitions

Definition 2.2. *(Maximal Function) For $f \in L^1_{loc}(X)$, the maximal function is*

$$Mf(x) := \sup_B \fint_{B(x,r)} |f(y)|\, d\mu(y), \tag{2.6}$$

where the supremum is taken over all balls B centered x.

Definition 2.3. *(Upper gradient) . A non-negative Borel function g on X is an upper gradient of an extended real-valued function f on X if for all paths $\gamma : [0, l_\gamma] \to X$,*

$$|f(\gamma(0)) - f(\gamma(l_\gamma))| \leq \int_\gamma g\, ds, \tag{2.7}$$

whenever both $f(\gamma(0))$ and $f(\gamma(l_\gamma))$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise.

If (2.7) holds for p-almost every path, then g is a p-weak upper gradient of f. By saying this, we mean the assertion fails only for a path family with zero p-modulus. Further more, if $g \in L^p(X)$ is a p-upper gradient of f, then there exist a sequence $\{g_i\}_{i=1}^{\infty}$ of upper gradients of f such that $g_i \to g$ in $L^p(X)$ of f. And if f has an upper gradient in $L^p(X)$, then it has a minimal p-weak upper gradient $g_f \in L^p(X)$ in the sense that for every p-weak upper gradient $g \in L^p(X)$ of f, $g_f \leq g$ a.e.

Definition 2.4. *(Newtonian Space) The Newtonian space or Sobolev space on a metric measure space X is the quotient space*

$$N^{1,p}(X) = \{u \in L^p(X) : \|u\|_{N^{1,p}(X)} < \infty\}/\sim, \tag{2.8}$$

where

$$\|u\|_{N^{1,p}(X)} = \Big(\int_X |u|^p \, d\mu + \inf_g \int_X g^p \, d\mu \Big)^{1/p}, \tag{2.9}$$

and $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$.

Definition 2.5. *A space X is said to support a $(1,p)$- Poincaré inequality if there exist a constants $C > 0$ and $\lambda \geq 1$ such that for all balls $B \subset X$, all integrable functions u on X and for all upper gradients g of u*

$$\fint_{B(x,r)} |u - u_B|^q \, d\mu \leq C(diamB)\Big(\fint_{B(x,\lambda r)} g_u^p \, d\mu \Big)^{1/p}. \tag{2.10}$$

By Hölder's inequality, one can show that if X is equipped with a doubling measure μ and X supports a $(1,p)$-Poincaré inequality, then X also supports (q,p)-Poincaré inequality for some $q > p$.

In particular, if X supports a $(1,p)$-Poincaré inequality, $1 < p < \infty$, and the measure is doubling, it follows that Lipschitz functions are dense in $N^{1,p}(X)$. This means that $N^{1,p}(X)$ can be characterized as the completion of $C(X) \cap N^{1,p}(X)$ with respect to the norm (2.9). In fact, the Sobolev space $N^{1,p}(X), 1 < p < \infty$ with the norm (2.9) is a Banach space. It is also worth noticing that this space is closed under taking maximum and minimum over finitely many functions. In general, a doubling space may not be complete.

Another result obtained from a doubling metric measure space supporting a $(1,p)$-Poincaré inequality is that it implies a Sobolev-Poincaré inequality. In particular, if $1 < p < Q, Q \geq 1$, there is a constant $C > 0$ and $\lambda \geq 1$ such that

$$\Big(\fint_{B(x,r)} |u - u_B|^{p^*} \, d\mu\Big)^{1/p^*} \leq C(diamB)\Big(\fint_{B(x,\lambda r)} g_u^p d\mu \Big)^{1/p}. \tag{2.11}$$

where $p^* = pQ/(Q-p)$ and the constant C depends on p, p^* and c_μ.

Definition 2.6. *(Sobolev p-Capacity) The Sobolev p-capacity of a set $E \subset X$ is the number*

$$C_p(E) = \inf\{\|u\|^p_{N^{1,p}(X)} : u \in \mathcal{A}(E)\}, \tag{2.12}$$

where

$$\mathcal{A}(E) = \{u \in N^{1,p}(X) : u \geq 1 \text{ on the neighbourhood of } E\}. \tag{2.13}$$

If $\mathcal{A}(E) = \emptyset$, we set $C_p(E) = \infty$. The Sobolev capacity is monotone and countably subadditive set function. It is easy to see that the Sobolev capacity is an outer capacity, which means that

$$C_p(E) = \inf\{C_p(V)V \supset E, V \text{ open}\}. \tag{2.14}$$

The capacity measures the exceptional sets for Sobolev functions and is obviously dependent on p. A property that holds everywhere but on a set of capacity zero is said to hold quasi-everywhere or q.e. for short.

Proposition 2.7. *Let $\{u_i\}$ be a sequence of functions in $N^{1,p}(X)$ with $\{g_{ui}\}$ a corresponding p-weak upper gradient sequence. If $u_i \to u$ in $L^p(X)$ and if $g_{u_i} \to g_u$ in $L^p(X)$ then u has a representative in $N^{1,p}(X)$ with each Borel representative of g_u as its p-weak upper gradient. Moreover, a subsequence of $\{u_i\}$ converges pointwise to this representative of u outside a set of p-capacity zero.*

Lemma 2.8. *If $0 < r < 1$ and $x \in X$ then*

$$C_p(B(x,r)) \le Cr^{-p}\mu(B(x,y)).$$

Proof. Let $u : X \to \mathbb{R}$ be the Lipschitz function given by

$$u(x) = \begin{cases} 1, & \text{if } y \in B(x,r), \\ 2 - \frac{|x-y|}{r}, & \text{if } y \in B(x,2r) \backslash B(x,r), \\ 0, & \text{if } y \in X \backslash B(x,2r), \end{cases}$$

then $0 \le u \le 1$ on X and u is supported on $B(x,2r)$. Moreover, u is $\frac{1}{r}-$Lipschitz. Thus

$$\begin{aligned} \mathrm{C}_p(B(x,r)) \le \|u\|^p_{N^{1,p}(X)} &= \int_{B(x,2r)} |u|^p \, d\mu + \inf \int_{B(x,2r)} g^p d\mu \\ &\le (1 + \frac{1}{r^p})\mu(B(x,2r)) \le (\frac{1}{r^p} + \frac{1}{r^p})\mu(B(x,2r)) \\ &\le Cr^{-p} \left| B(x,r) \right| . \end{aligned}$$

□

Lemma 2.9. *If f is a nonnegative function in $L^1_{loc}(X)$ and*

$$E = \{x \in X : \limsup_{r \to 0^+} r^p \fint_{B(x,r)} f d\mu > 0\}$$

then $C_p(E) = 0$.

Proof. We show the case $f \in L^1(X)$. Let $\epsilon > 0$ and

$$E_\epsilon = \{x \in X : \lim_{r \to 0^+} r^p \fint_{B(x,r)} f d\mu > \epsilon\}.$$

It suffices to show that $\mathrm{C}_p(E_\epsilon) = 0$ for every $\epsilon > 0$, then the claim follows by subadditivity. Recall that, by the absolute continuity of integrals, for every $\epsilon > 0$ there exists $\tau > 0$ such that whenever $A \subset X$ is a measurable set with $\mu(A) < \tau$, then $\int_A f d\mu < \epsilon$. Fix $\epsilon_1 > 0$, let τ be as above and chose $0 < \delta < 1/5$ such that

$$\frac{\delta^p}{\epsilon} \int_X f d\mu < \tau.$$

Note that for every $x \in E_\epsilon$ there is some r_x with $0 < r_x \leq \delta$ such that

$$r_x^p \fint_{B(x,r_x)} f d\mu < \epsilon.$$

We can cover E_ϵ by such balls and by the Vitali covering theorem, there exists a subfamily of countably many pairwise disjoint balls $B(x_i, r_i), i = 1, 2, ...,$ such that

$$E_\epsilon \subset \cup_i B(x_i, 5r_i).$$

By subadditivity of the capacity

$$\begin{aligned}\mathrm{C}_p(E_\epsilon) &\leq \sum_i \mathrm{C}_p(B(x_i, 5r_i)) \leq C \sum_i \frac{\mu(B_i)}{r_i^p} \\ &\leq \frac{C}{\epsilon} \sum_i \int_{B_i} f d\mu = \frac{C}{\epsilon} \int_{\cup_i B_i} f d\mu.\end{aligned}$$

On the other hand,

$$\mu(\cup_i B_i) = \sum_i \mu(B_i) \leq \sum_i \frac{r_i^p}{\epsilon} \int_{B_i} f d\mu \leq \frac{\delta^p}{\epsilon} \int_X f d\mu < \tau. \tag{2.15}$$

Therefore,

$$\mathrm{C}_p(E_\epsilon) \leq \frac{C}{\epsilon} \epsilon_1 \to 0 \text{ as } \epsilon_1 \to 0.$$

□

Lemma 2.10. *There is a constant $C > 1$, which depend only on the doubling constant c_μ, such that, for every measurable function $u : X \to Y$,*

$$C^{-1} Mu \leq M^* u \leq CMu. \tag{2.16}$$

Proof. We begin by proving the second inequality. Let r_j be a positive rational number and $x \in X$. Then

$$|u|_{r_j}(x) = \sum_i \varphi_{r_j,i}(x) \, |u|_{B(x_i,r_j)} \, . \tag{2.17}$$

Observe that if i is such that $\varphi_{r_j,i}(x) \neq 0$ then $x \in B(x_i, 2r_j)$ and $B(x_i, 2r_j) \subset B(x, 4r_j)$, we have by the doubling condition of μ that

$$|u|_{r_j}(x) = \sum_i \varphi_{r_j,i}(x) \, |u|_{B(x_i,4r_j)} \frac{\mu(B(x, 4r_j))}{\mu(B(x, r_j))} \leq C \, |u|_{B(x_i,4r_j)} \leq CMu(x), \tag{2.18}$$

Taking supremum over j on the left side yields the last inequality. Here C depends only on the doubling constant c_μ.

To prove the first inequality, we observe that if $r > 0$, there are some r_j such that $r_j/4 \le r \le r_j/2$. We denote the set

$$I_j(x) = \{i \in \mathbb{N} : B(x_i, r_j) \cap B(x, r) \neq \emptyset\}$$

By the doubling property of μ, I_j is a nonempty finite set for every $x \in X$. For every $i \in I_j(x)$ we have $B(x, r) \subset B(x_i, 2r_j)$ and $B(x_i, r_j) \subset B(x, 6r)$ which imply

$$|u|_{B(x,r)} \le \sum_{i \in I_j(x)} \varphi_{r_j,i}(x) \frac{B(x_i, 2r_j)}{B(x, r)} |u|_{B(x_i,2r_j)} = C\, |u|_{2r_j} \le cM^*u(x). \tag{2.19}$$

The claim follows by taking the supremum over all $r > 0$ on the left hand side, where C depends on the doubling constant c_μ.

□

Proposition 2.11. *Suppose that $p > 1$, $u \in N^{1,p}(X), g_u \in L^p(X)$ is the minimal p-weak upper gradient of u, and X supports a q-Poincaré inequality for some $1 \le q < p$. Then for every $r > 0$ we have that the discrete convolution $u_r \in N^{1,p}(X)$ and that there is a constant $C > 0$, independent of u and r, such that $C(Mg_u^q)^{1/q} \in L^p(X)$ is a p-weak upper gradient of u_r. Moreover, $M^*u \in N^{1,p}(X)$ with $C(Mg_u^q)^{1/q}$ as a p-weak upper gradient.*

Proof. We have

$$u_r(x) = \sum_{i=1} \varphi_{r,i}(x) u_{B(x_i,r)} = u(x) + \sum_{i=1} \varphi_{r,i}(x)(u_{B(x_i,r)} - u(x). \tag{2.20}$$

Since at each x the sum is only over finitely many balls then the series clearly converges. We want to show that

$$g_{u_r}(x) = g_u + \sum_{i=1} \big(\frac{C}{r} \left|u_{B(x_i,r)} - u\right| + g_u\big)\chi_{B(x_i,2r)} \tag{2.21}$$

is a p-weak upper gradient of u_r. Note that the sum is locally finite.

Let $x \in B(x_i, 2r)$, then by triangle inequality

$$\left|u(x) - u_{B(x_i,r)}\right| \le \left|u(x) - u_{B(x,4r)}\right| + \left|u_{B(x,4r)} - u_{B(x_i,r)}\right| \tag{2.22}$$

The second term on the right side is estimated by the Poincaré inequality and the doubling condition as

$$\begin{aligned} \left|u_{B(x,4r)} - u_{B(x_i,r)}\right| &= \left|\fint_{B(x_i,r)} (u - u_{B(x,4r)}) d\mu\right| \le \fint_{B(x_i,r)} \left|u - u_{B(x,4r)}\right| d\mu \\ &\le \frac{\mu(B(x, 4r))}{\mu(B(x_i, r))} \fint_{B(x,4r)} \left|u - u_{B(x,4r)}\right| d\mu \le Cr\big(\fint_{B(x,4\lambda r)} g_u^q d\mu\big)^{1/q} \\ &\le Cr\big(Mg_u^q(x)\big)^{1/q}. \end{aligned} \tag{2.23}$$

The first term on the right side is estimated by a standard telescoping argument. Write $B_j = B(x, 2^{2-j}r)$ for each nonnegative integer i. Since μ-almost every point is a Lebesgue point for u, $\lim_{j\to\infty} u_{B_j} = u(x)$. By doubling property of μ and Poincaré inequality we have

$$\begin{aligned} \left|u(x) - u_{B(x,4r)}\right| &\le \sum_{j=0}^{\infty} \left|u_{B_j} - u_{B_{j+1}}\right| \le \sum_{j=0}^{\infty} \fint_{B_{j+1}} \left|u - u_{B_j}\right| d\mu \\ &\le C \sum_{j=0}^{\infty} \fint_{B_j} \left|u - u_{B_j}\right| d\mu \\ &\le C \sum_{j=0}^{\infty} 2^{-j} r \big(\fint_{\lambda B_j} g^q d\mu \big)^{1/q} \le Cr\big(Mg_u^q(x)\big)^{1/q}. \end{aligned} \tag{2.24}$$

Therefore, for μ-a.e. $x \in B(x_i, 2r)$,

$$\left|u(x) - u_{B(x_i,r)}\right| \le Cr\big(Mg_u^q(x)\big)^{1/q}. \tag{2.25}$$

By Lebesgue differentiation theorem, we observe that $g_u(x) \le \big(Mg_u^q(x)\big)^{1/q}$ for μ-a.e. $x \in X$. Thus $C\big(Mg_u^q\big)^{1/q}$ is a p-weak upper gradient of u_r. Moreover, this function is p-integrable since the maximal function theorem shows that there is $C = c(p, c_\mu) > 0$ such that

$$\left\|(Mg_u^q)^{1/q}\right\|_{L^p(X)} \le C \left\|(g_u^q)^{1/q}\right\|_{L^p(X)} \le C \left\|g\right\|_{L^p(X)}. \tag{2.26}$$

Given k, there is at most C balls $B(x_i, 2r)$ intersect the ball $B(x_k, r)$,

$$\begin{aligned} \int_{B(x_k,r)} |u_r|^p \, d\mu &\le \int_{B(x_k,r)} \sum_{i=1} |\varphi_{r,i}(x)|^p \left|u_{B(x_i,r)}\right|^p d\mu \\ &\le C \sum_i \frac{\mu(B(x_k,r))}{\mu(B(x_i,r))} \int_{B(x_i,r)} |u|^p \, d\mu \\ &\le C \int_{B(x_k,3r)} |u|^p \, d\mu. \end{aligned} \tag{2.27}$$

The second inequality follows by Hölder inequality. Summing over k yields that $u_r \in N^{1,p}(X)$,

$$\begin{aligned} \int_X |u_r|^p \, d\mu &\le \sum_k \int_{B(x_k,r)} |u_r|^p \, d\mu \le \sum_k C \int_{B(x_k,3r)} |u|^p \, d\mu \\ &\le \int_X \sum_k C\chi_{B(x_k,3r)} |u|^p \, d\mu \le C \left\|u\right\|_{L^p(X)}^p. \end{aligned} \tag{2.28}$$

Towards the last claim, since $|u| \in N^{1,p}(X)$ with g_u as a p-weak upper gradient of $|u|$. From the first part of our claim, for each j we have that $|u|_{r_j} \in N^{1,p}(X)$

with $C\big(Mg_u^q\big)^{1/q}$ as a p-weak upper gradient. For $k \in \mathbb{N}$, define

$$v_k = \max_{1\leq j\leq k} |u|_{r_j} .$$

Then $v_k \in N^{1,p}(X)$ with the same p-weak upper gradient. By Lemma 2.10 we infer that $M^*u \in L^p(X)$ and hence, by the monotone convergence theorem, $v_k \to M^*u$ in $L^p(X)$. By the second part of proposition , $M^*u \in N^{1,p}(X)$ with $C(Mg_u^q)^{1/q}$ as a p-integrable p-weak upper gradient.

□

Lemma 2.12. *Suppose that $p > 1$ and that X supports a q-Poincaré inequality for some $1 \leq q < p$. If $u \in N^{1,p}(X)$ then, for every $\lambda > 0$,*

$$C_p(\{x \in X : Mu(x) > \lambda\}) \leq \frac{C}{\lambda^p} \|u\|^p_{N^{1,p}(X)} .$$

Proof. Let

$$E_\lambda = \{x \in X : CM^*u(x) \geq \lambda\},$$

where C is the comparison constant from Lemma 2.10. E_λ is open by lover semicontinuous of M^*u .Then

$$\{x \in X : Mu(x) \geq \lambda\} \subset E_\lambda, \tag{2.29}$$

hence the desired p-capacity is estimated from above by $\mathrm{C}_p(E_\lambda)$. Since $(C/\lambda M^*u) \in N^{1,p}(X)$ and hence is admissible for estimating the p-capacity of E_λ. With exponents $p > 1$ and $p/q > 1$,

$$\begin{aligned}\mathrm{C}_p(E_\lambda) &\leq \left\|\frac{C}{\lambda} M^*u\right\|^p_{N^{1,p}(X)} \leq \frac{C}{\lambda^p}\big(\, \|M^*u\|^p_{L^p(X)} + \left\|(Mg_u^q)^{1/q}\right\|^p_{L^p(X)}\big) \\ &\leq \frac{C}{\lambda^p}\big(\, \|u\|^p_{L^p(X)} + \|g_u\|^p_{L^p(X)}\big) \leq \frac{C}{\lambda^p} \|u\|^p_{N^{1,p}(X)} .\end{aligned}$$

By taking the infimum over all maximal gradients of u on the right hand side, the claim follows. □

Theorem 2.13. *Suppose that $p > 1$ and that X supports a q-Poincaré inequality for some $1 \leq q < p$, and $Q \geq 1$. If $u \in N^{1,p}(X)$, then p-q.e. point in X is a Lebesgue point of u. Furthermore, if $p < Q$ then, for p-q.e. $x \in X$,*

$$\lim_{r\to 0} \fint_{B(x,r)} |u - u(x)|^{p^*} \, d\mu = 0 \tag{2.30}$$

where

$$p^* = \frac{pQ}{Q-p}.$$

Proof. Let
$$A = \{x \in X : \limsup_{r\to 0} r^p \fint_{B(x,r)} g_u^p d\mu > 0\}$$
Since $g_u \in L^p(X)$, we have $g_u^p \in L^1(X)$ and hence by Lemma 2.9, $\mathrm{C}_p(A) = 0$. By Poincaré inequality, if $x \in X\backslash A$ then

$$(\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu)^p \le Cr^p \fint_{B(x,\lambda r)} g_u^p d\mu \to 0 \tag{2.31}$$

as $r \to 0$, that is,
$$\lim_{r\to 0} \fint_{B(x,r)} |u - u_{B(x,r)}| = 0 \tag{2.32}$$

whenever $x \in X\backslash A$. Since X supports a p-Poincaré inequality, it follows that Lipschitz functions are dense in $N^{1,p}(X)$. Let $\{u_j\}$ be a sequence of Lipschitz function in $N^{1,p}(X)$ such that

$$\|u - u_j\|^p_{N^{1,p}(X)} \le 2^{-j(p+1)},$$

for each j and there exists a set K with $\mathrm{C}_p(K) = 0$ for which $u_j \to u$ pointwise everywhere in $X\backslash K$. Such a sequence exists because of proposition 2.7. For $j \in \mathbb{N}$, let
$$A_j = \{x \in X : M(u - u_j)(x) > 2^{-j}\},$$
and set $E_j = A \cup K \cup \big(\cup_{k>j} A_k\big)$. By Lemma 2.12,

$$\mathrm{C}_p(A_j) \le \frac{C}{2^{-jp}} \|u - u_j\|^p_{N^{1,p}(X)} \le \frac{C}{2^{-jp}} 2^{-j(p+1)} = 2^{-j}C.$$

Then, by the subadditivity of the p-capacity

$$\mathrm{C}_p(E_j) \le 2 \times 2^{-j}C.$$

Note that

$$\begin{aligned} |u_k - u_{B(x,r)}| &\le \fint_{B(x,r)} |u - u_k(x)| \, d\mu \\ &\fint_{B(x,r)} |u_k - u| \, d\mu + \fint_{B(x,r)} |u_k - u_k(x)| \, d\mu \\ &\le M(u_k - u)(x) + \fint_{B(x,r)} |u_k - u_k(x)| \, d\mu. \end{aligned}$$

Hence, if $x \in X\backslash E_j$ and $k > j$ then

$$\begin{aligned} \limsup_{r\to 0} |u_k(x) - u_{B(x,r)}| &\le \limsup_{r\to 0} \fint_{B(x,r)} |u - u_k(x)| \, d\mu \\ &\le M(u_k - u)(x) \le 2^{-k}. \end{aligned} \tag{2.33}$$

Therefore, for every $x \in X \backslash E_j$ and for every $l \geq k \geq j$,

$$|u_k - u_l(x)| \leq \limsup_{r\to 0} \left|u_k(x) - u_{B(x,r)}\right| + \limsup_{r\to 0} \left|u_l(x) - u_{B(x,r)}\right| \leq 2^{1-k},$$

which shows that $\{x_k\}$ converges uniformly on $X \backslash E_j$ to u. (Note that, as $K \subset E_j, u_j \to u$ pointwise on $X \backslash E_j$). Thus, it follows that u is continuous on $X \backslash E_j$. Moreover, by the estimate in (2.33) if $x \in X \backslash E_j$ and $k \geq j$ then, for $l \geq k$,

$$\begin{aligned}
&\limsup_{r\to 0} ⨍_{B(x,r)} |u - u(x)|\, d\mu \\
&\qquad \leq \limsup_{r\to 0} ⨍_{B(x,r)} |u - u_k(x)|\, d\mu + |u_k(x) - u(x)| \\
&\qquad \leq 2^{-k} + |u_k(x) - u(x)|
\end{aligned}$$

and since $u_k(x) \to u(x)$ as $k \to \infty$, we see that

$$\limsup_{r\to 0} ⨍_{B(x,r)} |u - u(x)|\, d\mu \to 0.$$

Thus, each point $x \in X \backslash E_j$ is a Lebesgue point of u.

In the case where $p < Q$, for every $x \in X \backslash E_j$, we can apply the Sobolev-Poincaré inequality (2.11) instead of Poincaré inequality to estimate

$$⨍_{B(x,r)} |u - u(x)|^{p^*}\, d\mu \leq Cr\big(⨍_{B(x,\lambda r)} g_u^p d\mu\big)^{p^*/p} \to 0$$

as $r \to 0$. Thus we get

$$\limsup_{r\to 0} ⨍_{B(x,r)} |u - u(x)|^{p^*}\, d\mu = 0.$$

Hence, for $x \in X \backslash E_j$, using a fact that x is a Lebesgue point of u

$$\begin{aligned}
&\limsup_{r\to 0} ⨍_{B(x,r)} |u - u(x)|^{p^*}\, d\mu \\
&\qquad \leq 2^{p^*} \lim_{r\to 0} ⨍_{B(x,r)} \left|u - u_{B(x,r)}\right|^{p^*} d\mu + 2^{p^*} \lim_{r\to 0} \left|u(x) - u_{B(x,r)}\right|^{p^*} \\
&\qquad = 0.
\end{aligned}$$

By taking $E = \cap_j E_j$ we see $\mathrm{C}_p(E) = 0$ and the discussion holds for each $x \in X \backslash E$. This completes the proof.

□

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