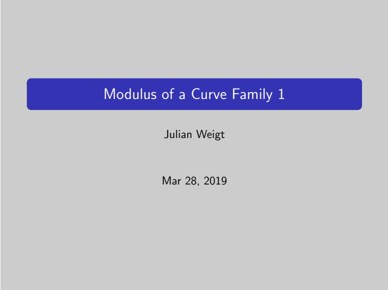
Modulus of a Curve Family 1

Julian Weigt

Mar 28, 2019



1. outer measure on the set of curves

Basics

Let X, d be a metric space. A map $\gamma : [a, b] \to X$ from an interval to the space is called a *curve* if it is continuous.

Basics

Let X, d be a metric space. A map $\gamma : [a, b] \to X$ from an interval to the space is called a *curve* if it is continuous.

Basics

Let X, d be a metric space. A map $\gamma : [a, b] \to X$ from an interval to the space is called a *curve* if it is continuous. The curve is called *rectifiable* if it has finite length, i.e.

$$\ell(\gamma) = \sup\left\{ \sum_{k=1}^{n} d[\gamma(t_k), \gamma(t_{k-1})] \;\middle|\; a \leq t_1 \leq \ldots \leq t_n \leq b \right\} < \infty.$$

Basics

Let X, d be a metric space. A map $\gamma : [a, b] \to X$ from an interval to the space is called a *curve* if it is continuous. The curve is called *rectifiable* if it has finite length, i.e.

$$\ell(\gamma) = \sup\left\{ \sum_{k=1}^{n} d[\gamma(t_k), \gamma(t_{k-1})] \;\middle|\; a \leq t_1 \leq \ldots \leq t_n \leq b \right\} < \infty.$$

Basics

Let X, d be a metric space. A map $\gamma : [a,b] \to X$ from an interval to the space is called a *curve* if it is continuous. The curve is called *rectifiable* if it has finite length, i.e.

$$\ell(\gamma) = \sup\left\{ \sum_{k=1}^{n} d[\gamma(t_k), \gamma(t_{k-1})] \;\middle|\; a \le t_1 \le \ldots \le t_n \le b \right\} < \infty.$$

For $t \in [a,b]$ denote by $s_\gamma(t)$ the length of the curve γ restricted to $[a,t]$.

Proposition

Every rectifiable curve γ has an *arc-length parametrization* $\tilde{\gamma} : [0, \ell(\gamma)] \to X$ with

$$\gamma = \tilde{\gamma} \circ s_\gamma.$$

For all $t \in [0, \ell(\gamma)]$ we have $\ell(\tilde{\gamma}, t) = t$.

Basics

Let X, d be a metric space. A map $\gamma : [a,b] \to X$ from an interval to the space is called a *curve* if it is continuous. The curve is called *rectifiable* if it has finite length, i.e.

$$\ell(\gamma) = \sup\left\{ \sum_{k=1}^{n} d[\gamma(t_k), \gamma(t_{k-1})] \;\middle|\; a \le t_1 \le \ldots \le t_n \le b \right\} < \infty.$$

For $t \in [a,b]$ denote by $s_\gamma(t)$ the length of the curve γ restricted to $[a,t]$.

Proposition

Every rectifiable curve γ has an *arc-length parametrization* $\tilde{\gamma} : [0, \ell(\gamma)] \to X$ with

$$\gamma = \tilde{\gamma} \circ s_\gamma.$$

For all $t \in [0, \ell(\gamma)]$ we have $\ell(\tilde{\gamma}, t) = t$.

1. We do not care about how fast the point moves along the curve thats why we introduce the arc-length parametrization.
2. s_γ is increasing, continuous $[a,b] \to [0, \ell(\gamma)]$. However the inverse is not necessarily well defined. But if s_γ is constant then so is γ, i.e. it does not matter which inverse image we take.
3. But we do not only care about the image of γ, it makes a difference if we move back and forth along the curve.

Basics

Let X, d be a metric space. A map $\gamma : [a, b] \to X$ from an interval to the space is called a *curve* if it is continuous. The curve is called *rectifiable* if it has finite length, i.e.

$$\ell(\gamma) = \sup\left\{ \sum_{k=1}^{n} d[\gamma(t_k), \gamma(t_{k-1})] \;\middle|\; a \leq t_1 \leq \ldots \leq t_n \leq b \right\} < \infty.$$

For $t \in [a, b]$ denote by $s_\gamma(t)$ the length of the curve γ restricted to $[a, t]$.

Proposition

Every rectifiable curve γ has an *arc-length parametrization* $\tilde{\gamma} : [0, \ell(\gamma)] \to X$ with

$$\gamma = \tilde{\gamma} \circ s_\gamma.$$

For all $t \in [0, \ell(\gamma)]$ we have $\ell(\tilde{\gamma}, t) = t$.

From now on all curves are assumed arc-length parametrized.

Basics

Let X, d be a metric space. A map $\gamma : [a, b] \to X$ from an interval to the space is called a *curve* if it is continuous. The curve is called *rectifiable* if it has finite length, i.e.

$$\ell(\gamma) = \sup\left\{ \sum_{k=1}^{n} d[\gamma(t_k), \gamma(t_{k-1})] \;\middle|\; a \leq t_1 \leq \ldots \leq t_n \leq b \right\} < \infty.$$

For $t \in [a, b]$ denote by $s_\gamma(t)$ the length of the curve γ restricted to $[a, t]$.

Proposition

Every rectifiable curve γ has an *arc-length parametrization* $\tilde{\gamma} : [0, \ell(\gamma)] \to X$ with

$$\gamma = \tilde{\gamma} \circ s_\gamma.$$

For all $t \in [0, \ell(\gamma)]$ we have $\ell(\tilde{\gamma}, t) = t$.

From now on all curves are assumed arc-length parametrized.

1. We do not care about how fast the point moves along the curve thats why we introduce the arc-length parametrization.
2. s_γ is increasing, continuous $[a, b] \to [0, \ell(\gamma)]$. However the inverse is not necessarily well defined. But if s_γ is constant then so is γ, i.e. it does not matter which inverse image we take.
3. But we do not only care about the image of γ, it makes a difference if we move back and forth along the curve.

Let $\gamma : I \to \mathbb{R}$ be a curve. For a nonnegative Borel function ρ on X we can define

$$\int_\gamma \rho = \int_I \rho(\gamma(t))\,\mathrm{d}t.$$

Let $\gamma : I \to \mathbb{R}$ be a curve. For a nonnegative Borel function ρ on X we can define

$$\int_\gamma \rho = \int_I \rho(\gamma(t))\,\mathrm{d}t.$$

1. ρ being Borel makes sure that $\rho \circ \gamma$ is measurable.

Let $\gamma : I \to \mathbb{R}$ be a curve. For a nonnegative Borel function ρ on X we can define

$$\int_\gamma \rho = \int_I \rho(\gamma(t))\,\mathrm{d}t.$$

Let μ be a Borel measure on (X, d).

Let $\gamma : I \to \mathbb{R}$ be a curve. For a nonnegative Borel function ρ on X we can define

$$\int_\gamma \rho = \int_I \rho(\gamma(t))\,\mathrm{d}t.$$

Let μ be a Borel measure on (X, d).

1. ρ being Borel makes sure that $\rho \circ \gamma$ is measurable.

Let $\gamma : I \to \mathbb{R}$ be a curve. For a nonnegative Borel function ρ on X we can define

$$\int_\gamma \rho = \int_I \rho(\gamma(t))\,\mathrm{d}t.$$

Let μ be a Borel measure on (X, d). For a family Γ of curves define

$$\mathrm{Mod}_p(\Gamma) = \inf\left\{ \int \rho^p \,\mathrm{d}\mu \;\middle|\; \forall \gamma \in \Gamma \ \int_\gamma \rho \geq 1 \right\}.$$

Such ρ are called *admissible* for Γ.

Let $\gamma : I \to \mathbb{R}$ be a curve. For a nonnegative Borel function ρ on X we can define

$$\int_\gamma \rho = \int_I \rho(\gamma(t))\,\mathrm{d}t.$$

Let μ be a Borel measure on (X, d). For a family Γ of curves define

$$\mathrm{Mod}_p(\Gamma) = \inf\left\{ \int \rho^p \,\mathrm{d}\mu \;\middle|\; \forall \gamma \in \Gamma \ \int_\gamma \rho \geq 1 \right\}.$$

Such ρ are called *admissible* for Γ.

1. ρ being Borel makes sure that $\rho \circ \gamma$ is measurable.
2. $\rho \geq 0$ or $|\rho|^p$.

Let $\gamma : I \to \mathbb{R}$ be a curve. For a nonnegative Borel function ρ on X we can define

$$\int_\gamma \rho = \int_I \rho(\gamma(t))\,\mathrm{d}t.$$

Let μ be a Borel measure on (X, d). For a family Γ of curves define

$$\mathrm{Mod}_p(\Gamma) = \inf\left\{ \int \rho^p \,\mathrm{d}\mu \;\middle|\; \forall \gamma \in \Gamma \; \int_\gamma \rho \geq 1 \right\}.$$

Such ρ are called *admissible* for Γ. The modulus is nonnegative, subadditive

$$\mathrm{Mod}_p\Big(\bigcup_i \Gamma_i\Big) \leq \sum_i \mathrm{Mod}_p(\Gamma_i)$$

and monotone, $\mathrm{Mod}_p(\Gamma) \leq \mathrm{Mod}_p(\Gamma \cup \Phi)$. I.e. Mod_p is an outer measure.

Let $\gamma : I \to \mathbb{R}$ be a curve. For a nonnegative Borel function ρ on X we can define

$$\int_\gamma \rho = \int_I \rho(\gamma(t))\,\mathrm{d}t.$$

Let μ be a Borel measure on (X, d). For a family Γ of curves define

$$\mathrm{Mod}_p(\Gamma) = \inf\left\{ \int \rho^p \,\mathrm{d}\mu \;\middle|\; \forall \gamma \in \Gamma \; \int_\gamma \rho \geq 1 \right\}.$$

Such ρ are called *admissible* for Γ. The modulus is nonnegative, subadditive

$$\mathrm{Mod}_p\Big(\bigcup_i \Gamma_i\Big) \leq \sum_i \mathrm{Mod}_p(\Gamma_i)$$

and monotone, $\mathrm{Mod}_p(\Gamma) \leq \mathrm{Mod}_p(\Gamma \cup \Phi)$. I.e. Mod_p is an outer measure.

1. ρ being Borel makes sure that $\rho \circ \gamma$ is measurable.
2. $\rho \geq 0$ or $|\rho|^p$.
3. For the proof of subadditivity just take the (ℓ^p-)sum of the $(\rho_k)_k$.
4. Mod_p is not a measure on a reasonable σ-algebra though: For a measure we want $\mathrm{Mod}_p(\Gamma \cup) = \mathrm{Mod}_p(\Gamma) + \mathrm{Mod}_p()$ if they are disjoint. But if Γ is a set of curves, and Γ_{-1} the set of curves that go the other way, then $\Gamma \cup \Gamma_{-1}$ can be disjoint however $\mathrm{Mod}_p(\Gamma \cup \Gamma_{-1}) = \mathrm{Mod}_p(\Gamma) = \mathrm{Mod}_p(\Gamma_{-1})$. So this may only be if $\mathrm{Mod}_p(\Gamma) \in \{0, \infty\}$.

Let $\gamma : I \to \mathbb{R}$ be a curve. For a nonnegative Borel function ρ on X we can define

$$\int_\gamma \rho = \int_I \rho(\gamma(t))\,\mathrm{d}t.$$

Let μ be a Borel measure on (X, d). For a family Γ of curves define

$$\mathrm{Mod}_p(\Gamma) = \inf\left\{ \int \rho^p \,\mathrm{d}\mu \;\middle|\; \forall \gamma \in \Gamma \ \int_\gamma \rho \geq 1 \right\}.$$

Such ρ are called *admissible* for Γ. The modulus is nonnegative, subadditive

$$\mathrm{Mod}_p\Big(\bigcup_i \Gamma_i\Big) \leq \sum_i \mathrm{Mod}_p(\Gamma_i)$$

and monotone, $\mathrm{Mod}_p(\Gamma) \leq \mathrm{Mod}_p(\Gamma \cup \Phi)$. I.e. Mod_p is an outer measure. A family Γ is called *p-exceptional* if $\mathrm{Mod}_p(\Gamma) = 0$. We say a property holds *p-almost everywhere* if the family of curves where it does not hold is *p*-exceptional.

Let $\gamma : I \to \mathbb{R}$ be a curve. For a nonnegative Borel function ρ on X we can define

$$\int_\gamma \rho = \int_I \rho(\gamma(t))\,\mathrm{d}t.$$

Let μ be a Borel measure on (X, d). For a family Γ of curves define

$$\mathrm{Mod}_p(\Gamma) = \inf\left\{ \int \rho^p \,\mathrm{d}\mu \;\middle|\; \forall \gamma \in \Gamma \ \int_\gamma \rho \geq 1 \right\}.$$

Such ρ are called *admissible* for Γ. The modulus is nonnegative, subadditive

$$\mathrm{Mod}_p\Big(\bigcup_i \Gamma_i\Big) \leq \sum_i \mathrm{Mod}_p(\Gamma_i)$$

and monotone, $\mathrm{Mod}_p(\Gamma) \leq \mathrm{Mod}_p(\Gamma \cup \Phi)$. I.e. Mod_p is an outer measure. A family Γ is called *p-exceptional* if $\mathrm{Mod}_p(\Gamma) = 0$. We say a property holds *p-almost everywhere* if the family of curves where it does not hold is *p*-exceptional.

1. ρ being Borel makes sure that $\rho \circ \gamma$ is measurable.
2. $\rho \geq 0$ or $|\rho|^p$.
3. For the proof of subadditivity just take the (ℓ^p-)sum of the $(\rho_k)_k$.
4. Mod_p is not a measure on a reasonable σ-algebra though: For a measure we want $\mathrm{Mod}_p(\Gamma \cup) = \mathrm{Mod}_p(\Gamma) + \mathrm{Mod}_p()$ if they are disjoint. But if Γ is a set of curves, and Γ_{-1} the set of curves that go the other way, then $\Gamma \cup \Gamma_{-1}$ can be disjoint however $\mathrm{Mod}_p(\Gamma \cup \Gamma_{-1}) = \mathrm{Mod}_p(\Gamma) = \mathrm{Mod}_p(\Gamma_{-1})$. So this may only be if $\mathrm{Mod}_p(\Gamma) \in \{0, \infty\}$.
5. In the lecture we only cared about if sets are exceptional or not. Here we will also prove some quantitative estimates.

From the lecture

Proposition (Fuglede)

Let $(g_i)_i$ be Borel, converging to a Borel g in $L^p(X,\mu)$. Then there is a subsequence $(g_{i_k})_k$ s.t. for p-a.e. curve γ we have

$$\int_\gamma |g_{i_k} - g| \to 0.$$

From the lecture

Proposition (Fuglede)

Let $(g_i)_i$ be Borel, converging to a Borel g in $L^p(X,\mu)$. Then there is a subsequence $(g_{i_k})_k$ s.t. for p-a.e. curve γ we have

$$\int_\gamma |g_{i_k} - g| \to 0.$$

From the lecture

Proposition (Fuglede)

Let $(g_i)_i$ be Borel, converging to a Borel g in $L^p(X, \mu)$. Then there is a subsequence $(g_{i_k})_k$ s.t. for p-a.e. curve γ we have

$$\int_\gamma |g_{i_k} - g| \to 0.$$

Example

Let γ be a constant curve. Then $\mathrm{Mod}_p(\{\gamma\}) = \infty$.

From the lecture

Proposition (Fuglede)

Let $(g_i)_i$ be Borel, converging to a Borel g in $L^p(X, \mu)$. Then there is a subsequence $(g_{i_k})_k$ s.t. for p-a.e. curve γ we have

$$\int_\gamma |g_{i_k} - g| \to 0.$$

Example

Let γ be a constant curve. Then $\mathrm{Mod}_p(\{\gamma\}) = \infty$.

1. This is because for each ρ we have $\int_\gamma \rho = 0$.

Proposition

Let $E \subset X$ Borel, $\mu(E) = 0$. Then for a.e. curve $\gamma : I \to X$ the set

$$\{t \mid \gamma(t) \in E\}$$

has zero measure; the length of γ in E is zero.

Proposition

Let $E \subset X$ Borel, $\mu(E) = 0$. Then for a.e. curve $\gamma : I \to X$ the set

$$\{t \mid \gamma(t) \in E\}$$

has zero measure; the length of γ in E is zero.

Proposition

Let $E \subset X$ Borel, $\mu(E) = 0$. Then for a.e. curve $\gamma : I \to X$ the set

$$\{t \mid \gamma(t) \in E\}$$

has zero measure; the length of γ in E is zero.

Proof: The function $\infty \cdot 1_E$ is admissible for all functions which have positive length in E and $\int (\infty \cdot 1_E)^p = 0$.

Proposition

Let $E \subset X$ Borel, $\mu(E) = 0$. Then for a.e. curve $\gamma : I \to X$ the set

$$\{t \mid \gamma(t) \in E\}$$

has zero measure; the length of γ in E is zero.

Proof: The function $\infty \cdot 1_E$ is admissible for all functions which have positive length in E and $\int (\infty \cdot 1_E)^p = 0$.

1. This is because

$$\int_\gamma \infty \cdot 1_E = \int_I \infty \cdot 1_{\{t|\gamma(t)\in E\}} = \int_{\{t|\gamma(t)\in E\}} \infty \in \{0, \infty\}.$$

2. This means that $\mu(E) = 0$ is also recognized by the curves. At least almost all of them.

Proposition

Let Γ be a set of curves that all have length at least L in a set A. Then

$$\mathrm{Mod}_p(\Gamma) \leq \mu(A)L^{-p}.$$

Proposition

Let Γ be a set of curves that all have length at least L in a set A. Then

$$\mathrm{Mod}_p(\Gamma) \leq \mu(A)L^{-p}.$$

1. Recall that constant (i.e. very short) curves have modulus ∞. This says that long curves have little modulus.

Proposition

Let Γ be a set of curves that all have length at least L in a set A. Then

$$\mathrm{Mod}_p(\Gamma) \leq \mu(A)L^{-p}.$$

Proof: The function $\frac{1}{L}1_A$ is admissible and $\int(\frac{1}{L}1_A)^p = \mu(A)L^{-p}$.

Proposition

Let Γ be a set of curves that all have length at least L in a set A. Then

$$\mathrm{Mod}_p(\Gamma) \leq \mu(A)L^{-p}.$$

Proof: The function $\frac{1}{L}1_A$ is admissible and $\int(\frac{1}{L}1_A)^p = \mu(A)L^{-p}$.

1. Recall that constant (i.e. very short) curves have modulus ∞. This says that long curves have little modulus.

Proposition

Let $p > 1$ and $(\Gamma_i)_i$ be an increasing family of paths. Then

$$\lim_{i\to\infty} \mathrm{Mod}_p(\Gamma_i) = \mathrm{Mod}_p\Big(\bigcup_i \Gamma_i\Big).$$

Proposition

Let $p > 1$ and $(\Gamma_i)_i$ be an increasing family of paths. Then

$$\lim_{i\to\infty} \mathrm{Mod}_p(\Gamma_i) = \mathrm{Mod}_p\Big(\bigcup_i \Gamma_i\Big).$$

1. First observe that $\leq$ is obvious because $\Gamma_k \subset \bigcup_i \Gamma_i$. However it is not clear if the right hand side is maybe strictly larger.

Proposition

Let $p > 1$ and $(\Gamma_i)_i$ be an increasing family of paths. Then

$$\lim_{i\to\infty} \mathrm{Mod}_p(\Gamma_i) = \mathrm{Mod}_p\left(\bigcup_i \Gamma_i\right).$$

Proof: Let $(\rho_i)_i$ be a corresponding sequence of admissible functions with $\int \rho_i^p \leq \mathrm{Mod}_p(\Gamma_i) + \frac{1}{i}$. It is bounded and hence has a weakly convergent subsequence. Its weak limit ρ can be written as a strong limit of convex combinations of the $(\rho_i)_i$ with arbitrarily large indeces i. Since $(\Gamma_i)_i$ are increasing, these convex combinations are again admissible functions.

Proposition

Let $p > 1$ and $(\Gamma_i)_i$ be an increasing family of paths. Then

$$\lim_{i\to\infty} \mathrm{Mod}_p(\Gamma_i) = \mathrm{Mod}_p\left(\bigcup_i \Gamma_i\right).$$

Proof: Let $(\rho_i)_i$ be a corresponding sequence of admissible functions with $\int \rho_i^p \leq \mathrm{Mod}_p(\Gamma_i) + \frac{1}{i}$. It is bounded and hence has a weakly convergent subsequence. Its weak limit ρ can be written as a strong limit of convex combinations of the $(\rho_i)_i$ with arbitrarily large indeces i. Since $(\Gamma_i)_i$ are increasing, these convex combinations are again admissible functions.

1. First observe that $\leq$ is obvious because $\Gamma_k \subset \bigcup_i \Gamma_i$. However it is not clear if the right hand side is maybe strictly larger.
2. Let $\gamma \in \Gamma_i$. Then for each $k \geq i$ also $\gamma \in \Gamma_k$. Thus

$$\int_\gamma \sum_{k\geq i} c_k \rho_k = \sum_{k\geq i} c_k \int_\gamma \rho_k \geq \sum_{k \geq i} c_k = 1.$$

Proposition

Let $p > 1$ and $(\Gamma_i)_i$ be an increasing family of paths. Then

$$\lim_{i\to\infty} \operatorname{Mod}_p(\Gamma_i) = \operatorname{Mod}_p\Big(\bigcup_i \Gamma_i\Big).$$

Proof: Let $(\rho_i)_i$ be a corresponding sequence of admissible functions with $\int \rho_i^p \leq \operatorname{Mod}_p(\Gamma_i) + \frac{1}{i}$. It is bounded and hence has a weakly convergent subsequence. Its weak limit ρ can be written as a strong limit of convex combinations of the $(\rho_i)_i$ with arbitrarily large indeces i. Since $(\Gamma_i)_i$ are increasing, these convex combinations are again admissible functions. By Fuglede so is their L^p-limit.

Proposition

Let $p > 1$ and $(\Gamma_i)_i$ be an increasing family of paths. Then

$$\lim_{i\to\infty} \operatorname{Mod}_p(\Gamma_i) = \operatorname{Mod}_p\Big(\bigcup_i \Gamma_i\Big).$$

Proof: Let $(\rho_i)_i$ be a corresponding sequence of admissible functions with $\int \rho_i^p \leq \operatorname{Mod}_p(\Gamma_i) + \frac{1}{i}$. It is bounded and hence has a weakly convergent subsequence. Its weak limit ρ can be written as a strong limit of convex combinations of the $(\rho_i)_i$ with arbitrarily large indeces i. Since $(\Gamma_i)_i$ are increasing, these convex combinations are again admissible functions. By Fuglede so is their L^p-limit.

1. First observe that $\leq$ is obvious because $\Gamma_k \subset \bigcup_i \Gamma_i$. However it is not clear if the right hand side is maybe strictly larger.
2. Let $\gamma \in \Gamma_i$. Then for each $k \geq i$ also $\gamma \in \Gamma_k$. Thus

$$\int_\gamma \sum_{k\geq i} c_k \rho_k = \sum_{k\geq i} c_k \int_\gamma \rho_k \geq \sum_{k\geq i} c_k = 1.$$

3. We have

$$\lim_{i\to\infty} \sum_{k\geq i} c_k^i \rho_k = \rho \qquad \text{in } L^p$$

By Fuglede this gives for all i and a.e. $\gamma \in \Gamma_i$ that

$$\int_\gamma \rho = \lim_{i\to\infty} \int_\gamma \sum_{k\geq i} c_k^i \rho_k \geq 1.$$

Curves connecting two sets

It fails for $p = 1$:

Curves connecting two sets

It fails for $p = 1$:

Curves connecting two sets

It fails for $p = 1$: For two sets $A, B \subset X$ denote by $\mathrm{Mod}_p(A, B)$ the modulus of the collection of curves that start in A and end in B.

Curves connecting two sets

It fails for $p = 1$: For two sets $A, B \subset X$ denote by $\mathrm{Mod}_p(A, B)$ the modulus of the collection of curves that start in A and end in B.

Curves connecting two sets

It fails for $p = 1$: For two sets $A, B \subset X$ denote by $\mathrm{Mod}_p(A, B)$ the modulus of the collection of curves that start in A and end in B.

$$\mathrm{Mod}_1\left[\overline{B_1}, \mathbb{R}^n \setminus B_{1+\varepsilon}\right] \lesssim 1 \tag{1}$$

$$\mathrm{Mod}_1\left[\overline{B_1}, \mathbb{R}^n \setminus \overline{B_1}\right] = \infty \tag{2}$$

Curves connecting two sets

It fails for $p = 1$: For two sets $A, B \subset X$ denote by $\mathrm{Mod}_p(A, B)$ the modulus of the collection of curves that start in A and end in B.

$$\mathrm{Mod}_1\left[\overline{B_1}, \mathbb{R}^n \setminus B_{1+\varepsilon}\right] \lesssim 1 \tag{1}$$

$$\mathrm{Mod}_1\left[\overline{B_1}, \mathbb{R}^n \setminus \overline{B_1}\right] = \infty \tag{2}$$

1. Take Γ_i to be the curves going from $\overline{B_1}$ to $\mathbb{R}^n \setminus B_{1+\frac{1}{i}}$. Draw a picture.

Curves connecting two sets

It fails for $p = 1$: For two sets $A, B \subset X$ denote by $\mathrm{Mod}_p(A, B)$ the modulus of the collection of curves that start in A and end in B.

$$\mathrm{Mod}_1\left[\overline{B_1}, \mathbb{R}^n \setminus B_{1+\varepsilon}\right] \lesssim 1 \tag{1}$$

$$\mathrm{Mod}_1\left[\overline{B_1}, \mathbb{R}^n \setminus \overline{B_1}\right] = \infty \tag{2}$$

The function $\rho_\varepsilon = \frac{1}{\varepsilon}1_{\{x:1\le|x|\le1+\varepsilon\}}$ is admissible for (1) and $\int \rho_\varepsilon = \frac{1}{\varepsilon}\omega_n[(1+\varepsilon)^n - 1^n] \to \omega_{n-1}$.

Curves connecting two sets

It fails for $p = 1$: For two sets $A, B \subset X$ denote by $\mathrm{Mod}_p(A, B)$ the modulus of the collection of curves that start in A and end in B.

$$\mathrm{Mod}_1\left[\overline{B_1}, \mathbb{R}^n \setminus B_{1+\varepsilon}\right] \lesssim 1 \tag{1}$$

$$\mathrm{Mod}_1\left[\overline{B_1}, \mathbb{R}^n \setminus \overline{B_1}\right] = \infty \tag{2}$$

The function $\rho_\varepsilon = \frac{1}{\varepsilon}1_{\{x:1\le|x|\le1+\varepsilon\}}$ is admissible for (1) and $\int \rho_\varepsilon = \frac{1}{\varepsilon}\omega_n[(1+\varepsilon)^n - 1^n] \to \omega_{n-1}$.

1. Take Γ_i to be the curves going from $\overline{B_1}$ to $\mathbb{R}^n \setminus B_{1+\frac{1}{i}}$. Draw a picture.
2. Draw a picture.

Curves connecting two sets

It fails for $p = 1$: For two sets $A, B \subset X$ denote by $\mathrm{Mod}_p(A, B)$ the modulus of the collection of curves that start in A and end in B.

$$\mathrm{Mod}_1\left[\overline{B_1}, \mathbb{R}^n \setminus B_{1+\varepsilon}\right] \lesssim 1 \tag{1}$$

$$\mathrm{Mod}_1\left[\overline{B_1}, \mathbb{R}^n \setminus \overline{B_1}\right] = \infty \tag{2}$$

The function $\rho_\varepsilon = \frac{1}{\varepsilon} 1_{\{x : 1 \le |x| \le 1+\varepsilon\}}$ is admissible for (1) and $\int \rho_\varepsilon = \frac{1}{\varepsilon}\omega_n[(1+\varepsilon)^n - 1^n] \to \omega_{n-1}$.
However if ρ is admissible for (2) then for each ε and each unit vector e we must have

$$\int_1^{1+\varepsilon} \rho(te)\,\mathrm{d}t \ge 1.$$

This requires $\int_1^2 \rho(te)\,\mathrm{d}t = \infty$ and by integration over e thus $\int \rho = \infty$.

Curves connecting two sets

It fails for $p = 1$: For two sets $A, B \subset X$ denote by $\mathrm{Mod}_p(A, B)$ the modulus of the collection of curves that start in A and end in B.

$$\mathrm{Mod}_1\left[\overline{B_1}, \mathbb{R}^n \setminus B_{1+\varepsilon}\right] \lesssim 1 \tag{1}$$

$$\mathrm{Mod}_1\left[\overline{B_1}, \mathbb{R}^n \setminus \overline{B_1}\right] = \infty \tag{2}$$

The function $\rho_\varepsilon = \frac{1}{\varepsilon} 1_{\{x : 1 \le |x| \le 1+\varepsilon\}}$ is admissible for (1) and $\int \rho_\varepsilon = \frac{1}{\varepsilon}\omega_n[(1+\varepsilon)^n - 1^n] \to \omega_{n-1}$.
However if ρ is admissible for (2) then for each ε and each unit vector e we must have

$$\int_1^{1+\varepsilon} \rho(te)\,\mathrm{d}t \ge 1.$$

This requires $\int_1^2 \rho(te)\,\mathrm{d}t = \infty$ and by integration over e thus $\int \rho = \infty$.

1. Take Γ_i to be the curves going from $\overline{B_1}$ to $\mathbb{R}^n \setminus B_{1+\frac{1}{i}}$. Draw a picture.
2. Draw a picture.
3. Draw a picture
4. So is this counterexample a bug? It could be fixed by also allowing measures and not only L^1-functions. Except it's maybe not clear how to integrate a measure along a curve? Or is it? Idk.

Bound on $\mathrm{Mod}_p\left[\overline{B(x,r)}, X \setminus B(x,R)\right]$

Proposition

Let $x \in X$ and $n \geq 1$. If for all $0 < r < R_0$ we have

$$\mu(B(x,r)) \lesssim r^n$$

then for all $0 < 2r < R < R_0$ we have

$$\mathrm{Mod}_p\left[\overline{B(x,r)}, X \setminus B(x,R)\right] \lesssim_{p,n} \left(\log \frac{R}{r}\right)^{-p} \begin{cases} \log \frac{R}{r} & p = n, \\ r^{n-p} & p > n, \\ R^{n-p} & p < n. \end{cases}$$

Bound on $\mathrm{Mod}_p\left[\overline{B(x,r)}, X \setminus B(x,R)\right]$

Proposition

Let $x \in X$ and $n \geq 1$. If for all $0 < r < R_0$ we have

$$\mu(B(x,r)) \lesssim r^n$$

then for all $0 < 2r < R < R_0$ we have

$$\mathrm{Mod}_p\left[\overline{B(x,r)}, X \setminus B(x,R)\right] \lesssim_{p,n} \left(\log \frac{R}{r}\right)^{-p} \begin{cases} \log \frac{R}{r} & p = n, \\ r^{n-p} & p > n, \\ R^{n-p} & p < n. \end{cases}$$

1. Upper bounds tend to be easier because it suffices to provide an admissible function.
2. In $\mathbb{R}^n$:

$$\sim \begin{cases} \log\left(\frac{R}{r}\right)^{1-p} & p = n \\ \frac{1}{\left|R^{\frac{p-n}{p-1}} - r^{\frac{p-n}{p-1}}\right|^{p-1}} & p \neq n \end{cases}$$

   Coincidence for $n = p$, about a $\log()^{-p}$ too much for $p \neq n$.
3. Note that in case $p = n$ it only depends on $\frac{R}{r}$. That means here the modulus is scaling invariant. In $\mathbb{R}^n$ this corresponds to $p = n$. Maybe keep that in mind, it will appear again in the second talk on this topic in a few weeks.

Bound on $\mathrm{Mod}_p\left[\overline{B(x,r)}, X \setminus B(x,R)\right]$

Proof: Take the smallest s.t. $2^k r \geq R$ i.e. $k \approx \log \frac{R}{r}$. On $B(x,R) \setminus B(x,r)$ set

$$\rho(y) = \frac{2}{k} \frac{1}{\mathrm{d}(x,y)}.$$

Bound on $\mathrm{Mod}_p\left[\overline{B(x,r)}, X \setminus B(x,R)\right]$

Proof: Take the smallest s.t. $2^k r \geq R$ i.e. $k \approx \log \frac{R}{r}$. On $B(x,R) \setminus B(x,r)$ set

$$\rho(y) = \frac{2}{k} \frac{1}{\mathrm{d}(x,y)}.$$

1. Draw graph of ρ

Bound on $\mathrm{Mod}_p\left[\overline{B(x,r)}, X \setminus B(x,R)\right]$

Proof: Take the smallest s.t. $2^k r \geq R$ i.e. $k \approx \log \frac{R}{r}$. On $B(x,R) \setminus B(x,r)$ set

$$\rho(y) = \frac{2}{k}\frac{1}{\mathrm{d}(x,y)}.$$

Then for each curve γ its subcurve γ_j in $B(x, 2^{j+1}r) \setminus B(x, 2^j r)$ has length at least $2^j r$, if $0 \leq j \leq k-1$.

Bound on $\mathrm{Mod}_p\left[\overline{B(x,r)}, X \setminus B(x,R)\right]$

Proof: Take the smallest s.t. $2^k r \geq R$ i.e. $k \approx \log \frac{R}{r}$. On $B(x,R) \setminus B(x,r)$ set

$$\rho(y) = \frac{2}{k}\frac{1}{\mathrm{d}(x,y)}.$$

Then for each curve γ its subcurve γ_j in $B(x, 2^{j+1}r) \setminus B(x, 2^j r)$ has length at least $2^j r$, if $0 \leq j \leq k-1$.

1. Draw graph of ρ
2. Image of concentric balls with doubling radius here.

Bound on $\mathrm{Mod}_p\left[\overline{B(x,r)}, X \setminus B(x,R)\right]$

Proof: Take the smallest s.t. $2^k r \geq R$ i.e. $k \approx \log \frac{R}{r}$. On $B(x,R) \setminus B(x,r)$ set

$$\rho(y) = \frac{2}{k} \frac{1}{\mathrm{d}(x,y)}.$$

Then for each curve γ its subcurve γ_j in $B(x, 2^{j+1}r) \setminus B(x, 2^j r)$ has length at least $2^j r$, if $0 \leq j \leq k-1$. Thus

$$\int_\gamma \rho \geq \sum_{j=0}^{k-1} \int_{\gamma_j} \rho \geq \sum_{j=0}^{k-1} 2^j r \frac{2}{k} \frac{1}{2^{j+1} r} = 1$$

Bound on $\mathrm{Mod}_p\left[\overline{B(x,r)}, X \setminus B(x,R)\right]$

Proof: Take the smallest s.t. $2^k r \geq R$ i.e. $k \approx \log \frac{R}{r}$. On $B(x,R) \setminus B(x,r)$ set

$$\rho(y) = \frac{2}{k} \frac{1}{\mathrm{d}(x,y)}.$$

Then for each curve γ its subcurve γ_j in $B(x, 2^{j+1}r) \setminus B(x, 2^j r)$ has length at least $2^j r$, if $0 \leq j \leq k-1$. Thus

$$\int_\gamma \rho \geq \sum_{j=0}^{k-1} \int_{\gamma_j} \rho \geq \sum_{j=0}^{k-1} 2^j r \frac{2}{k} \frac{1}{2^{j+1} r} = 1$$

1. Draw graph of ρ
2. Image of concentric balls with doubling radius here.

Bound on $\mathrm{Mod}_p\left[\overline{B(x,r)}, X \setminus B(x,R)\right]$

Proof: Take the smallest s.t. $2^k r \geq R$ i.e. $k \approx \log \frac{R}{r}$. On $B(x,R) \setminus B(x,r)$ set

$$\rho(y) = \frac{2}{k}\frac{1}{\mathrm{d}(x,y)}.$$

Then for each curve γ its subcurve γ_j in $B(x, 2^{j+1}r) \setminus B(x, 2^j r)$ has length at least $2^j r$, if $0 \leq j \leq k-1$. Thus

$$\int_\gamma \rho \geq \sum_{j=0}^{k-1} \int_{\gamma_j} \rho \geq \sum_{j=0}^{k-1} 2^j r \frac{2}{k}\frac{1}{2^{j+1} r} = 1$$

$$\int \rho^p = \sum_{j=0}^{k-1} \int_{B(x,2^{j+1}r)\setminus B(x,2^j r)} \rho \leq \sum_{j=0}^{k-1} \mu(B(x, 2^{j+1}r)) \left(\frac{2}{k}\frac{1}{2^j r}\right)^p$$

Bound on $\mathrm{Mod}_p\left[\overline{B(x,r)}, X \setminus B(x,R)\right]$

Proof: Take the smallest s.t. $2^k r \geq R$ i.e. $k \approx \log \frac{R}{r}$. On $B(x,R) \setminus B(x,r)$ set

$$\rho(y) = \frac{2}{k}\frac{1}{\mathrm{d}(x,y)}.$$

Then for each curve γ its subcurve γ_j in $B(x, 2^{j+1}r) \setminus B(x, 2^j r)$ has length at least $2^j r$, if $0 \leq j \leq k-1$. Thus

$$\int_\gamma \rho \geq \sum_{j=0}^{k-1} \int_{\gamma_j} \rho \geq \sum_{j=0}^{k-1} 2^j r \frac{2}{k}\frac{1}{2^{j+1} r} = 1$$

$$\int \rho^p = \sum_{j=0}^{k-1} \int_{B(x,2^{j+1}r)\setminus B(x,2^j r)} \rho \leq \sum_{j=0}^{k-1} \mu(B(x, 2^{j+1}r)) \left(\frac{2}{k}\frac{1}{2^j r}\right)^p$$

1. Draw graph of ρ
2. Image of concentric balls with doubling radius here.

Bound on $\mathrm{Mod}_p\left[\overline{B(x,r)}, X \setminus B(x,R)\right]$

Proof: Take the smallest s.t. $2^k r \geq R$ i.e. $k \approx \log \frac{R}{r}$. On $B(x,R) \setminus B(x,r)$ set

$$\rho(y) = \frac{2}{k}\frac{1}{\mathrm{d}(x,y)}.$$

Then for each curve γ its subcurve γ_j in $B(x,2^{j+1}r) \setminus B(x,2^j r)$ has length at least $2^j r$, if $0 \leq j \leq k-1$. Thus

$$\int_\gamma \rho \geq \sum_{j=0}^{k-1} \int_{\gamma_j} \rho \geq \sum_{j=0}^{k-1} 2^j r \frac{2}{k}\frac{1}{2^{j+1}r} = 1$$

$$\int \rho^p = \sum_{j=0}^{k-1} \int_{B(x,2^{j+1}r)\setminus B(x,2^j r)} \rho \leq \sum_{j=0}^{k-1} \underbrace{\mu(B(x,2^{j+1}r))}_{\lesssim (2^{j+1}r)^n} \left(\frac{2}{k}\frac{1}{2^j r}\right)^p$$

Bound on $\mathrm{Mod}_p\left[\overline{B(x,r)}, X \setminus B(x,R)\right]$

Proof: Take the smallest s.t. $2^k r \geq R$ i.e. $k \approx \log \frac{R}{r}$. On $B(x,R) \setminus B(x,r)$ set

$$\rho(y) = \frac{2}{k}\frac{1}{\mathrm{d}(x,y)}.$$

Then for each curve γ its subcurve γ_j in $B(x,2^{j+1}r) \setminus B(x,2^j r)$ has length at least $2^j r$, if $0 \leq j \leq k-1$. Thus

$$\int_\gamma \rho \geq \sum_{j=0}^{k-1} \int_{\gamma_j} \rho \geq \sum_{j=0}^{k-1} 2^j r \frac{2}{k}\frac{1}{2^{j+1}r} = 1$$

$$\int \rho^p = \sum_{j=0}^{k-1} \int_{B(x,2^{j+1}r)\setminus B(x,2^j r)} \rho \leq \sum_{j=0}^{k-1} \underbrace{\mu(B(x,2^{j+1}r))}_{\lesssim (2^{j+1}r)^n} \left(\frac{2}{k}\frac{1}{2^j r}\right)^p$$

1. Draw graph of ρ
2. Image of concentric balls with doubling radius here.

Bound on $\mathrm{Mod}_p\left[\overline{B(x,r)}, X \setminus B(x,R)\right]$

Proof: Take the smallest s.t. $2^k r \geq R$ i.e. $k \approx \log \frac{R}{r}$. On $B(x,R) \setminus B(x,r)$ set

$$\rho(y) = \frac{2}{k}\frac{1}{\mathrm{d}(x,y)}.$$

Then for each curve γ its subcurve γ_j in $B(x,2^{j+1}r) \setminus B(x,2^j r)$ has length at least $2^j r$, if $0 \leq j \leq k-1$. Thus

$$\int_\gamma \rho \geq \sum_{j=0}^{k-1} \int_{\gamma_j} \rho \geq \sum_{j=0}^{k-1} 2^j r \frac{2}{k}\frac{1}{2^{j+1} r} = 1$$

$$\int \rho^p = \sum_{j=0}^{k-1} \int_{B(x,2^{j+1}r)\setminus B(x,2^j r)} \rho \leq \sum_{j=0}^{k-1} \underbrace{\mu(B(x,2^{j+1}r))}_{\lesssim (2^{j+1}r)^n} \left(\frac{2}{k}\frac{1}{2^j r}\right)^p$$

$$= \frac{1}{k^p} r^{n-p} \sum_{j=0}^{k-1} 2^{(n-p)j}$$

Bound on $\mathrm{Mod}_p\left[\overline{B(x,r)}, X \setminus B(x,R)\right]$

Proof: Take the smallest s.t. $2^k r \geq R$ i.e. $k \approx \log \frac{R}{r}$. On $B(x,R) \setminus B(x,r)$ set

$$\rho(y) = \frac{2}{k}\frac{1}{\mathrm{d}(x,y)}.$$

Then for each curve γ its subcurve γ_j in $B(x,2^{j+1}r) \setminus B(x,2^j r)$ has length at least $2^j r$, if $0 \leq j \leq k-1$. Thus

$$\int_\gamma \rho \geq \sum_{j=0}^{k-1} \int_{\gamma_j} \rho \geq \sum_{j=0}^{k-1} 2^j r \frac{2}{k}\frac{1}{2^{j+1} r} = 1$$

$$\int \rho^p = \sum_{j=0}^{k-1} \int_{B(x,2^{j+1}r)\setminus B(x,2^j r)} \rho \leq \sum_{j=0}^{k-1} \underbrace{\mu(B(x,2^{j+1}r))}_{\lesssim (2^{j+1}r)^n} \left(\frac{2}{k}\frac{1}{2^j r}\right)^p$$

$$= \frac{1}{k^p} r^{n-p} \sum_{j=0}^{k-1} 2^{(n-p)j}$$

1. Draw graph of ρ
2. Image of concentric balls with doubling radius here.

$$\frac{1}{k^p} r^{n-p} \sum_{j=0}^{k-1} 2^{(n-p)j} \lesssim_{n,p} \frac{1}{k^p} \left\{ \vphantom{\sum_{j=0}^{k-1}} \right.$$

$$\frac{1}{k^p} r^{n-p} \sum_{j=0}^{k-1} 2^{(n-p)j} \lesssim_{n,p} \frac{1}{k^p} \left\{ \vphantom{\sum_{j=0}^{k-1}} \right.$$

$$\frac{1}{k^p} r^{n-p} \sum_{j=0}^{k-1} 2^{(n-p)j} \lesssim_{n,p} \frac{1}{k^p} \begin{cases} k & n = p, \\ & \end{cases}$$

$$\frac{1}{k^p} r^{n-p} \sum_{j=0}^{k-1} 2^{(n-p)j} \lesssim_{n,p} \frac{1}{k^p} \begin{cases} k & n = p, \\ & \end{cases}$$

1. Geometric sums with nonzero exponent are always bounded by their largest summand.

$$\frac{1}{k^p} r^{n-p} \sum_{j=0}^{k-1} 2^{(n-p)j} \lesssim_{n,p} \frac{1}{k^p} \begin{cases} k & n = p, \\ r^{n-p} & n < p, \end{cases}$$

$$\frac{1}{k^p} r^{n-p} \sum_{j=0}^{k-1} 2^{(n-p)j} \lesssim_{n,p} \frac{1}{k^p} \begin{cases} k & n = p, \\ r^{n-p} & n < p, \end{cases}$$

1. Geometric sums with nonzero exponent are always bounded by their largest summand.

$$\frac{1}{k^p} r^{n-p} \sum_{j=0}^{k-1} 2^{(n-p)j} \lesssim_{n,p} \frac{1}{k^p} \begin{cases} k & n = p, \\ r^{n-p} & n < p, \\ R^{n-p} & n > p. \end{cases}$$

$$\frac{1}{k^p} r^{n-p} \sum_{j=0}^{k-1} 2^{(n-p)j} \lesssim_{n,p} \frac{1}{k^p} \begin{cases} k & n = p, \\ r^{n-p} & n < p, \\ R^{n-p} & n > p. \end{cases}$$

1. Geometric sums with nonzero exponent are always bounded by their largest summand.

$$\frac{1}{k^p} r^{n-p} \sum_{j=0}^{k-1} 2^{(n-p)j} \lesssim_{n,p} \frac{1}{k^p} \begin{cases} k & n = p, \\ r^{n-p} & n < p, \\ R^{n-p} & n > p. \end{cases}$$

$$\mathrm{Mod}_p \left[\overline{B(x,r)}, X \setminus B(x,R) \right] \lesssim_{p,n} \left(\log \frac{R}{r} \right)^{-p} \begin{cases} \log \frac{R}{r} & n = p, \\ r^{n-p} & n < p, \\ R^{n-p} & n > p. \end{cases}$$

$$\frac{1}{k^p} r^{n-p} \sum_{j=0}^{k-1} 2^{(n-p)j} \lesssim_{n,p} \frac{1}{k^p} \begin{cases} k & n = p, \\ r^{n-p} & n < p, \\ R^{n-p} & n > p. \end{cases}$$

$$\mathrm{Mod}_p \left[\overline{B(x,r)}, X \setminus B(x,R) \right] \lesssim_{p,n} \left(\log \frac{R}{r} \right)^{-p} \begin{cases} \log \frac{R}{r} & n = p, \\ r^{n-p} & n < p, \\ R^{n-p} & n > p. \end{cases}$$

1. Geometric sums with nonzero exponent are always bounded by their largest summand.

$$\frac{1}{k^p} r^{n-p} \sum_{j=0}^{k-1} 2^{(n-p)j} \lesssim_{n,p} \frac{1}{k^p} \begin{cases} k & n = p, \\ r^{n-p} & n < p, \\ R^{n-p} & n > p. \end{cases}$$

$$\mathrm{Mod}_p\left[\overline{B(x,r)}, X \setminus B(x,R)\right] \lesssim_{p,n} \left(\log \frac{R}{r}\right)^{-p} \begin{cases} \log \frac{R}{r} & n = p, \\ r^{n-p} & n < p, \\ R^{n-p} & n > p. \end{cases}$$

Corollary

If there is a $n > p$ or $n \geq p, p > 1$ s.t. for small r we have

$$\mu(B(x,r)) \lesssim r^n$$

then the set of all nonconstant curves through x has 0 modulus.

$$\frac{1}{k^p} r^{n-p} \sum_{j=0}^{k-1} 2^{(n-p)j} \lesssim_{n,p} \frac{1}{k^p} \begin{cases} k & n = p, \\ r^{n-p} & n < p, \\ R^{n-p} & n > p. \end{cases}$$

$$\mathrm{Mod}_p\left[\overline{B(x,r)}, X \setminus B(x,R)\right] \lesssim_{p,n} \left(\log \frac{R}{r}\right)^{-p} \begin{cases} \log \frac{R}{r} & n = p, \\ r^{n-p} & n < p, \\ R^{n-p} & n > p. \end{cases}$$

Corollary

If there is a $n > p$ or $n \geq p, p > 1$ s.t. for small r we have

$$\mu(B(x,r)) \lesssim r^n$$

then the set of all nonconstant curves through x has 0 modulus.

1. Geometric sums with nonzero exponent are always bounded by their largest summand.

$$\frac{1}{k^p} r^{n-p} \sum_{j=0}^{k-1} 2^{(n-p)j} \lesssim_{n,p} \frac{1}{k^p} \begin{cases} k & n = p, \\ r^{n-p} & n < p, \\ R^{n-p} & n > p. \end{cases}$$

$$\mathrm{Mod}_p \left[\overline{B(x,r)}, X \setminus B(x,R) \right] \lesssim_{p,n} \left(\log \frac{R}{r} \right)^{-p} \begin{cases} \log \frac{R}{r} & n = p, \\ r^{n-p} & n < p, \\ R^{n-p} & n > p. \end{cases}$$

Corollary

If there is a $n > p$ or $n \geq p, p > 1$ s.t. for small r we have

$$\mu(B(x,r)) \lesssim r^n$$

then the set of all nonconstant curves through x has 0 modulus.

Proof: Letting $r \to 0$ shows that all curves that go through x and $B(x,R)$ have zero modulus. Summing over all $R_n = \frac{1}{n}$ proves the Corollary.

$$\frac{1}{k^p} r^{n-p} \sum_{j=0}^{k-1} 2^{(n-p)j} \lesssim_{n,p} \frac{1}{k^p} \begin{cases} k & n = p, \\ r^{n-p} & n < p, \\ R^{n-p} & n > p. \end{cases}$$

$$\mathrm{Mod}_p \left[\overline{B(x,r)}, X \setminus B(x,R) \right] \lesssim_{p,n} \left(\log \frac{R}{r} \right)^{-p} \begin{cases} \log \frac{R}{r} & n = p, \\ r^{n-p} & n < p, \\ R^{n-p} & n > p. \end{cases}$$

Corollary

If there is a $n > p$ or $n \geq p, p > 1$ s.t. for small r we have

$$\mu(B(x,r)) \lesssim r^n$$

then the set of all nonconstant curves through x has 0 modulus.

Proof: Letting $r \to 0$ shows that all curves that go through x and $B(x,R)$ have zero modulus. Summing over all $R_n = \frac{1}{n}$ proves the Corollary.

1. Geometric sums with nonzero exponent are always bounded by their largest summand.
2. Actually only those that start in x, but every curve that goes through x has a subcurve that starts in x and it suffices to estimate the modulus of subcurves because an admissible function for a subcurve is an admissible function for the curve.

Modulus and Capacity

For two sets A, B define

$$\mathrm{Cap}_p(A, B) = \inf\left\{ \int g_u^p \;\middle|\; u\,|_A = 0,\;\; u\,|_B = 1 \right\}.$$

Modulus and Capacity

For two sets A, B define

$$\mathrm{Cap}_p(A, B) = \inf\left\{ \int g_u^p \;\middle|\; u\,|_A = 0,\;\; u\,|_B = 1 \right\}.$$

Modulus and Capacity

For two sets A, B define

$$\mathrm{Cap}_p(A, B) = \inf\left\{ \int g_u^p \;\middle|\; u\,|_A = 0,\;\; u\,|_B = 1 \right\}.$$

Proposition

$$\mathrm{Cap}_p(A, B) = \mathrm{Mod}_p(A, B).$$

Modulus and Capacity

For two sets A, B define

$$\mathrm{Cap}_p(A, B) = \inf\left\{ \int g_u^p \;\middle|\; u\,|_A = 0,\;\; u\,|_B = 1 \right\}.$$

Proposition

$$\mathrm{Cap}_p(A, B) = \mathrm{Mod}_p(A, B).$$

Modulus and Capacity

For two sets A, B define

$$\mathrm{Cap}_p(A,B) = \inf\left\{ \int g_u^p \;\middle|\; u\mid_A = 0,\;\; u\mid_B = 1 \right\}.$$

Proposition

$$\mathrm{Cap}_p(A,B) = \mathrm{Mod}_p(A,B).$$

Proof of $\geq$: Let u be an admissible function for the capacity and γ a curve from $x_0 \in A$ to $x_1 \in B$. Then

$$\int_\gamma g_u \geq u(x_1) - u(x_0) = 1,$$

g_u is admissible for $\mathrm{Cap}_p(A,B)$.

Modulus and Capacity

For two sets A, B define

$$\mathrm{Cap}_p(A,B) = \inf\left\{ \int g_u^p \;\middle|\; u\mid_A = 0,\;\; u\mid_B = 1 \right\}.$$

Proposition

$$\mathrm{Cap}_p(A,B) = \mathrm{Mod}_p(A,B).$$

Proof of $\geq$: Let u be an admissible function for the capacity and γ a curve from $x_0 \in A$ to $x_1 \in B$. Then

$$\int_\gamma g_u \geq u(x_1) - u(x_0) = 1,$$

g_u is admissible for $\mathrm{Cap}_p(A,B)$.

$\leq$: let ρ be an admissible function for the modulus. Define

$$u(x) = \inf\left\{ \int_\gamma \rho \;\middle|\; \gamma(0) \in A,\ \gamma(a) = x \right\}$$

Then cap u at 1.

$\leq$: let ρ be an admissible function for the modulus. Define

$$u(x) = \inf\left\{ \int_\gamma \rho \;\middle|\; \gamma(0) \in A,\ \gamma(a) = x \right\}$$

Then cap u at 1.

1. $\gamma : [0, a] \to X$ curve

$\leq$: let ρ be an admissible function for the modulus. Define

$$u(x) = \inf\left\{ \int_\gamma \rho \;\middle|\; \gamma(0) \in A,\ \gamma(a) = x \right\}$$

Then cap u at 1. If $x \in A$ then $u(x) = 0$. If $x \in B$ then $u(x) = 1$. We have to show that ρ is an upper gradient for u. Let γ be a curve connecting $x, y \in X$. Let γ_x and γ_y be curves that begin in $x_0, y_0 \in A$ and where the infimum is almost attained. Then

$$u(y) - u(x) \leq \int_{\gamma_y} \rho - \int_{\gamma_x} \rho + \varepsilon \leq \int_{\gamma_x \cup \gamma} \rho - \int_{\gamma_x} \rho + \varepsilon = \int_\gamma \rho + \varepsilon.$$

$\leq$: let ρ be an admissible function for the modulus. Define

$$u(x) = \inf\left\{ \int_\gamma \rho \;\middle|\; \gamma(0) \in A,\ \gamma(a) = x \right\}$$

Then cap u at 1. If $x \in A$ then $u(x) = 0$. If $x \in B$ then $u(x) = 1$. We have to show that ρ is an upper gradient for u. Let γ be a curve connecting $x, y \in X$. Let γ_x and γ_y be curves that begin in $x_0, y_0 \in A$ and where the infimum is almost attained. Then

$$u(y) - u(x) \leq \int_{\gamma_y} \rho - \int_{\gamma_x} \rho + \varepsilon \leq \int_{\gamma_x \cup \gamma} \rho - \int_{\gamma_x} \rho + \varepsilon = \int_\gamma \rho + \varepsilon.$$

1. $\gamma : [0, a] \to X$ curve

Thanks!

Thanks!