Assoud Embedding Theorem, MS-E1991 Calculus of Variations, Project work

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Theorem (Assouad embedding theorem, N independent of α)

For each $C_0 \geq 1$, there is an integer N and, for $1/2 < \alpha < 1$, a constant $C = C(C_0, \alpha)$ such that if (E, d) is a metric space hat admits the metric doubling constant C_0, we can find an injection $F : E \to \mathbb{R}^N$ such that

$$C^{-1} d(x,y)^\alpha \leq |F(x) - F(y)| \leq C d(x,y)^\alpha$$

for $x, y \in E$.

The embedding problem for metric spaces

An *embedding* is a map that is a homeomorphism onto its image.

An embedding $f : X \to Y$ is called:

- *quasisymmetric* if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ so that
  $$|x - a| \leq t|x - b| \text{ implies } |f(x) - f(a)| \leq \eta(t)|f(x) - f(b)|$$
  for all triples a, b, x of points in X, and for all $t > 0$. Quasisymmetric maps distort *relative* distances by a bounded amount.
- *bi-Lipschitz* if both f and f^{-1} are Lipschitz. bi-Lipschitz maps distort *absolute* distances by a bounded amount, a much stricter condition.
- The snowflaking identity map $(X, |x - y|) \to (X, |x - y|^{\alpha})$ is t^{α}-quasisymmetric. C-bi-Lipschitz maps are $C^2 t$-quasisymmetric.
- Thus, Assouad's snowflaked bi-Lipshitz embedding is quasisymmetric.

When is a metric space bi-Lipshitz embeddable to $\mathbb{R}^n$?

A *necessary condition* is that the space is doubling.
But this is *not sufficient*. Common counterexamples include:

1. The Heisenberg group with its Carnot metric (Py Pansu's thm: every lipschitz map is differentiable a.e., would get an algebra homomorphism, incompatible with the Abelian structure of $\mathbb{R}^n$. Because of non-commutativity of the Heisenberg group has necessarily a kernel, but bilipschitzness implies injectivity).
2. Laakso spaces ($[0,1] \times K$, where K is a Cantor set with identification of points through wormholes).
3. Bourdon-Pajot spaces (related to certain hyperbolic buildings).

However, if we alter our metric *slightly* by mapping the metric space $(M.d(x,y))$ with an identity mapping to the metric space $(M, d(x,y)^\alpha), 0 < \alpha < 1$, we obtain the desired embedding into an Euclidean space $\mathbb{R}^n$ for some $n \in \mathbb{N}$. This modification of the metric is called *snowflaking*.

Assouad implies quasisymmetric embeddings

Which metric spaces are quasisymmetrically equivalent? Difficult. Which spaces can be quasisymmetrically embedded in some Euclidean space? Easier:

Theorem (Quasisymmetric embeddability)

A metric space is quasisymmetrically embeddable in some Euclidean space if and only if it is doubling, quantitatively.

Necessity follows from the following

Theorem (Conservation of the doubling property)

A quasisymmetric image of a doubling space is doubling, quantitatively.

Sufficiency is implied by the Assouad embedding theorem as snowflaking is t^α-quasisymmetric.

Theorem (Assouad Embedding Theorem)

Each snowflaked version of a doubling metric space admits a bi-Lipschitz embedding in some Euclidean space.

Assoud Embedding Theorem

Theorem (Assouad embedding theorem, original)

Let $(E, d(x,y))$ be a doubling metric space. Then for every $\alpha \in (0,1)$ there exist $N, C > 0$ and $F : E \to \mathbb{R}^N$ such that

$$C^{-1} d(x,y)^\alpha \leq |F(x) - F(y)| \leq C d(x,y)^\alpha$$

for all $x, y \in M$.

We prove a version with dimension independent of snowflaking.

Theorem (Assouad embedding theorem, N independent of α)

For each $C_0 \geq 1$, there is an integer N and, for $1/2 < \alpha < 1$, a constant $C = C(C_0, \alpha)$ such that if (E, d) is a metric space hat admits the metric doubling constant C_0, we can find an injection $F : E \to \mathbb{R}^N$ such that

$$C^{-1} d(x,y)^\alpha \leq |F(x) - F(y)| \leq C d(x,y)^\alpha$$

for $x, y \in E$.

Snowflake spaces

Definition

Let $(M, d(x,y))$ be a metric space. Its *snowlaked version* is a metric space $(M, d(x,y)^\alpha), 0 < \alpha < 1$.

The name stems from the fact that $(\mathbb{R}, |x-y|^\alpha), 1/2 < \alpha < 1$ admits a bi-Lipschitz embedding in $\mathbb{R}^2$, and the image resembles the boundary of a snowflake.

> *(P. Assouad, Plongements lipschitziens dans $\mathbb{R}^n$)*
> *- la troisième méthode utilise des courbes de Von KOCH généralisées (de fait cette méthode étend l'observation, due à GLAESER [5] p. 57, que la courbe classique de H. Von KOCH [7] réalise un plongement Lipschitzien de l'espace $([0,1], ||\cdot||^{\log 3/\log 4})$ dans $(\mathbb{R}^2, ||\cdot||)$).*
> ...
> *La construction que nous allons écrire généralise, comme on le verra, celle de la courbe de Von Koch classique [7]; c'est ce qui justifie notre terminologie.*

Figure: Glaeser: Koch curve is a bi-Lipschitz image of $([0,1], |\cdot|^{\log 3/\log 4})$ on $(\mathbb{R}^2, ||\cdot||)$.

Un exemple plus élémentaire est fourni par la célèbre courbe de von Koch ([1]).

Exemple 8.

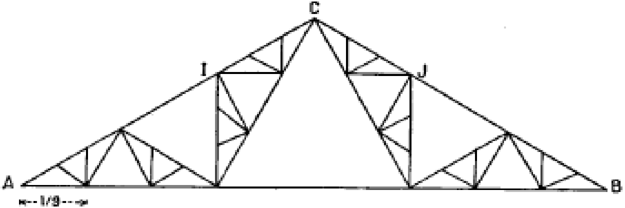

Figure 2

On sait que cette courbe C n'est pas rectifiable; mais on peut définir sur C une mesure: sa "pseudo-longueur" (Cf. Bouligand [1] p. 218) possédant l'"exposant de similitude" $\alpha = \log 4/\log 3$. Cela veut dire que si l'on soumet C à une similitude de rapport k, sa pseudo-longueur est multipliée par k^α.

Figure: Glaeser: Koch curve is a bi-Lipschitz image of $([0,1], |\cdot|^{\log 3/\log 4})$ on $(\mathbb{R}^2, |\cdot|)$).

On sait que cette courbe est localement semblable à elle-même.

Lemme. Il existe deux constantes K et K' telles que, pour tout couple de points $M, N \in C$, on ait :

$0 \leq K'. \|MN\|^{\alpha} \leq$ pseudo-longueur de l'arc $\widehat{MN} \leq K . \|MN\|^{\alpha}$.

L'existence des constantes est trivialement assurée si l'on se borne à envisager des couples M, N satisfaisant à $\|MN\| \geq 1/9$ (si l'on prend $1 = \|AE\|$ pour unité de longueur).

Remarquons alors qui si $\|MN\| \leq 1/9$, l'arc $\widehat{MN}$ est entièrement intérieur à l'un des trois arcs suivants : $\widehat{AC}$, $\widehat{IJ}$, ou $\widehat{BC}$ (cf. fig.). Il en résulte qu'il existe une similitude de rapport $\sqrt{3}$ ou 3 (suivant le cas) qui transforme l'arc $\widehat{MN}$ en un autre arc de C.

Ceci posé, quels que soient M et N, il existe un entier k tel que $1/(\sqrt{3})^{k+1} \leq \|MN\| \leq 1/(\sqrt{3})^{k}$. Tant que $k \geq 4$, la remarque précédente permet des substituer à l'arc $\widehat{MN}$ un arc plus grand semblable au premier. Dans une telle similitude, le rapport $\dfrac{\text{ps-long } \widehat{MN}}{\|MN\|^{\alpha}}$ n'est pas modifié. On peut donc toujours se ramener au cas où $\|MN\| \geq 1/9$. Le lemme est ainsi établi.

Glaeser: Koch curve is a bi-Lipschitz image of $([0,1], |\cdot|^{\log 3/\log 4})$ on $(\mathbb{R}^2, |\cdot|)$.

Lemma (Glaeser, 1958)

For each point $x, y \in C$, there exist constants K and K' such that

$$0 \leq K'|x-y|^{\alpha} \leq \text{ pseudo-length of arc } \widehat{xy} \leq K|x-y|^{\alpha}, \alpha = \log 3/\log 4$$

Proof.

- If $|x-y| \geq 1/9$, existence of constants K, K' is easy to see.
- If $|x-y| \leq 1/9$, then the arc $\widehat{xy}$ is entirely within one of the following segments of the curve (cf. figure) $\widehat{AC}$, $\widehat{IJ}$ or $\widehat{BC}$.
- Depending on the case, there exists a similarity mapping of ratio $\sqrt{3}$ or 3 between the segment $\widehat{xy}$ and another arc of C.
- Thus, there exist a $k \in \mathbb{N}$, $1/(\sqrt{3})^{k+1} \leq |x-y| \leq 1/(\sqrt{3})^{k}$.
- For $k \geq 4$ we can substitute the arc with a bigger similar arc without changing $\frac{pseudolength(\widehat{xy})}{|x-y|^{\alpha}}$. Can reduce to $|x-y| \geq 1/9$.

Snowflake space is a metric space

- If $(M, d(x,y))$ is any metric space and $0 < \alpha < 1$, then $d(x,y)^\alpha$ also defines a metric on M.
- We need to check that the *triangle inequality* still holds. If a, b are nonnegative real numbers and $0 < \alpha < 1$, then

$$(a+b)^\alpha \le a^\alpha + b^\alpha.$$

Observe, that

$$\max(a,b) \le (a^\alpha + b^\alpha)^{1/\alpha}.$$

Then

$$a + b \le (a^\alpha + b^\alpha)\max(a,b)^{1-\alpha} \le (a^\alpha + b^\alpha)^{1/\alpha}.$$

- The snowflaked metric $d(x,y)^\alpha$ still generates the same topology as $d(x,y)$.

Snowflaking turns all continuous paths of finite length unrectifiable.

- Let $(E, d(x,y))$ be a metric space and $x, y \in E, x \neq y$.
- Let $\gamma : [0,1] \to E$ be a rectifiable path joining x and y.
- Then, for every $\epsilon > 0$, there exists a partition $0 = t_0 < t_1 < t_2 \cdots \leq t_n = 1$ of the interval $[0,1]$ such that $d(\gamma(t_i), \gamma(t_{i-1})) \leq \epsilon$ for all $i \in 1, \ldots, n$.
- For length $l(\gamma)$ of the path γ in the snowflaked metric $d(x,y)^\alpha, 0 < \alpha < 1$, the following holds:

$$\begin{aligned} l(\gamma)_{(E,d^\alpha)} &\geq \sum_{i=1}^{n} d(\gamma(t_i), \gamma(t_{i-1}))^\alpha \\ &\geq \sum_{i=1}^{n} d(\gamma(t_i), \gamma(t_{i-1}))\ \epsilon^{\alpha-1} \\ &\geq \epsilon^{\alpha-1} d(x,y) \to \infty \text{ as } \epsilon \to 0 \end{aligned}$$

Snowflaked spaces have plenty of nonconstant functions with "gradient" 0

- Suppose that f is a Lipschitz function on M with respect to $d(x, y)$.
- Then
$$lim_{p \to x} = \frac{|f(x) - f(p)|}{d(x, p)^\alpha} = 0.$$
- Thus there are plenty of nonconstant functions with "gradient" 0 with respect to the snowflaked distance. Every Lipschitz function with respect to the original meric will do.
- See Semmes, "Calculus, fractals, and analysis on metric spaces", for more.

Problems with the original proof

For values of α bounded away from both 0 and 1 we can use the standard Assouad proof to show that the resulting snowflake spaces can all be embedded in a fixed dimensional Euclidean space. For α close to 0, the dimension independence fails necessarily. Even when $E = \mathbb{R}$ with the usual distance, we need many dimensions to construct an α-snowflake. However, for α close to 1, we can get dimension independence, but the standard proof does not give it. In fact $N \to \infty$, as $\alpha \to 1$, by the standard proof of Assouad's theorem.

Assouad with dimension independent of snowflaking

A probabilistic proof of the following theorem was given by Naor and Neiman. Their proof constructs a random embedding to $\mathbb{R}^N$ with a limited dependency structure of events and proves that the desired bi-Lipschitz embedding exists with positive probability using a generalized version of Lovasz Local Lemma.
We present a simpler non-probabilistic proof by David and Snipes (2012) that modifies the standard proof. We use an adaptive argument and work at *small relative scales* to use the fact that there is *a lot of space* in $\mathbb{R}^n$. A very *sparse collection of scales* helps control the residual terms.

Assouad Embedding Theorem, dimension independent of snowflaking, statement

Definition

A metric space (E, d) is metrically doubling if there is an integer $C_0 \geq 1$ such that for every $r > 0$, every (closed) ball of radius $2r$ in E can be covered with no more than C_0 balls of radius r. We call C_0 a metric doubling constant for (E, d).

Theorem (Naor,Neiman(2012), David,Snipes(2012))

For each $C_0 \geq 1$, there is an integer N and, for $1/2 < \alpha < 1$, a constant $C = C(C_0, \alpha)$, such that if (E, d) is a metric space that admits the metric doubling constant C_0, we can find an injection $F : E \to \mathbb{R}^N$ such that

$$C^{-1} d(x, y)^{\alpha} \leq |F(x) - F(y)| \leq C d(x, y)^{\alpha}$$

for $x, y \in E$.

Assouad Embedding Theorem, dimension independent of snowflaking, proof

Proof.(David,Snipes(2012))

We need to consider only α very close to 1. We use a small parameter $\tau > 0$, with $\tau \leq 1 - \alpha$, and work at the scales

$$r_k = \tau^{2k}, k \in \mathbb{Z}. \tag{1}$$

Suppose first that E has finite diameter. Then we can choose an initial scale $k = k_0$ such that $r_{k_0} \geq \operatorname{diam} E$. Our construction and main constants will not depend on this choice.

For each $k \geq k_0$, select a maximal collection $\{x_j\}, j \in J_k$, of points of E, with $d(x_i, x_j) \geq r_k$ for $i \neq j$. Thus, by maximality

$$E \subset \bigcup_{j \in J_k} B(x_j, r_k). \tag{2}$$

Assouad Embedding Theorem, dimension independent of snowflaking, proof (2)

Let $N(x)$ denote the number of indices $j \in J_k$ such that $d(x_j, x) \leq 10r_k$. Then

$$N(x) \leq C_0^5 \text{ for } x \in E. \tag{3}$$

This holds, since we can cover $B(x, 10r_k)$ with fewer than C_0^5 balls D_l of radius $r_k/3$.

Each D_l contains at most one x_j, because $d(x_i, x_j) \geq r_k$ for $i \neq j$.

All the x_j that lie in $B(x, 10r_k)$ are contained in some D_l, so (3) follows.

From (3) and the assumption that E is bounded, we get that J_k is finite.

Assouad Embedding Theorem, proof (3)

Set $\Xi = \{1, 2, \dots, C_0^5\}$ (a set of colors).

Enumerate J_k, and for $j \in J_k$ let $\xi(j)$ be the first color not taken by an earlier close neighbor, i.e. earlier $i \in J_k$ such that $d(x_i, x_j) \leq 10r_k$. By construction

$$\xi(i) \neq \xi(j) \text{ for } i, j \in J_k \text{ with } i \neq j \text{ and } d(x_i, x_j) \leq 10r_k.$$

For each color $\xi \in \Xi$, define the set $J_k(\xi) := \{j \in J_k : \xi(j) = \xi\}$. Thus

$$d(x_i, x_j) > 10r_k \text{ for } i, j \in J_k(\xi) \text{ such that } i \neq j. \qquad (4)$$

For each $j \in J_k$, set $\varphi_j(x) = max\{0, 1 - r_k^{-1} dist(x, B_j)\}$. This makes sure that

$$\begin{aligned} 0 \leq &\varphi_j(x) \leq 1 \text{ everywhere}, \\ &\varphi_j(x) = 1 \text{ for } x \in B_j, \\ &\varphi_j(x) = 0 \text{ for } x \in E \setminus 2B_j. \end{aligned} \qquad (5)$$

and

$$\varphi_j \text{ is Lipschitz, with } ||\varphi_j||_{lip} \leq r_k^{-1}. \qquad (6)$$

Assouad Embedding Theorem, proof (4)

For each color $\xi \in \Xi$, we will construct two mappings: $F^{\xi} : E \to \mathbb{R}^M$ and a slightly modified version $\widetilde{F^{\xi}} : E \to \mathbb{R}^M$. Here M is a very large integer depending only on the metric doubling constant.

Our final mapping will be the tensor product of these $2C_0^5$ mappings. Thus the dimension N is $2C_0^5 M$. The mapping F^{ξ} will be of the form:

$$F^{\xi}(x) = \sum_{k \geq k_0} r_k^{\alpha} f_k^{\xi}(x),$$

where

$$f_k^{\xi}(x) = \sum_{j \in J_k(\xi)} v_j \varphi_j(x), \tag{7}$$

with vectors $v_j \in \mathbb{R}^M$ that will be carefully chosen later.

Assouad Embedding Theorem, proof (5)

The extra room in $\mathbb{R}^M$ will be used to give lots of different choices of v_j.

The other mapping $\widetilde{F}^\xi$ will have the same form, but with a different choice of the vectors $\{v_j\}$.

For both functions we will choose the v_j inductively, and so that

$$v_j \in B(0, \tau^2) \subset \mathbb{R}^M,$$

with the same very small $\tau > 0$ as in the definition of the scales $r_k = \tau^{2k}$. τ will be chosen later. ($\tau \leq 1 - \alpha$ will do, as we'll see.) With this choice, we see that

$$||f_k^\xi||_\infty \leq \tau^2$$

because the $\varphi_j, j \in J_k(\xi)$ have disjoint supports by (4) and(5). Hence, the series in (7) converges.

Assouad Embedding Theorem, proof (6)

Moreover, if we set

$$F_k^{\xi} = \sum_{k_0 \le \ell \le k} r_\ell^{\alpha} f_\ell^{\xi}(x), \tag{8}$$

we get that

$$||F^{\xi} - F_k^{\xi}||_\infty \le \sum_{\ell \ge k} r_\ell^{\alpha} \tau^2 = r_{k+1}^{\alpha} \tau^2 \sum_{\ell \ge 0} \tau^{2\ell\alpha} \le 2\tau^2 r_{k+1}^{\alpha} \tag{9}$$

because $r_\ell = \tau^{2\ell}$ and $\tau^{2\alpha} < 1/2$ when τ is small.

Assouad Embedding Theorem, proof (7), bound $||\cdot||_{lip}$

The Lipschitz norm of f_k^ξ is

$$||f_k^\xi||_{lip} \leq \tau^2 r_k^{-1}$$

by (6) and because the φ_j are supported on disjoint balls. Thus

$$\begin{aligned} ||F_k^\xi||_{lip} &\leq \sum_{\ell \leq k} r_\ell^\alpha ||f_\ell^\xi||_{lip} \\ &\leq \tau^2 \sum_{\ell \leq k} r_\ell^{\alpha-1} \\ &= \tau^2 r_k^{\alpha-1} \sum_{\ell \leq k} \tau^{2(\ell-k)(\alpha-1)} \\ &= \tau^2 r_k^{\alpha-1} (1 - \tau^{2(1-\alpha)})^{-1} \end{aligned}$$

Assouad Embedding Theorem, proof (8), bound $||\cdot||_{lip}$

Now, take $\tau \leq 1-\alpha$. Then

$$\ln(\tau^{2(1-\alpha)}) = 2(1-\alpha)\ln(\tau) = -2(1-\alpha)\ln(\frac{1}{\tau}) \leq -2\tau\ln(\frac{1}{\tau}).$$

By exponentiating, we get that

$$\tau^{2(1-\alpha)} \leq e^{-2\tau\ln(\frac{1}{\tau})} \leq 1-\tau\ln(\frac{1}{\tau})$$

if τ is small enough, hence

$$1-\tau^{2(1-\alpha)} \geq \tau\ln(\frac{1}{\tau}),$$

and thus

$$||F_k^{\xi}||_{lip} \leq \frac{\tau}{\ln(\frac{1}{\tau})} r_k^{\alpha-1} \leq r_k^{\alpha-1} \tag{10}$$

if τ is small enough, for example $\tau < 1/2$ works.

Assouad Embedding Theorem, proof (9), choose vectors v_j

- We want to choose the vectors $v_j, j \in J_k$ so that the differences $|F_k^\xi(x) - F_k^\xi(y)|$, will be as large as possible in order to allow us to prove that the inverse $(F_k^\xi)^{-1}$ is Lipschitz.
- Fix $k \geq k_0$, suppose that the F_{k-1} were already constructed, and fix a color $\xi \in \Xi$.
- Put any order $<$ on the finite set $J_k(\xi)$.
- We will choose the v_j for F_k^ξ in Lemma 3 using the order $<$.
- We will use the reverse order for $\widetilde{F}_k^\xi$.

Assouad Embedding Theorem, proof (10), choose v_j

Recall that we defined

$$F_k^\xi(y) = F_{k-1}^\xi + r_k^\alpha f_k^\xi(y) = F_{k-1}^\xi + r_k^\alpha \sum_{i \in J_k(\xi)} v_i \varphi_i(y). \tag{11}$$

For each $j \in J_k(\xi)$, we shall also consider the *partial sum* $G_{k,j}^\xi$ defined by

$$G_{k,j}^\xi(y) = F_{k-1}^\xi(y) + r_k^\alpha \sum_{i \in J_k : i < j} v_i \varphi_i(y), \tag{12}$$

which we therefore assume to be known before we choose v_j.

Assouad Embedding Theorem, proof (11), choose v_j

Lemma (3)

For each $j \in J_k(\xi)$, we can choose $v_j \in B(0, \tau^2)$ so that

$$|F_k^\xi(x) - G_{k,j}^\xi(y)| \geq \tau^3 r_k^\alpha \tag{13}$$

for $x \in B_j$ and $y \in B(x_j, 10\tau^{-2} r_k) \setminus 2B_j$.

For $x \in B_j$, we have $\varphi_j(x) = 1$ and $\varphi_i(x) = 0$ for *all* the other indices $i \neq j \in J_k(\xi)$. Thus

$$F_k^\xi(x) = F_{k-1}\xi(x) + r_k^\alpha f_k^\xi(x) = F_{k-1}^\xi(x) + r_k^\alpha v_j \tag{14}$$

by definitions of the functions (7) and (8).

Assouad Embedding Theorem, proof (12): We use $||\cdot||_{lip}$ bounds to work with discrete sets

By the Lipschitz bound for F_k^ξ (10), and its proof we have that

$$||F_k^\xi||_{lip} \leq r_k^{\alpha-1} \text{ and also } ||G_{k,j}^\xi||_{lip} \leq r_k^{\alpha-1}$$

since we just add fewer terms.

- We use this to replace B_j and $B(x_j, 10\tau^{-2}r_k) \setminus 2B_j$ with *discrete sets*.
- Set $\eta = \tau^3 r_k$, and pick an η-dense set X in B_j and an η-dense set Y in $B(x_j, 10\tau^{-2}r_k) \setminus 2B_j$
- We shall prove that we can choose the vectors v_j so that

$$|F_k^\xi(x) - G_{k,j}^\xi(y)| \geq 3\tau^3 r_k^\alpha \tag{15}$$

  for $x' \in X$ and $y' \in Y$.
- We check next that the lemma will follow.

Assouad Embedding Theorem, proof (13): Proof of Lemma 3 assuming that the discrete bound holds

For $x \in B_j$, we can find $x' \in X$ such that

$$|F_k^\xi(x') - F_k^\xi(x)| \leq ||F_k^\xi||_{lip}\eta \leq r_k^{\alpha-1} \cdot \tau^3 r_k = \tau^3 r_k^\alpha.$$

Similarly, for $y \in B(x_j, 10\tau^{-2}r_k) \setminus 2B_j$ we can find $y' inY$ such that

$$|G_{k,j}^\xi(y) - G_{k,j}^\xi(y')| \leq ||G_{k,j}^\xi||_{lip}\eta \leq \tau^3 r_k^\alpha.$$

Then (13) follows. In other words, we can find a vector v_j such that the points considered map to points sufficiently far away from each other.

Assoud Embedding Theorem, proof (14): The discrete sets X and Y are small.

Remark 1.
In a doubling metric space, for $\lambda > 0$, every ball of radius λr can be covered by $C_0\lambda^{N_0}$ balls of radius r, where $N_0 = \log_2 C_0$. To see this replace λ by the next power of 2.

- First we bound $|X|$, the number of elements in X.
- Here we can use the doubling property, as above.
- Thus, we can cover $B_j = B(x_j, r_k)$ by $C_0(2\tau^{-3})^{N_0}$ balls of radius $\eta/2$, since $\eta = \tau^3 r_k$.
- We keep those that meet B_j, pick an element of B_j in each such ball and get an η-dense net X, with $|X| \leq C_0(2\tau^{-3})^{N_0}$ balls of radius $\eta/2$.

Assouad Embedding Theorem, proof (15): The discrete sets X and Y are small.

- Similarly, we can find a discrete η-dense net Y such that

$$|Y| \leq C_0 \left(2\frac{10\tau^{-2}r_k}{\eta}\right)^{N_0} = C_0 \left(\frac{20\tau^{-2}r_k}{\tau^3 r_k}\right)^{N_0} = C_0(20\tau^{-5})^{N_0}.$$

- The *total number of pairs* (x', y') for which we have to check that discrete points map far (15), is thus

$$|X||Y| \leq C_0^2(40\tau^{-8})^{N_0}$$

Assouad Embedding Theorem, proof (16):

A ball $B(0,\tau^2) \subset \mathbb{R}^M$ for sufficiently large M contains a bigger ($|V| > |X||Y|$), maximal finite set V with points separated by at least $7\tau^3$.

- Pick such a set.
- For each pair (x', y'), the different choices of $v_j \in V$ yield the same value of $G^{\xi}_{k,j}(y')$
- The values of $F^{\xi}_k(x')$ differ by at least $7\tau^3 r_k^{\alpha}$, because the only $F^{\xi}_k(x')$ changes by precisely v_j.
- Thus the discrete bound (15) cannot for this pair (x', y') can fail for at most one choice of $v_j \in V$.
- Thus it is enough to show that V has more than $|X||Y|$ elements.
- Take C_M to be the doubling constant of $\mathbb{R}^M$.
- Remark 1 implies that $|V| \geq C_M(\frac{1}{7\tau})^{log_2 C_M}$.
- This is larger than $|X||Y|$ if $C_M > C_0^8$ and τ is small enough, depending on M. Lemma 3 follows.

Assouad Embedding Theorem, proof (17): Define F.

- For each color ξ, choose the vectors v_j, and hence the mapping F_k^{ξ}, as in Lemma (3).
- Also define a second version $\widetilde{F}_k^{\xi}$ using the opposite order on $J_k(\xi)$.
- By (9), $F^{\xi} : E \to \mathbb{R}^M$ is the limit $F^{\xi} = lim_{k \to \infty} F_k^{\xi}$.
- We are ready to check that F, the tensor product of the maps $\{F^{\xi}, \widetilde{F}^{\xi} : \xi \in \Xi\}$, is bilipschitz from the snowflaked space (E, d^{α}).

Assouad Embedding Theorem, proof (18): Lemma 4.

Lemma (4)

We have that

$$\frac{\tau^2}{8} d(x,y)^\alpha \leq |F(x) - F(y)| \leq 5N\tau^{-2(1-\alpha)} d(x,y)^\alpha \text{ for } x, y \in E.$$

Proof.

Let $x, y \in E$ be given. We may assume that $x \neq y$. Let k be such that

$$4r_k \leq d(x,y) \leq 4r_{k-1} = 4\tau^{-2} r_k. \tag{16}$$

Then $r_k \leq 4r_k \leq d(x,y) \leq diam(E) \leq r_{k_0}$ by our definition of k_0 and so $k \geq k_0$.

Assouad Embedding Theorem, proof (19): Lemma 4, proof. Upper bound.

By the choice of r_k as above (16) and the Lipschitz bound (10), we get that

$$|F_k^\xi(x) - F_k^\xi(y)| \leq ||F_k^\xi||_{lip} d(x,y) \leq r_k^{\alpha-1} d(x,y) \leq \left(\frac{d(x,y)}{4\tau^{-2}}\right)^{\alpha-1} d(x,y) \tag{17}$$

But by the estimate (9) we get for sufficiently small τ, that

$$\begin{aligned} ||F^\xi(x) - F^\xi(y)| - |F_k^\xi(x) - F_k^\xi(y)|| &\leq 2||F^\xi - F_k^\xi||_\infty \\ &\leq 4\tau^2 r_{k+1}^\alpha \\ &= 4\tau^2 \tau^{2\alpha} r_k \\ &\leq 2\tau^2 d(x,y)^\alpha. \end{aligned} \tag{18}$$

These inequalities give the upper bound in Lemma (4). Similarly, for $\widetilde{F}_k^\xi$.

Assouad Embedding Theorem, proof (20): Lemma 4, proof. Lower bound.

- For the lower bound, consider the same fixed $x, y \in E$.
- Since we have a covering, by (2), we can find $j \in J_k$ such that $x \in B_j$.
- Let $\xi \in \Xi$ be the color such that $j \in J_k(\xi)$, i.e. the color of the ball B_j.
- consider two cases separately:
  1. $y \in 2B_i$ for some $i \in J_k(\xi)$. Then we need to be able to choose the v_j's as in Lemma (3).
  2. $y \notin 2B_i$ for all $i \in J_k(\xi)$. Then all the terms $\varphi_i(x)$ vanish.
- Note that now we have

$$y \in B(x_j, 10\tau^{-2} r_k) \setminus 2B_j. \tag{19}$$

- $y \in B(x_j, 10\tau^{-2} r_k) \setminus 2B_j$, follows from our choice of scale (16), because $d(x, x_j) \leq r_k$, since $x \in B_j$.
- Moreover, $y \notin 2B_j$, since if $y \in 2B_j$, then $d(x, y) \leq d(x, x_j) + d(x_j, y) \leq 3r_k$, which would contradict our choice of scale (16).

Assouad Embedding Theorem, proof (21): Lemma 4, proof. Lower bound, case $y \in 2B_i$ for some $i \in J_k(\xi)$.

- We have $i \neq j$, by (19).
- Let us assume that $i < j$. Otherwise, we would use $\widetilde{F}_k^\xi$ instead of F_k^ξ.
- All the $\varphi_l(y), l \neq i$, are equal to 0, by the definition of φ_l and the coloring $J_k(\xi)$.
- Then
$$F_k^\xi(y) = F_{k-1}^\xi(y) + r_k^\alpha v_i \varphi_i(y) = G_{k,j}^\xi(y)$$
with $F_{k-1}^\xi(y) = 0$ if $k = k_0$.
- Now we can apply Lemma (3), because of the fact that $y \in B(x_j, 10\tau^{-2} r_k) \setminus 2B_j$. Thus
$$|F_k^\xi(x) - F_k^\xi(y)| = |F_k^\xi(x) - G_{k,j}(y)| \geq \tau^3 r_k^\alpha .$$

Assouad Embedding Theorem, proof (22): Lemma 4, proof. Lower bound, case $y \in 2B_i$ for some $i \in J_k(\xi)$. (2)

- We combine this with (18) and get that

$$|F^\xi(x) - F^\xi(y)| \geq |F_k^\xi(x) - F_k^\xi(y)| - 4\tau^2\tau^{2\alpha} r_k^\alpha$$
$$\geq \tau^3 r_k^\alpha - 4\tau^2 r_{k+1}^\alpha = \tau^3 r_k^\alpha (1 - 4\tau^{-1}\tau^{2\alpha}) \geq \frac{\tau^3 r_k^\alpha}{2}$$

  since by definition, $r_k = \tau^{2k}$ and because we can take $\alpha > 2/3$ and τ small.
- When $1/2 \leq \alpha \leq 2/3$ we could simply use the standard Assouad proof.
- Now

$$|F(x) - F(y)| \geq |F^\xi(x) - F^\xi(y)| \geq \frac{\tau^3 r_k^\alpha}{2}$$
$$\geq \frac{\tau^3}{2}\left(\frac{d(x,y)}{4\tau^{-2}}\right)^\alpha$$
$$\geq \frac{\tau^5}{8} d(x,y)^\alpha$$

Assouad Embedding Theorem, proof (23): Lemma 4, proof. Conclusion.

- This proves the lemma when $y \in 2B_i$ for some $i \in J_k(\xi)$.
- Thus $F_k^\xi(y) = G_{k,j}^\xi(y) = F_{k-1}^\xi(y)$ for all j by the definition of the functions F_k^ξ and $G_{k,j}^\xi$.
- In the other case, all the $\varphi_i(x)$ vanish by the definition of the functions φ_i.
- Thus $F_k^\xi(y) = G_{k,j}^\xi(y) = F_{k-1}^\xi(y)$ as in the first case and we can continue similarly.
- This proves Lemma(4).

Assouad Embedding Theorem, proof (24): E unbounded.

- Now suppose E is an unbounded metric space with doubling constant C_0.
- Fix an origin x_0.
- Apply the construction above to sets $E_m = E \cap B(x_0, 2^m)$.
- The set E_m is itself a doubling metric space with constant C_0^2.
- Why? If $x \in E_m$ and $r > 0$, we can cover the set $E_m \cap B(x, 2r)$ with C_0^2 balls of radius $r/2$ which, when they meet E_m, we can replace with balls of radius r whose centers are in E_m.
- From the proof above, we get a mapping F_m such that

  $$C^{-1} d(x,y)^\alpha \leq |F_m(x) - F_m(y)| \leq C d(x,y)^\alpha$$

  for $x, y \in E_m$, where C depends only on C_0 and α, but not on m.
- We may assume hat $F_m(x_0) = 0$, after possibly adding a constant, which would not destroy the bilipshitz estimate above.

Assouad Embedding Theorem, proof (25): E unbounded. Conclusion.

- Define for each $k \in \mathbb{Z}$, a maximal collection $\{x_j\} \subset E, j \in J_k$, with $d(x_i, x_j) \geq r_k$ for $i \neq j$. This is still at most countable.
- For each x_j, the sequence $\{F_m(x_j)\}$ is bounded. Hence we can extract a subsequence, so that the sequence F_{m_j} converges for each x_j.
- The convergence is uniform on each bounded subset of E and thus the bilipschitz estimate above passes to the limit.
- This completes our proof of the Assouad Embedding Theorem.

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