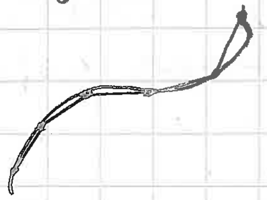
1. (X,d) metric space. (won't need a measure!)

Path: A cts map $\gamma:[a,b]\to X$.

Def The length $\ell(\gamma)$ of a path is

$$\ell(\gamma)=\sup\left\{\sum_{j=1}^{m} d(\gamma(t_j),\gamma(t_{j-1}))\ \middle|\ a=t_0<t_1<\dots<t_m=b\right\}$$



γ is rectifiable if $\ell(\gamma)<\infty$.

For a rectifiable path $\gamma:[a,b]\to X$ we define its length function

$S_\gamma:[a,b]\to[0,\ell(\gamma)]$ as

$$S_\gamma(t)=\ell(\gamma|[a,t])$$

- S_γ well-def and increasing: If $a\le s<t\le b$ then for any partition $a=t_0<t_1<\dots<t_m=s$ of $[a,s]$ we have that $\{t_0,t_1,\dots,t_m,t\}$ is a partition of $[a,t]$, so

$$\sum_{j=1}^{m} d(\gamma(t_j),\gamma(t_{j-1}))\le\sum_{j=1}^{m} d(\gamma(t_j),\gamma(t_{j-1}))+d(\gamma(s),\gamma(t))$$

OR: $\ell([a,b])=\ell([a,\tau])+\ell([\tau,b])$ $\le \ell(\gamma|[a,t])=S_\gamma(t)$

Sup over partitions: $S_\gamma(s)=\ell(\gamma|[a,s])\le S_\gamma(t)$.

Especially $S_\gamma(t)\le\ell(\gamma)\ \forall t\in[a,b]$.

Lemma For $\gamma:[a,b]\to X$ rectifiable, the function S_γ is cts.

Pf: We prove cont. on (a,b). The reasoning for endpts is similar.

Sp S_γ not cts on (a,b): $\exists\ \tau\in(a,b)$ s.t.

$$\eta:=\underbrace{\lim_{t\to\tau+}S_\gamma(t)-\lim_{t\to\tau-}S_\gamma(t)}_{\text{exist since } S_\gamma \text{ is increasing}}>0 \qquad (\ddot\smile)$$

Take part. $a=t_0<t_1<\dots<t_m=b$ s.t. $\displaystyle\sum_{j=1}^{m} d(\gamma(t_j),\gamma(t_{j-1}))>\ell(\gamma)-\eta/3$ (1)

We may also assume that $d(\gamma(t_j),\gamma(t_{j-1}))<\eta/3$

– Why? – $[a,b]$ compact and γ cts so γ is unifly cts.

May assume $\tau\notin\{t_0,\dots,t_m\}$ by (1) and the cont of γ: $\tau\in(t_{j-1},t_j)$ for some j.

$$\ell(\gamma|[t_{j-1},t_j])=\ell(\gamma|[a,t_j])-\ell(\gamma|[a,t_{j-1}])=S_\gamma(t_j)-S_\gamma(t_{j-1})\underset{(\ddot\smile)}{\ge}\eta$$

$\therefore$ Can pick a partition $t_{j-1}=s_0<s_1<\dots<s_N=t_j$ s.t.

$$\sum_{i=1}^{N} d(\gamma(s_i),\gamma(s_{i-1}))>\frac{3}{4}\eta.$$

Now, $t_0<t_1\dots t_{j-1}=s_0<s_1<\dots<s_N=t_j<\dots<t_m$ is a part. of $[a,b]$ and

$$\sum_{\substack{k=1\\k\ne j}}^{m} d(\gamma(t_k),\gamma(t_{k-1}))+\sum_{i=1}^{N} d(\gamma(s_i),\gamma(s_{i-1}))>\sum_{\substack{k=1\\k\ne j}}^{m} d(\gamma(t_k),\gamma(t_{k-1}))+\frac{3}{4}\eta$$

$$=\sum_{k=1}^{m} d(\gamma(t_k),\gamma(t_{k-1}))+\frac{3}{4}\eta-\underbrace{d(\gamma(t_j),\gamma(t_{j-1}))}_{<\eta/3}>\ell(\gamma)-\eta/3+\frac{3}{4}\eta-\eta/3>\ell(\gamma)$$

This contradicts the def of $\ell(\gamma)$ as the sup over partitions! $\square$

Def. If $\gamma:[a,b]\to X$ is a path and if $\alpha:[c,d]\to[a,b]$ is increasing <u>cts</u> (or decreasing) (surjection) we say that $\gamma\circ\alpha$ is a <u>reparametrization</u> of γ.

2

Observation In the case above we have $\ell(\gamma\circ\alpha)=\ell(\gamma)$. $\quad(\ddot{\frown})$
We could define γ for arbitrary (not necessarily cts) maps $[a,b]\to X$.
In this case $(\ddot{\frown})$ remains valid.

Thm1. If $\gamma:[a,b]\to X$ is a rectifiable path, then there is a unique path $\tilde\gamma:[0,\ell(\gamma)]\to X$ s.t. $\gamma=\tilde\gamma\circ s_\gamma$.
Moreover, $\ell(\tilde\gamma|[0,t])=t \quad \forall\ t\in[0,\ell(\gamma)]$. Esp. $\tilde\gamma$ is 1-Lipschitz.
$\ell(\tilde\gamma|[s,t])=t-s$

Pf Uniqueness: Follows from the surjectivity of s_γ.
Def now $h:[0,\ell(\gamma)]\to[a,b]$, $\quad h(t)=\inf s_\gamma^{-1}\{t\}=\inf\{\tau\in[a,b]\mid s_\gamma(\tau)=t\}$. (not necessarily cts.)
s_γ cts so $s_\gamma^{-1}\{t\}$ closed and thus $h(t)\in s_\gamma^{-1}\{t\}$ i.e. $s_\gamma(h(t))=t$
Define $\tilde\gamma:=\gamma\circ h$
For $t\in[a,b]$, $h(s_\gamma(t))\le t$ so
$$d(\gamma(t),\gamma(h(s_\gamma(t))))\le\ell(\gamma|_{[h(s_\gamma(t)),t]})=s_\gamma(t)-s_\gamma(h(s_\gamma(t)))$$
$$=s_\gamma(t)-s_\gamma(t)=0$$
$$\therefore\ \overbrace{\gamma\circ h}^{\tilde\gamma}\circ s_\gamma=\gamma.$$
For $\tau\in[0,\ell(\gamma)]$ we have (use the obs)
$$\ell(\tilde\gamma|[0,\tau])\underset{s_\gamma\circ h=id}{=}\ell(\tilde\gamma|[0,s_\gamma(h(\tau))])=\ell((\underbrace{\tilde\gamma\circ s_\gamma}_{\gamma})|[a,h(\tau)])=\ell(\gamma|[a,h(\tau)])$$
$$\underset{\text{Def of } s_\gamma}{=}s_\gamma(h(\tau))=\tau$$
Thus ($\tau_1<\tau_2$)
$$d(\tilde\gamma(\tau_1),\tilde\gamma(\tau_2))\le\ell(\tilde\gamma|[\tau_1,\tau_2])$$
$$=\ell(\tilde\gamma|[0,\tau_2])-\ell(\tilde\gamma|[0,\tau_1])$$
$$=\tau_2-\tau_1.$$
$\therefore\ \tilde\gamma$ is 1-Lipschitz. $\qquad\square$

Def For a path $\gamma:[a,b]\to X$ we define the (metric) speed at $t\in(a,b)$ as
$$|\dot\gamma|_d(t):=\lim_{h\to0}\frac{d(\gamma(t+h),\gamma(t))}{|h|}$$
Provided that the limit exists.

Thm2 For every Lipschitz curve $\gamma:[a,b]\to X$, the metric speed exists a.e. in (a,b) and
$$\ell(\gamma)=\int_a^b|\dot\gamma|_d(t)\,dt.$$

Pf: Let $\{x_n\mid n\in\mathbb N\}$ be a dense subset of $\gamma([a,b])$. (E.g. the image of $\gamma([a,b]\cap\mathbb Q)$)
Def $\varphi_n:[a,b]\to\mathbb R$, $\quad\varphi_n(t)=d(\gamma(t),x_n)$
$s,t\in[a,b]$:
$$|\varphi_n(t)-\varphi_n(s)|\underset{\Delta\text{-ineq}}{\le}d(\gamma(t),\gamma(s))\underset{\gamma\text{ is Lip.}}{\le}C_\gamma|t-s| \qquad\therefore\ \varphi_n \text{ is Lip.}$$

φ_n Lip ∴ φ_n diff'able a.e. Also: $\|\varphi_n'\|_\infty \leq C_r$
(Why? φ_n Lip $\Rightarrow$ φ_n has bdd var $\Rightarrow$ φ_n the sum of two monotone functions $\Rightarrow$ φ_n diff'able a.e.)
φ_n diff'able outside A_n, $|A_n| = 0$. $A = \bigcup_{n\in\mathbb{N}} A_n$ $\quad |A| = 0$

$m(t) := \sup_{n\in\mathbb{N}} |\varphi_n'(t)|$ — Defined outside A.

$$\liminf_{h\to 0} \frac{d(\gamma(t+h),\gamma(t))}{|h|} \overset{\Delta\text{-ineq}}{\geq} \liminf_{h\to 0} \left|\frac{\varphi_n(t+h)-\varphi_n(t)}{h}\right| = |\varphi_n'(t)| \quad \text{for } t\notin A$$

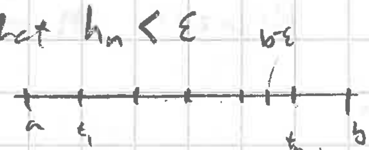
Sup over n: $$\liminf_{h\to 0} \frac{d(\gamma(t+h),\gamma(t))}{|h|} \geq m(t), \quad t\notin A,$$

For $s\leq t$,
$$d(\gamma(t),\gamma(s)) \overset{\{x_n\}_{n\in\mathbb{N}} \text{ dense in } \gamma([a,b])}{=} \sup_{n\in\mathbb{N}} |\underbrace{d(\gamma(t),x_n)}_{\varphi_n(t)} - \underbrace{d(\gamma(s),x_n)}_{\varphi_n(s)}| \overset{\varphi_n \text{ Lip} \Rightarrow \varphi_n \text{ abs cts}}{=} \sup_{n\in\mathbb{N}} \left|\int_s^t \varphi_n'(\tau)\,d\tau\right|$$
$$\leq \sup_{n\in\mathbb{N}} \int_s^t \underbrace{|\varphi_n'(\tau)|}_{\leq m(\tau)}\,d\tau \leq \int_s^t m(\tau)\,d\tau \qquad \left(\substack{0\,0\\0}\right)$$

$\|\varphi_n'\|_\infty \leq C_r$ so $m \leq C_r$. Thus m is integrable. By the Lebesgue diff thm,
$$\limsup_{h\to 0} \frac{d(\gamma(t+h),\gamma(t))}{|h|} \leq \limsup_{h\to 0} \frac{1}{|h|}\int_t^{t+h} m(\tau)\,d\tau = m(t) \text{ for a.e. } t.$$

∴ $|\dot\gamma|_d(t) = m(t)$ for a.e. t.
It is now suff to show that $\ell(\gamma) = \int_a^b m(t)\,dt$.
For any partition $a = t_0 < t_1 < \dots < t_m = b$ we have
$$\sum_{j=1}^m d(\gamma(t_j),\gamma(t_{j-1})) \overset{\left(\substack{0\,0\\0}\right)}{\leq} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} m(t)\,dt = \int_a^b m(t)\,dt$$
sup over partitions $\Rightarrow$ $\ell(\gamma) \leq \int_a^b m(t)\,dt$.

Let $\varepsilon > 0$. Define $h_n := \frac{1}{n}(b-a)$ for $n\in\mathbb{N}$ so large that $h_n < \varepsilon$
Consider the part $t_j := a + j h_n$, $j\in\{0,1,\dots,n\}$



$$h_n^{-1}\int_a^{b-\varepsilon} d(\gamma(t+h_n),\gamma(t))\,dt \leq h_n^{-1}\sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j} d(\gamma(t+h_n),\gamma(t))\,dt$$
$$= h_n^{-1}\sum_{j=1}^{n-1}\int_0^{h_n} d(\gamma(t+t_j),\gamma(t+t_{j-1}))\,dt = h_n^{-1}\int_0^{h_n} \underbrace{\sum_{j=1}^{n-1} d(\gamma(t+t_j),\gamma(t+t_{j-1}))}_{\substack{\text{sum corresp. to a partition of } [a+t,\, t+t_{n-1}] \\ \leq \ell(\gamma|_{[a+t,\, t+t_{n-1}]}) \leq \ell(\gamma)}}\,dt$$
$$\leq \ell(\gamma).$$

$$\int_a^{b-\varepsilon} \underbrace{\frac{d(\gamma(t+h_n),\gamma(t))}{h_n}}_{0 \leq\ \leq C_r}\,dt$$ can use DCT, take $n\to\infty$
$$\int_a^{b-\varepsilon} m(t)\,dt \leq \ell(\gamma).$$

Taking $\varepsilon\to 0$ yields the claim $\square$

4 Applications of Thm 2.

Cor 2 If $\gamma:[a,b]\to X$ is any rect. path, and $\tilde\gamma$ is its arc-length parametrization, then $|\dot{\tilde\gamma}|_d = 1$ a.e. in $[0, \ell(\gamma)]$.

Pf $\tilde\gamma$ is 1-Lip, so by Thm 2 $|\dot{\tilde\gamma}|_d$ exists a.e. $\tilde\gamma$ 1-Lip so

$$\frac{d(\tilde\gamma(t+h), \tilde\gamma(t))}{|h|} \leq 1 \qquad \therefore\ |\dot{\tilde\gamma}|_d(t) \leq 1 \text{ whenever defined.}$$

$$\ell(\gamma) = \ell(\tilde\gamma) \underset{\text{Thm 2}}{=} \int_0^{\ell(\gamma)} \underbrace{|\dot{\tilde\gamma}|_d(t)}_{\leq 1}\,dt$$

– If we had $|\dot{\tilde\gamma}|_d(t) < 1$ on a set of positive measure, we could estimate $< \ell(\gamma)$, a contradiction! $\square$

(use instead $|s_\gamma(t+h) - s_\gamma(t)| = |h|$

Cor 2 first! Cor 1 simple case.

Cor 1 If $\gamma:[a,b]\to X$ is Lipschitz then s_γ is Lipschitz and $s_\gamma'(t) = |\dot\gamma_d|(t)$ for a.e. $t\in(a,b)$.

Pf: For $a \leq t_1 \leq t_2 \leq b$ we have

$$|s_\gamma(t_1) - s_\gamma(t_2)| = \ell(\gamma|[t_1,t_2]) \overset{\text{Thm 2}}{=} \int_{t_1}^{t_2} \underbrace{|\dot\gamma|_d(\tau)}_{\leq L}\,d\tau \leq L|t_1 - t_2|$$

classical def!

$\therefore$ s_γ is Lipschitz. (could also be seen using the def of ℓ and partitions)

$\therefore$ $s_\gamma'(t)$ exists a.e. and

$$\int_a^b s_\gamma'(t)\,dt = s_\gamma(b) - s_\gamma(a) = \ell(\gamma) = \int_a^b |\dot\gamma|_d(t)\,dt$$

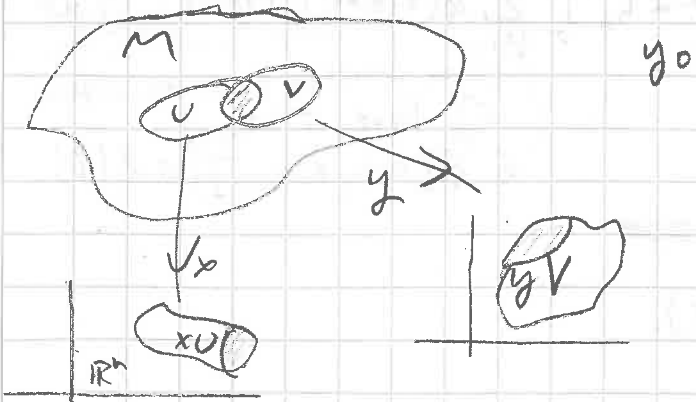
This and $s_\gamma'(t) \geq |\dot\gamma|_d$ implies that $s_\gamma'(t) = |\dot\gamma|_d(t)$ a.e. $\square$

– OR: use Lebesgue diff Thm

Applications of Thm 2 to mflds.

Recall: M a topological n-mfld if
1) Hausdorff
2) Second countable
3) Locally homeomorphic to $\mathbb{R}^n$

M is a smooth manifold if in addition it has a smooth structure: Can cover M by charts (U,x) that are smoothly compatible.



$$y\circ x^{-1}: x[U\cap V] \to y[U\cap V] \text{ is } C^\infty.$$

(M,g) is a Riemannian mfld if M a smooth manifold and $g\in \mathcal{T}^0_2(M)$ is a metric tensor

"A smooth function assigning an inner product to each tangent space T_pM – the vector space of "directions"

5

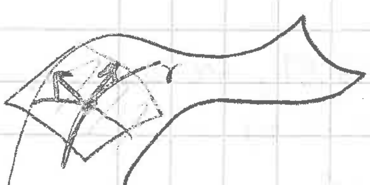
The tangent space $\quad f \in C^\infty(M) \overset{def}{\iff} f$ cont. and $f \circ x^{-1}$ smooth for all smooth charts

$p \in M \quad C^\infty(p) = \{$eq. classes of smooth functions defined in a nbhood of $p\}$
($f \sim g$ if $f = g$ in a nbhood of p.)

T_pM – the set of linear maps $C^\infty(p) \to \mathbb{R}$ respecting the Leibniz rule: $\quad v \in T_pM \iff$ i) $v(af + bg) = avf + bvg \quad a, b \in \mathbb{R},\ f, g \in C^\infty(p)$
ii) $v(fg) = f(p)vg + g(p)vf$

Any smooth path $\gamma : I \to M$ (i.e. $x \circ \gamma$ smooth for any coords x "$I \to \mathbb{R}^n$")
defines an elem $\dot\gamma_t \in T_{\gamma(t)}M$ by
$$\dot\gamma_t f = (f \circ \gamma)'(t)$$

(In fact, all elements of T_pM can be represented like this by smooth real curves.)
$$\dim T_pM = \dim M$$



The metric tensor g assigns an innerproduct
$g_p : T_pM \times T_pM \to \mathbb{R}$.
(Induces a norm on T_pM: $\|v\|_g = (g_p(v,v))^{1/2} \quad \forall v \in T_pM$

For a smooth path $\gamma : I \to M$ we now define the Riemannian length $\quad L(\gamma) := \int_I \|\dot\gamma_t\|_g \, dt$

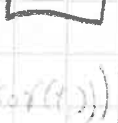
If M connected, this allows us to construct a metric
$d(p,q) := \inf \{ L(\gamma) \mid \gamma$ is piecewise smooth and joins p and $q\}$

Using this metric, we can again consider the (metric) length $\ell(\gamma)$ – sup over partitions.

- Question: Is $\ell(\gamma) = L(\gamma)$ for (piecewise) smooth γ?

Lemma Any smooth $\gamma : I \to M$ is Lipschitz wrt the metric d.

Pf: For $t_1, t_2 \in I$ we have
$$d(\gamma(t_1), \gamma(t_2)) \leq L(\gamma|[t_1,t_2]) = \int_{t_1}^{t_2} \|\dot\gamma_t\|_g \, dt \leq \int_{t_1}^{t_2} C \, dt = C|t_2 - t_1|$$
$t \mapsto \|\dot\gamma_t\|_g$ cts and hence bdd on the compact set I: $\|\dot\gamma_t\|_g \leq C$



6 * For a path $\alpha : I \to xU$ and $\sigma = x^{-1}\circ\alpha : I \to M$

we have $\dot\sigma_t f = (f\circ\sigma)'(t) = (f\circ x^{-1}\circ\alpha)'(t)$

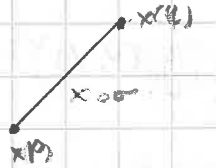
$$= (\alpha^i)'(t)(\partial_i(f\circ x^{-1}))(\alpha(t))$$

$$= (\alpha^i)'(t)\left(\frac{\partial f}{\partial x^i}\right)_{x^{-1}(\alpha(t))}$$

$$\dot\sigma_t = (\alpha^i)'(t)\left(\frac{\partial}{\partial x^i}\right)_{x^{-1}(\alpha(t))} \qquad \left(\frac{\partial}{\partial x^i}\right)_p \in T_pM$$

def is $\left(\frac{\partial f}{\partial x^i}\right)_p = \partial_i(f\circ x^{-1})(x(p))$

<u>Normal coordinates</u> For each $p\in M$ $\exists\, r>0$ and chart (U,x) with $xU = B(\bar 0, r)\subset\mathbb{R}^n$ s.t. for each $q\in U$, the map $\sigma(t) = x^{-1}(t\,x(q) + (1-t)\,x(p))$ is the unique length-minimizing path from p to q.



<u>Thm 3</u> For a smooth $\gamma : [a,b]\to M$, we have $|\dot\gamma|_d(t) = \|\dot\gamma_t\|_g$ for all $t\in(a,b)$

and $\ell(\gamma) = L(\gamma)$.

<u>Pf</u>: Let $t\in(a,b)$. Pick normal coordinates (U,x) at $\gamma(t)$. may assume defined on $\bar U$

For h sufficiently small, $\gamma(t+h)\in U$. Then

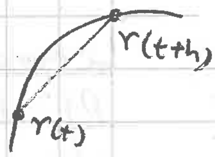
$\sigma^h(s) := x^{-1}(x\circ\gamma(t) + s(x\circ\gamma(t+h) - x\circ\gamma(t)))$ is the unique length-minim curve from $\gamma(t)$ to $\gamma(t+h)$.

Thus $\dfrac{d(\gamma(t+h),\gamma(t))}{|h|} = \displaystyle\int_0^1 \frac{\|\dot\sigma^h_s\|_g}{|h|}\,ds \overset{\text{as in prev. lemma}}{\underset{(*)}{=}} \int_0^1 \left\| \frac{(x^i\circ\gamma(t+h) - x^i\circ\gamma(t))}{h}\left(\frac{\partial}{\partial x^i}\right)_{\sigma^h(s)}\right\|_g ds$

$$\left|\frac{x^i\circ\gamma(t+h) - x^i\circ\gamma(t)}{h}\right| = |(x^i\circ\gamma)'(\tau^t_h)| \le \|(x^i\circ\gamma)'\|_{L^\infty([a,b])}$$

$\tau^t_h \in (t,t+h)$

and $\left\|\left(\frac{\partial}{\partial x^i}\right)_p\right\|_g \le C$ for $p\in\bar U$



Thus we may apply the DCT and take $h\to 0$ in the integral

$$h\to 0: \int_0^1 \underbrace{\left\|(x^i\circ\gamma)'(t)\left(\frac{\partial}{\partial x^i}\right)_{\gamma(t)}\right\|_g}_{\|\dot\gamma_t\|_g} ds \overset{\text{no } s\text{-dep}}{=} \|\dot\gamma_t\|_g.$$

γ smooth $\Rightarrow$ γ Lipschitz so $\ell(\gamma) \overset{\text{Thm 2}}{=} \displaystyle\int_a^b |\dot\gamma|_d(t)\,dt = \int_a^b \|\dot\gamma_t\|_g\,dt = L(\gamma)$ $\square$

– Generalizations: Finsler mflds should be possible: still C^1 locally as above.