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Research Article

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Regularity of minimizers of the area functional in metric spaces

Abstract: In this article we study minimizers of functionals of linear growth in metric measure spaces. We introduce the generalized problem in this setting, and prove existence and local boundedness of the minimizers. We give counterexamples to show that in general, minimizers are not continuous and can have jump discontinuities inside the domain.

Keywords: Bounded variation, area integral, analysis on metric measure spaces

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1 Introduction

This paper studies minimizers of the nonparametric area integral

$$\mathcal{F}(u,\Omega)=\int_{\Omega}\sqrt{1+|Du|^2}\,dx$$

in metric measure spaces equipped with a doubling measure and a Poincaré inequality. In the Euclidean case the minimizers satisfy the corresponding minimal surface equation

$$\sum_{j=1}^{n}D_j\frac{D_ju}{\sqrt{1+|Du|^2}}=0$$

in an open and bounded subset Ω of $\mathbb{R}^n$. It is well known that an equivalent concept can be obtained as the relaxed area integral

$$\mathcal{F}(u,\Omega)=\inf\left\{\liminf_{i\to\infty}\int_{\Omega}\sqrt{1+|Du_i|^2}\,dx\right\},$$

where the infimum is taken over all sequences of functions $u_i\in C^1(\Omega)$ with $u_i\to u$ in $L^1(\Omega)$ as $i\to\infty$, see [3, Section 5.5] and [15, Chapter 6]. Minimizers are functions of bounded variation (BV) with prescribed boundary values, see [7], [10], [14], [15, Chapter 6], [16] and [21]. The advantage of the variational approach is that it can be adapted to the very general context of metric measure spaces, and it also applies to more general integrals and quasiminimizers with linear growth. Indeed, functions of bounded variation are defined through relaxation in the metric setting, see [1, 2, 4, 22].

Boundary values of BV-functions are a delicate issue already for domains with a smooth boundary in the Euclidean case, since the trace operator fails to be continuous with respect to the weak*-topology in BV. A standard approach is to consider extensions of boundary values to a slightly larger reference domain. Minimizers with the extended boundary values are the same as for the original problem, and they turn out to be independent of the extension. The larger reference domain is also natural in the sense that in general the total variation measure of the minimizer charges the boundary. By using the structure theorem for BV-functions in the Euclidean case it is possible to obtain an integral representation of the area integral with a penalty term for the boundary values, see [15, Chapter 6]. In the metric setting such a formula remains an open question.

We give a definition of the minimizer of a relaxed area integral with prescribed boundary values in metric measure spaces. The direct methods in the calculus of variations can be applied to show that a minimizer exists for an arbitrary bounded domain with BV-boundary values. The necessary compactness result can be found in [22], and the lower semicontinuity property of the area integral is shown in this work.

In the Euclidean case with the Lebesgue measure, minimizers can be shown to be smooth. However, it is somewhat unexpected that the regularity fails even for continuously differentiable weights in the Euclidean case. We give an explicit example of a minimizer that is discontinuous at an interior point of the domain. Similar examples for slightly different functionals are presented in [13, Example 3.1] and [7, p. 132]. This phenomenon occurs only in the case when the variational integral has linear growth. For variational integrals with superlinear growth, the minimizers are locally Hölder continuous by [20]. In particular, these examples show that there does not seem to be hope to extend the regularity theory of minimizers to the metric setting.

Our main result shows that the minimizers are locally bounded, and the previously mentioned examples show that this result is essentially the best possible that can be obtained in this generality. We prove the main result by purely variational techniques without referring to the minimal surface equation. Indeed, the minimizers satisfy a De Giorgi type energy estimate, and the local boundedness follows from an iteration scheme. This point of view may be interesting already in the Euclidean case, because it also applies to quasiminimizers.

2 Preliminaries

In this paper, (X, d, μ) is a complete metric measure space with a Borel regular outer measure μ, and $\operatorname{diam}(X) = \infty$. The measure is assumed to be nontrivial in the sense that $0 < \mu(B(x, r)) < \infty$ for every ball with center $x \in X$ and radius $r > 0$. It is also assumed to be doubling, meaning that there exists a constant $C_D > 0$ such that

$$\mu(B(x, 2r)) \le C_D \mu(B(x, r))$$

for all $x \in X$ and $r > 0$. This implies that

$$\frac{\mu(B(y, R))}{\mu(B(x, r))} \le C\left(\frac{R}{r}\right)^Q$$

for every $r \le R$ and $x \in B(y, R)$, and some $Q > 1$ and $C \ge 1$ that only depend on C_D. Later on the symbol Q will always refer to this exponent, which in a certain manner represents the dimension of the space X. We recall that a complete metric space endowed with a doubling measure is proper, that is, closed and bounded sets are compact.

A nonnegative Borel measurable function g on X is an upper gradient of an extended real valued function u on X if for all paths γ in X, we have

$$|u(x) - u(y)| \le \int_\gamma g\, ds \tag{2.1}$$

whenever both $u(x)$ and $u(y)$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise. Here x and y are the end points of γ. If g is a nonnegative measurable function on X and (2.1) holds for almost every path with respect to the 1-modulus, then g is a 1-weak upper gradient of u. By saying that (2.1) holds for 1-almost every path we mean that it fails only for a path family with zero 1-modulus. A family Γ of paths is of zero 1-modulus if there is a nonnegative Borel measurable function $\rho \in L^1(X)$ such that for all paths $\gamma \in \Gamma$, the path integral $\int_\gamma \rho\, ds$ is infinite.

The collection of all upper gradients, together, play the role of the modulus of the weak gradient of a Sobolev function in the metric setting. We consider the following norm:

$$\|u\|_{N^{1,1}(X)} = \|u\|_{L^1(X)} + \inf_g \|g\|_{L^1(X)},$$

with the infimum taken over all upper gradients g of u. The Newton–Sobolev space considered is the space

$$N^{1,1}(X) = \{u : \|u\|_{N^{1,1}(X)} < \infty\}/\sim,$$

where the equivalence relation $\sim$ is given by $u \sim v$ if and only if

$$\|u - v\|_{N^{1,1}(X)} = 0,$$

see [23]. For more on Newtonian spaces, we refer to [6].

Next we recall the definition and basic properties of functions of bounded variation on metric spaces, see [22].

Definition 2.1. For $u \in L^1_{\mathrm{loc}}(X)$, we define the total variation as

$$\|Du\|(X) = \inf\Big\{\liminf_{i\to\infty} \int_X g_{u_i}\, d\mu : u_i \in \mathrm{Lip}_{\mathrm{loc}}(X),\ u_i \to u \text{ in } L^1_{\mathrm{loc}}(X)\Big\},$$

where g_{u_i} is a 1-weak upper gradient of u_i and $\mathrm{Lip}_{\mathrm{loc}}(X)$ denotes the class of functions that are Lipschitz continuous on compact subsets of X. We say that a function $u \in L^1(X)$ is of bounded variation, and denote $u \in \mathrm{BV}(X)$, if $\|Du\|(X) < \infty$.

By replacing X with an open set $U \subset X$ in the definition of the total variation, we can define $\|Du\|(U)$. For an arbitrary set $A \subset X$, we define

$$\|Du\|(A) = \inf\{\|Du\|(U) : U \supset A,\ U \subset X \text{ is open}\}.$$

If $u \in \mathrm{BV}(X)$, $\|Du\|(\cdot)$ is a finite Borel outer measure by [22, Theorem 3.4].

We say that X supports a $(1,1)$-Poincaré inequality if there exist constants $C_P > 0$ and $\tau \ge 1$ such that for all balls $B(x,r)$, all locally integrable functions u, and all 1-weak upper gradients g of u, we have

$$\fint_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \le C_P r \fint_{B(x,\tau r)} g\, d\mu,$$

where

$$u_{B(x,r)} = \fint_{B(x,r)} u\, d\mu = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u\, d\mu.$$

If the space supports a $(1,1)$-Poincaré inequality, by an approximation argument we get for every $u \in L^1_{\mathrm{loc}}(X)$

$$\fint_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \le C_P r \frac{\|Du\|(B(x,\tau r))}{\mu(B(x,\tau r))},$$

where the constant C_P and the dilation factor τ are the same as in the $(1,1)$-Poincaré inequality. *We assume, without further notice, that the measure μ is doubling and that the space supports a $(1,1)$-Poincaré inequality. For brevity, the $(1,1)$-Poincaré inequality will be called the Poincaré inequality later on.*

The Poincaré inequality implies the Sobolev–Poincaré inequality

$$\Big(\fint_{B(x,r)} |u - u_{B(x,r)}|^{Q/(Q-1)}\, d\mu\Big)^{(Q-1)/Q} \le Cr \fint_{B(x,2\tau r)} g\, d\mu$$

for every $u \in L^1_{\mathrm{loc}}(X)$ and every 1-weak upper gradient g of u [6, Theorem 4.21]. Here the constant $C > 0$ depends only on the doubling constant and the constants in the Poincaré inequality. We will use the following version of the Sobolev inequality for BV-functions.

Lemma 2.2. *There exists a constant $C > 0$, depending only on the doubling constant and the constants in the Poincaré inequality, such that if $B(x,r)$ is a ball in X with $0 < r < \mathrm{diam}(X)$ and $u \in L^1_{\mathrm{loc}}(X)$ with a compact support in $B(x,r)$, then*

$$\Big(\fint_{B(x,r)} |u|^{Q/(Q-1)}\, d\mu\Big)^{(Q-1)/Q} \le \frac{C r}{\mu(B(x,r))} \|Du\|(B(x,r)).$$

Proof. This result follows from Sobolev's inequality ([6, Theorem 5.51]) by an approximation argument. □

We also specify what we mean by boundary values of BV-functions.

Definition 2.3. Let Ω and Ω^* be open subsets of X such that $\Omega \Subset \Omega^*$, and assume that $f \in \mathrm{BV}(\Omega^*)$. We define the space $\mathrm{BV}_f(\Omega)$ as the space of functions $u \in \mathrm{BV}(\Omega^*)$ such that $u = f$ μ-almost everywhere in $\Omega^* \setminus \Omega$.

In particular, when $f = 0$, we get the BV space with zero boundary values $\mathrm{BV}_0(\Omega)$. It is obvious that $u \in \mathrm{BV}_f(\Omega)$ if and only if $u - f \in \mathrm{BV}_0(\Omega)$. Note that we could choose $\Omega^* = X$. However, the approach used here is consistent with that of the Euclidean case in [15, pp. 584–585].

3 Area functional in metric spaces

In this section we consider the existence of a BV-minimizer of a nonparametric area integral subject to given boundary values. Instead of the classical definition, which is not suited to the metric space setting, we introduce a definition based on a relaxation of the area functional, which also takes into account the boundary values in an appropriate way. First we briefly recall the classical Euclidean definition with the Lebesgue measure.

Example 3.1. In the Euclidean case with the Lebesgue measure and an open $\Omega \subset \mathbb{R}^n$ with a Lipschitz boundary, the classical BV-version of the Dirichlet problem for nonparametric minimal surfaces is the following, see [16] and [15, Chapter 6]. Given a function $f \in L^1(\partial\Omega)$ with respect to the $(n-1)$-dimensional Hausdorff measure, find a function $u \in \mathrm{BV}(\Omega)$ minimizing the nonparametric area integral

$$\int_\Omega \sqrt{1 + |Du|^2}\, dx$$

with the boundary values f on $\partial\Omega$ in the sense of traces as in [15, pp. 584–585]. The area functional is defined for BV-functions as

$$\int_\Omega \sqrt{1 + |Du|^2}\, dx = \sup \int_\Omega \left(\psi_{n+1} + \sum_{i=1}^n u \frac{\partial \psi_i}{\partial x_i} \right) dx,$$

where the supremum is taken over all functions $\psi \in C_0^1(\Omega; \mathbb{R}^{n+1})$ satisfying $\|\psi\|_\infty \le 1$. The area functional corresponds to the total variation of the vector valued measure $(\mathcal{L}^n, Du)$.

Let Ω^* be a bounded open set such that $\Omega \Subset \Omega^*$. By Gagliardo's extension theorem in [12], every $f \in L^1(\partial\Omega)$ can be extended to $\Omega^* \setminus \overline{\Omega}$ as $\Phi \in W^{1,1}(\Omega^* \setminus \overline{\Omega})$ satisfying $\Phi = 0$ on $\partial\Omega^*$. The space $\mathrm{BV}_f(\Omega)$ is defined as the space of functions $u \in \mathrm{BV}(\Omega^*)$ such that $u = \Phi$ in $\Omega^* \setminus \Omega$, where Φ is an extension of f.

For functions $u \in \mathrm{BV}_f(\Omega)$ we define the extension of the area integral as

$$\int_\Omega \sqrt{1 + |Du|^2}\, dx + \int_{\partial\Omega} |u - f|\, d\mathcal{H}^{n-1} + \int_{\Omega^* \setminus \overline{\Omega}} \sqrt{1 + |D\Phi|^2}\, dx.$$

By [15, Theorem 8], this integral has a minimizer in $\mathrm{BV}_f(\Omega)$, and since the last term depends only on f, we find that for every $f \in L^1(\partial\Omega)$ there is a minimizer in $\mathrm{BV}(\Omega)$ of the integral

$$\int_\Omega \sqrt{1 + |Du|^2}\, dx + \int_{\partial\Omega} |u - f|\, d\mathcal{H}^{n-1}.$$

Observe that the minimizer is independent of Ω^* and Φ. This problem is the same as the minimization problem in the following definition, see [15, pp. 582–585].

Definition 3.2. Let Ω and Ω^* be bounded open subsets of X such that $\Omega \Subset \Omega^*$, and assume that $f \in \mathrm{BV}(\Omega^*)$. For every $u \in \mathrm{BV}_f(\Omega)$, we define the generalized surface area functional by

$$\mathcal{F}(u, \Omega) = \inf \left\{ \liminf_{i\to\infty} \int_{\Omega^*} \sqrt{1 + g_{u_i}^2}\, d\mu \right\},$$

where g_{u_i} is a 1-weak upper gradient of u_i, and the infimum is taken over sequences of functions $u_i \in \mathrm{Lip}_{\mathrm{loc}}(\Omega^*)$ such that $u_i \to u$ in $L^1_{\mathrm{loc}}(\Omega^*)$. A function $u \in \mathrm{BV}_f(\Omega)$ is a minimizer of the generalized surface area functional with the boundary values f if

$$\mathcal{F}(u,\Omega) = \inf \mathcal{F}(v,\Omega),$$

where the infimum is taken over all $v \in \mathrm{BV}_f(\Omega)$.

Remark 3.3. (1) It is possible to define a local concept of a minimizer by requiring that $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ is a minimizer with the boundary values u in every $\Omega' \Subset \Omega$. It is clear that a minimizer $u \in \mathrm{BV}_f(\Omega)$ of the generalized surface area functional with the boundary values $f \in \mathrm{BV}(\Omega^*)$ is also a local minimizer. We shall only consider minimizers with boundary values in this work.

(2) The set Ω^* is merely a reference set and it does not have an important role for us. Observe that the minimizers do not depend on Ω^*, but the value of the generalized area functional does. However, we are interested in local regularity of the minimizers and in this respect the value of the area functional is irrelevant. Because of these reasons we do not include Ω^* in the notation. We could also require that f is compactly supported in Ω^* by using a simple cutoff function.

(3) The interpretation of the boundary trace of a BV-function is a delicate issue already in the Euclidean case with the Lebesgue measure. We have chosen an approach that generalizes the existing Euclidean results.

Remark 3.4. We could consider more general variational integrals

$$\mathcal{F}(u,\Omega) = \inf\left\{\liminf_{i\to\infty} \int_{\Omega^*} I(g_{u_i})\,d\mu\right\},$$

where $I(p)$ is a continuous and convex function with the linear growth condition

$$\alpha|p| \le I(p) \le \beta(1+|p|)$$

for some $0 < \alpha \le \beta < \infty$. Otherwise the arguments are essentially the same, but in the De Giorgi type estimate, i.e. Theorem 4.1 below, we apply a hole filling technique as in the proof of [17, Theorem 6.5]. For simplicity, we only consider the model case $I(p) = \sqrt{1+|p|^2}$ here.

First we give a useful lower semicontinuity result.

Lemma 3.5. *Let Ω and Ω^* be bounded open subsets of X such that $\Omega \Subset \Omega^*$, and assume that $f \in \mathrm{BV}(\Omega^*)$. If $u, u_i \in \mathrm{BV}_f(\Omega)$, $i = 1,2,\dots$, and $u_i \to u$ in $L^1(\Omega)$ as $i \to \infty$, then*

$$\mathcal{F}(u,\Omega) \le \liminf_{i\to\infty} \mathcal{F}(u_i,\Omega).$$

Proof. For $k = 1,2,\dots$, denote

$$\Omega_k = \left\{y \in \Omega^* : \mathrm{dist}(y, X\setminus\Omega^*) > \frac{1}{k}\right\}.$$

For every $i = 1,2,\dots$, we choose a sequence $(v_{i,j})$, with $v_{i,j} \in \mathrm{Lip}_{\mathrm{loc}}(\Omega^*)$, such that $v_{i,j} \to u_i$ in $L^1_{\mathrm{loc}}(\Omega^*)$ and

$$\int_{\Omega^*} \sqrt{1+g^2_{v_{i,j}}}\,d\mu \to \mathcal{F}(u_i,\Omega)$$

as $j \to \infty$. We choose indices $j(i)$ such that

$$\int_{\Omega_i} |u_i - v_{i,j(i)}|\,d\mu < \frac{1}{i}$$

and

$$\int_{\Omega^*} \sqrt{1+g^2_{v_{i,j(i)}}}\,d\mu < \mathcal{F}(u_i,\Omega) + \frac{1}{i}.$$

We set $\tilde{v}_i = v_{i,j(i)}$ and notice that for every $k = 1,2,\dots$, we have

$$\int_{\Omega_k} |u-\tilde{v}_i|\,d\mu \le \int_{\Omega_k} |u-u_i|\,d\mu + \int_{\Omega_k} |u_i-\tilde{v}_i|\,d\mu \to 0$$

as $i \to \infty$. Hence $\tilde{v}_i \to u$ in $L^1_{\mathrm{loc}}(\Omega^*)$ and, by the definition of the generalized surface area functional, this implies that

$$\mathcal{F}(u, \Omega) \le \liminf_{i\to\infty} \int_{\Omega^*} \sqrt{1 + g_{\tilde{v}_i}^2}\, d\mu \le \liminf_{i\to\infty}\Big(\mathcal{F}(u_i, \Omega) + \frac{1}{i}\Big) = \liminf_{i\to\infty} \mathcal{F}(u_i, \Omega).$$

This completes the proof. □

The following result shows that a minimizer with given boundary values exists. Even in the Euclidean case we cannot expect uniqueness of the minimizer without further assumptions, see [16, 15.12] and [15, Chapter 6].

Theorem 3.6. *Let Ω and Ω^* be bounded open subsets of X such that $\Omega \Subset \Omega^*$. Then for every $f \in \mathrm{BV}(\Omega^*)$ there exists a minimizer $u \in \mathrm{BV}_f(\Omega)$ of the generalized surface area functional with the boundary values f.*

Proof. Denote $m = \inf \mathcal{F}(v, \Omega)$, where the infimum is taken over all $v \in \mathrm{BV}_f(\Omega)$. We can pick a minimizing sequence $u_i \in \mathrm{BV}_f(\Omega)$ such that $\mathcal{F}(u_i, \Omega) \to m$ as $i \to \infty$. Since $a \le \sqrt{1 + a^2}$, we see that

$$\|Du_i\|(\Omega^*) \le \mathcal{F}(u_i, \Omega) \quad \text{for every } i = 1, 2, \dots,$$

and consequently $(\|Du_i\|(\Omega^*))$ is a bounded sequence of real numbers. Since $u_i - f \in \mathrm{BV}_0(\Omega)$, we have

$$\int_{\Omega} |u_i - f|\, d\mu \le C \operatorname{diam}(\Omega) \|D(u_i - f)\|(\overline{\Omega}),$$

where the constant C depends only on the doubling constant and the constants in the Poincaré inequality. For a proof of this fact, we refer to [19, Corollary 2.4]. Now we can estimate

$$\begin{aligned}
\int_{\Omega^*} |u_i|\, d\mu &\le \int_{\Omega^*} |f|\, d\mu + \int_{\Omega} |u_i - f|\, d\mu \\
&\le \int_{\Omega^*} |f|\, d\mu + C \operatorname{diam}(\Omega) \|D(u_i - f)\|(\overline{\Omega}) \\
&\le \int_{\Omega^*} |f|\, d\mu + C \operatorname{diam}(\Omega) \big(\|Du_i\|(\Omega^*) + \|Df\|(\Omega^*)\big).
\end{aligned}$$

This implies that the sequence (u_i) is bounded in $\mathrm{BV}(\Omega^*)$. Thus there is a subsequence, still denoted (u_i), such that $u_i \to u$ as $i \to \infty$ in $L^1_{\mathrm{loc}}(\Omega^*)$ for some $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega^*)$. We refer to [22, Theorem 3.7] for this compactness result. By passing to a subsequence, if necessary, we may assume that $u_i \to u$ pointwise μ-almost everywhere in Ω^*. We see that

$$|u(x) - f(x)| \le |u(x) - u_i(x)| + |u_i(x) - f(x)| \to 0$$

for μ-almost every $x \in \Omega^* \setminus \Omega$ as $i \to \infty$, since the latter term on the right-hand side is identically zero there. This implies that $u = f$ μ-almost everywhere in $\Omega^* \setminus \Omega$, and consequently that $u \in \mathrm{BV}(\Omega^*)$ and $u_i \to u$ in $L^1(\Omega^*)$ as $i \to \infty$. Thus $u \in \mathrm{BV}_f(\Omega)$, and by Lemma 3.5 we conclude that

$$m \le \mathcal{F}(u, \Omega) \le \liminf_{i\to\infty} \mathcal{F}(u_i, \Omega) = m,$$

and this proves the claim. □

4 Local boundedness of minimizers

In this section we study regularity of the minimizers of the generalized surface area functional. We will later see that minimizers may fail to be continuous, and in fact, they may have jump discontinuities inside the domain even for very nice domains and measures. Nonetheless, here we apply the De Giorgi method to show that minimizers are locally bounded. First we derive a De Giorgi type energy estimate for a minimizer of the generalized surface area functional. For convenience, in this chapter we assume that the 1-weak upper gradients of functions are minimal 1-weak upper gradients, see [6, Theorem 2.5, Theorem 2.25].

Theorem 4.1. *Let Ω and Ω^* be bounded open subsets of X such that $\Omega \Subset \Omega^*$, and assume that $f \in \mathrm{BV}(\Omega^*)$. Let $u \in \mathrm{BV}_f(\Omega)$ be a minimizer of the generalized surface area functional with the boundary values f. Assume that $B(x,R) \subset \Omega$, and let $0 < r < R$. Then for every $k \in \mathbb{R}$, we have*

$$\|D(u-k)_+\|(B(x,r)) \le \frac{2}{R-r} \int_{B(x,R)} (u-k)_+ \, d\mu + \mu(A_{k,R}),$$

where $A_{k,R} = B(x,R) \cap \{u > k\}$.

Proof. Let $u_i \in \mathrm{Lip}_{\mathrm{loc}}(\Omega^*)$ be a minimizing sequence such that $u_i \to u$ in $L^1_{\mathrm{loc}}(\Omega^*)$ and that

$$\mathcal{F}(u,\Omega) = \lim_{i\to\infty} \int_{\Omega^*} \sqrt{1+g_{u_i}^2}\, d\mu.$$

Let $k \in \mathbb{R}$ and let us denote $A_{k,R,i} = B(x,R) \cap \{u_i > k\}$. Now for $y \in B(x,R)$, we have

$$\int_{-\infty}^{\infty} |\chi_{A_{k,R}}(y) - \chi_{A_{k,R,i}}(y)| dk = \int_{\min\{u(y),u_i(y)\}}^{\max\{u(y),u_i(y)\}} 1\, dk = |u(y) - u_i(y)|.$$

Thus

$$\int_{B(x,R)} |u-u_i|\, d\mu = \int_{-\infty}^{\infty} \Bigg(\int_{B(x,R)} |\chi_{A_{k,R}} - \chi_{A_{k,R,i}}|\, d\mu \Bigg) dk$$

and hence there exists a subsequence (u_i) such that $\chi_{A_{k,R,i}} \to \chi_{A_{k,R}}$ in $L^1(B(x,R))$ as $i \to \infty$ for $\mathcal{L}^1$-almost every $k \in \mathbb{R}$, see [11, p. 188]. In particular, this implies that $\mu(A_{k,R,i}) \to \mu(A_{k,R})$ as $i \to \infty$ for $\mathcal{L}^1$-almost every $k \in \mathbb{R}$.

Let $k \in \mathbb{R}$ be such that the above convergence takes place. Let $\eta \in \mathrm{Lip}(\Omega)$, $0 \le \eta \le 1$ be a Lipschitz cutoff function such that η has a compact support in $B(x,R)$, $\eta = 1$ in $B(x,r)$ and $g_\eta \le 2/(R-r)$. Let $\varphi_i = -\eta(u_i - k)_+$ and $\varphi = -\eta(u-k)_+$. Since $u + \varphi \in \mathrm{BV}_f(\Omega^*)$, it is an admissible test function for the minimization problem and thus

$$\mathcal{F}(u,\Omega) \le \mathcal{F}(u+\varphi,\Omega).$$

Let $\varepsilon > 0$. Since also $u_i + \varphi_i \to u + \varphi$ in $L^1_{\mathrm{loc}}(\Omega^*)$ as $i \to \infty$, there exists an N_ε such that for every $i \ge N_\varepsilon$, we have

$$\int_{\Omega^*} \sqrt{1+g_{u_i}^2}\, d\mu < \mathcal{F}(u,\Omega) + \frac{\varepsilon}{2}$$

and

$$\mathcal{F}(u+\varphi,\Omega) < \int_{\Omega^*} \sqrt{1+(g_{u_i+\varphi_i})^2}\, d\mu + \frac{\varepsilon}{2}.$$

By combining the three previous inequalities we obtain that

$$\int_{\Omega^*} \sqrt{1+g_{u_i}^2}\, d\mu < \int_{\Omega^*} \sqrt{1+(g_{u_i+\varphi_i})^2}\, d\mu + \varepsilon$$

for $i \ge N_\varepsilon$. Furthermore, since $u_i = u_i + \varphi_i$ in $\Omega^* \setminus A_{k,R,i}$, the locality of the minimal weak upper gradient, see [6, Corollary 2.21], implies that $g_{u_i} = g_{u_i+\varphi_i}$ μ-almost everywhere in $\Omega^* \setminus A_{k,R,i}$. Thus

$$\int_{A_{k,R,i}} \sqrt{1+g_{u_i}^2}\, d\mu < \int_{A_{k,R,i}} \sqrt{1+(g_{u_i+\varphi_i})^2}\, d\mu + \varepsilon$$

for $i \ge N_\varepsilon$. We note that $g_{(u_i-k)_+} = g_{u_i}$ in $A_{k,R,i}$ (see for instance [6, Corollary 2.20]). By combining this with [6, Lemma 2.18], which is based on the fact that functions in $N^{1,1}(\Omega)$ are absolutely continuous on paths outside a family of 1-modulus zero, we conclude that the perturbed function $u_i + \varphi_i$ has the minimal 1-weak upper gradient $g_{u_i+\varphi_i}$ satisfying

$$g_{u_i+\varphi_i} \le (1-\eta) g_{(u_i-k)_+} + g_\eta (u_i - k)_+$$

in $A_{k,R,i}$. By using the previous two estimates and the elementary inequality $a \le \sqrt{1+a^2} \le 1+a$ for $a \ge 0$, we obtain that for $i \ge N_\varepsilon$

$$\begin{aligned}\int_{A_{k,R,i}} g_{(u_i-k)_+}\,d\mu = \int_{A_{k,R,i}} g_{u_i}\,d\mu &\le \int_{A_{k,R,i}} \sqrt{1+g_{u_i}^2}\,d\mu\\ &\le \int_{A_{k,R,i}} \sqrt{1+(g_{u_i+\varphi_i})^2}\,d\mu + \varepsilon\\ &\le \int_{A_{k,R,i}} (1+g_{u_i+\varphi_i})\,d\mu + \varepsilon\\ &\le \int_{A_{k,R,i}} (1-\eta)g_{(u_i-k)_+}\,d\mu + \int_{A_{k,R,i}} g_\eta (u_i-k)_+\,d\mu + \mu(A_{k,R,i}) + \varepsilon.\end{aligned}$$

Thus

$$\int_{A_{k,R,i}} \eta g_{(u_i-k)_+}\,d\mu \le \frac{2}{R-r}\int_{A_{k,R,i}} (u_i-k)_+\,d\mu + \mu(A_{k,R,i}) + \varepsilon,$$

and since $\eta = 1$ in $B(x,r)$ and $A_{k,R,i} \subset B(x,R)$, we obtain

$$\int_{B(x,r)} g_{(u_i-k)_+}\,d\mu \le \frac{2}{R-r}\int_{B(x,R)} (u_i-k)_+\,d\mu + \mu(A_{k,R,i}) + \varepsilon.$$

Since $(u_i-k)_+ \to (u-k)_+$ in $L^1_{\mathrm{loc}}(\Omega^*)$ as $i \to \infty$, the lower semicontinuity of the total variation measure implies that

$$\begin{aligned}\|D(u-k)_+\|(B(x,r)) &\le \liminf_{i\to\infty} \int_{B(x,r)} g_{(u_i-k)_+}\,d\mu\\ &\le \frac{2}{R-r}\int_{B(x,R)} (u-k)_+\,d\mu + \lim_{i\to\infty}\mu(A_{k,R,i}) + \varepsilon\\ &= \frac{2}{R-r}\int_{B(x,R)} (u-k)_+\,d\mu + \mu(A_{k,R}) + \varepsilon.\end{aligned}$$

The claim for $\mathcal{L}^1$-almost every $k \in \mathbb{R}$ follows from this by letting $\varepsilon \to 0$. For an arbitrary $k \in \mathbb{R}$, we take a sequence of numbers $k_i \searrow k$ for which the above estimate holds. Then the lower semicontinuity of the total variation measure implies that

$$\begin{aligned}\|D(u-k)_+\|(B(x,r)) &\le \liminf_{i\to\infty} \|D(u-k_i)_+\|(B(x,r))\\ &\le \liminf_{i\to\infty}\bigg(\frac{2}{R-r}\int_{B(x,R)} (u-k_i)_+\,d\mu + \mu(A_{k_i,R})\bigg)\\ &\le \frac{2}{R-r}\int_{B(x,R)} (u-k)_+\,d\mu + \mu(A_{k,R}).\end{aligned}$$

This completes the proof. □

The following result shows that minimizers are locally bounded.

Theorem 4.2. *Let Ω and Ω^* be bounded open subsets of X such that $\Omega \Subset \Omega^*$, and assume that $f \in \mathrm{BV}(\Omega^*)$. Let $u \in \mathrm{BV}_f(\Omega)$ be a minimizer of the generalized surface area functional with the boundary values f. Assume that $B(x,R) \subset \Omega$ with $R > 0$, and let $k_0 \in \mathbb{R}$. Then*

$$\operatorname*{ess\,sup}_{B(x,R/2)} u \le k_0 + C \fint_{B(x,R)} (u-k_0)_+\,d\mu + R,$$

where the constant C depends only on the doubling constant of the measure and the constants in the Poincaré inequality.

Proof. Let $d > 0$ be a constant that will be fixed later, and let

$$k_i = k_0 + d(1 - 2^{-i})$$

for $i = 0, 1, \ldots$. Moreover, let

$$r_0 = R, \quad r_i = \frac{R}{2} + 2^{-i-1}R, \quad \widetilde{r}_i = \frac{r_i + r_{i+1}}{2},$$

and $A_{k_{i+1},r_i} = B(x, r_i) \cap \{u > k_{i+1}\}$. Denote $B_i = B(x, r_i)$ and $\widetilde{B}_i = B(x, \widetilde{r}_i)$, and observe that $B_{i+1} \subset \widetilde{B}_i \subset B_i$. For every index $i = 0, 1, \ldots$, we define a Lipschitz cutoff function η_i with a compact support in $\widetilde{B}_i$ such that $0 \le \eta_i \le 1$, $\eta_i = 1$ in B_{i+1}, and the weak upper gradient satisfies

$$g_{\eta_i} \le \frac{2}{\widetilde{r}_i - r_{i+1}} = \frac{2^{i+4}}{R}.$$

Let $q = Q/(Q-1)$ as in Lemma 2.2, and take a sequence of nonnegative locally Lipschitz functions (v_j) such that $v_j \to (u - k_{i+1})_+$ in $L^1_{\mathrm{loc}}(\widetilde{B}_i)$ and

$$\|D(u - k_{i+1})_+\|(\widetilde{B}_i) = \lim_{j\to\infty} \int_{\widetilde{B}_i} g_{v_j}\, d\mu.$$

Observe that now $\eta_i v_j \to \eta_i (u - k_{i+1})_+$ in $L^1(\widetilde{B}_i)$ as $j \to \infty$.

We use Hölder's inequality and Lemma 2.2 to obtain

$$\begin{aligned}
\int_{B_{i+1}} (u - k_{i+1})_+\, d\mu &\le \int_{\widetilde{B}_i} \eta_i (u - k_{i+1})_+\, d\mu \\
&\le \mu(\widetilde{B}_i)^{1/q} \bigg(\fint_{\widetilde{B}_i} |\eta_i (u - k_{i+1})_+|^q\, d\mu \bigg)^{1/q} \mu(A_{k_{i+1},r_i})^{1-1/q} \\
&\le CR\mu(A_{k_{i+1},r_i})^{1-1/q} \mu(\widetilde{B}_i)^{1/q-1} \|D(\eta_i (u - k_{i+1})_+)\|(\widetilde{B}_i) \\
&\le CR \bigg(\frac{\mu(A_{k_{i+1},r_i})}{\mu(\widetilde{B}_i)} \bigg)^{1-1/q} \liminf_{j\to\infty} \int_{\widetilde{B}_i} g_{\eta_i v_j}\, d\mu \\
&\le CR \bigg(\frac{\mu(A_{k_{i+1},r_i})}{\mu(\widetilde{B}_i)} \bigg)^{1-1/q} \bigg(\limsup_{j\to\infty} \int_{\widetilde{B}_i} g_{\eta_i} v_j\, d\mu + \limsup_{j\to\infty} \int_{\widetilde{B}_i} g_{v_j}\, d\mu \bigg).
\end{aligned}$$

The last inequality follows from the Leibniz rule [6, Theorem 2.15] for weak upper gradients. The estimate for the weak upper gradient g_{η_i} gives

$$\limsup_{j\to\infty} \int_{\widetilde{B}_i} g_{\eta_i} v_j\, d\mu \le \frac{2^{i+4}}{R} \int_{\widetilde{B}_i} (u - k_{i+1})_+\, d\mu,$$

and Theorem 4.1 implies that

$$\begin{aligned}
\limsup_{j\to\infty} \int_{\widetilde{B}_i} g_{v_j}\, d\mu &= \|D(u - k_{i+1})_+\|(\widetilde{B}_i) \\
&\le \frac{2}{r_i - \widetilde{r}_i} \int_{B_i} (u - k_{i+1})_+\, d\mu + \mu(A_{k_{i+1},r_i}) \\
&= \frac{2^{i+4}}{R} \int_{B_i} (u - k_{i+1})_+\, d\mu + \mu(A_{k_{i+1},r_i}).
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_{B_{i+1}} (u - k_{i+1})_+\, d\mu &\le CR \bigg(\frac{\mu(A_{k_{i+1},r_i})}{\mu(\widetilde{B}_i)} \bigg)^{1-1/q} \bigg(\frac{2^{i+5}}{R} \int_{B_i} (u - k_{i+1})_+\, d\mu + \mu(A_{k_{i+1},r_i}) \bigg) \\
&\le C2^{i+5} \bigg(\frac{\mu(A_{k_{i+1},r_i})}{\mu(\widetilde{B}_i)} \bigg)^{1-1/q} \bigg(\int_{B_i} (u - k_{i+1})_+\, d\mu + R\mu(A_{k_{i+1},r_i}) \bigg). \qquad (4.1)
\end{aligned}$$

In order to estimate the term $R\mu(A_{k_{i+1},r_i})$, we note that $k_{i+1} - k_i = 2^{-(i+1)}d$ for $i = 0, 1, \dots$, which implies that

$$u - k_i > k_{i+1} - k_i = 2^{-(i+1)}d$$

in A_{k_{i+1},r_i}. Thus

$$\mu(A_{k_{i+1},r_i}) \le \frac{2^{i+1}}{d} \int_{B_i} (u - k_i)_+ \, d\mu. \tag{4.2}$$

Let

$$Y_i = \frac{1}{d} \fint_{B_i} (u - k_i)_+ \, d\mu.$$

The doubling property of the measure μ and the fact that $R/2 \le r_{i+1} < \tilde{r}_i < r_i \le R$ imply that

$$\mu(B_{i+1}) \approx \mu(\widetilde{B}_i) \approx \mu(B_i),$$

where the constants of comparison depend only on the doubling constant of the measure. Thus, by combining (4.1) and (4.2) and observing that $k_{i+1} > k_i$, we arrive at

$$\begin{aligned}
Y_{i+1} = \frac{1}{d} \fint_{B_{i+1}} (u - k_{i+1})_+ \, d\mu &\le C2^{i+5} \left(\frac{\mu(A_{k_{i+1},r_i})}{\mu(\widetilde{B}_i)} \right)^{1-1/q} \left(\frac{1}{d} \fint_{B_i} (u - k_i)_+ \, d\mu + \frac{R}{d} \frac{\mu(A_{k_{i+1},r_i})}{\mu(B_i)} \right) \\
&\le C2^{i+5} \left(\frac{\mu(A_{k_{i+1},r_i})}{\mu(\widetilde{B}_i)} \right)^{1-1/q} \left(1 + \frac{2^{i+1}R}{d} \right) \frac{1}{d} \fint_{B_i} (u - k_i)_+ \, d\mu \\
&\le C2^{i+5} \left(\frac{\mu(A_{k_{i+1},r_i})}{\mu(B_i)} \right)^{1-1/q} \left(1 + \frac{2^{i+1}R}{d} \right) Y_i \\
&\le C2^{i+5} (2^{i+1}Y_i)^{1-1/q} \left(\frac{1}{2^{i+1}} + \frac{R}{d} \right) 2^{i+1} Y_i \\
&= C2^{i+5} \left(\frac{1}{2^{i+1}} + \frac{R}{d} \right) (2^{i+1}Y_i)^{1+\alpha} \\
&\le C \left(\frac{1}{2} + \frac{R}{d} \right) (2^{2+\alpha})^i Y_i^{1+\alpha},
\end{aligned}$$

where $\alpha = 1 - 1/q = 1/Q > 0$ and the constant C depends only on the doubling constant of the measure and the constants in the Poincaré inequality. If we now assume that $d \ge R$, then we have that

$$Y_{i+1} \le C(2^{2+\alpha})^i Y_i^{1+\alpha}$$

for $i = 0, 1, \dots$, and the dependencies of the constant C remain the same. We now apply [17, Lemma 7.1] to conclude that $Y_i \to 0$ as $i \to \infty$ provided $d \ge R$ and

$$\frac{1}{d} \fint_{B(x,R)} (u - k_0)_+ \, d\mu \le C^{-1/\alpha} (2^{2+\alpha})^{-1/\alpha^2} = \widetilde{C}^{-1}.$$

Both conditions for d are satisfied if we choose

$$d = \widetilde{C} \fint_{B(x,R)} (u - k_0)_+ \, d\mu + R.$$

Since $B(x, R/2) \subset B_i$ and $(u - k_0 - d)_+ \le (u - k_i)_+$ for $i = 0, 1, \dots$, we have that

$$\int_{B(x,R/2)} (u - k_0 - d)_+ \, d\mu \le \lim_{i\to\infty} \int_{B_i} (u - k_i)_+ \, d\mu = 0.$$

Thus

$$\operatorname*{ess\,sup}_{B(x,R/2)} u \le k_0 + d = k_0 + \widetilde{C} \fint_{B(x,R)} (u - k_0)_+ \, d\mu + R.$$

This proves the result. □

5 Counterexamples on regularity

The classical treatment of the area functional begins with results concerning the existence of Lipschitz continuous minimizers. An important step in the argument is the so-called reduction to the boundary principle, which follows from the maximum principle. The reduction to the boundary principle states that in order to estimate the Lipschitz-constant of a minimizer, it suffices to consider its behavior on the boundary of the domain. More precisely, if u is a Lipschitz continuous minimizer, then

$$\operatorname{Lip}(u) = \sup_{\substack{x\in\Omega \\ y\in\partial\Omega}} \frac{|u(x)-u(y)|}{|x-y|}.$$

The same principle appears also in the study of more general functionals, and it is known that the geometry of the domain plays an important role in the theory, see e.g. [17].

However, in the general context of metric measure spaces, the reduction to the boundary principle fails even for nice domains. In fact, in the weighted one-dimensional case it is relatively easy to construct examples of minimization problems which give minimizers that do not satisfy this principle. This phenomenon gives rise to an example of a discontinuous minimizer of the area functional. Hence the local boundedness result obtained in the last section is essentially optimal in this generality. We begin with a motivating example.

Example 5.1. Let $\mathbb{R}$ be equipped with the usual Euclidean distance and the weighted measure

$$d\mu = \min\{e, e^{x^2}\}\,dx,$$

and let $\Omega = (-1,1)$. Then the area functional takes the form

$$\mathcal{F}(u,\Omega) = \int_{-1}^{1} \sqrt{1+(u'(x))^2}\, e^{x^2}\,dx.$$

Let us suppose that this problem has a Lipschitz continuous minimizer u with boundary values $u(-1)=0$ and $u(1)=1$. The smoothness and convexity of the integrand imply that u is smooth and satisfies the strong form of the Euler–Lagrange equation

$$\frac{\partial}{\partial x}\left(\frac{u'(x)e^{x^2}}{\sqrt{1+(u'(x))^2}}\right) = 0$$

for every $x \in (-1,1)$, see e.g. [7, Theorem 4.6], [10, Theorem 4.12] and [9]. This implies that

$$|u'(x)| = \left(\frac{e^{2x^2}}{C^2} - 1\right)^{-1/2},$$

where the constant C satisfies $0 < |C| < 1$. The largest values of $|u'|$ are obtained near the point $x = 0$, and it is then a straightforward application of the Fundamental Theorem of Calculus to conclude that the reduction to the boundary principle fails.

The next example shows that in general, our problem does not have minimizers in the Sobolev class. In fact, even in the one-dimensional case, the minimizer can be discontinuous inside the domain. This example is in strict contrast with the unweighted case, where the minimizer is a straight line segment joining the boundary values. A similar example for a slightly different functional is presented in [13, Example 3.1]. See also [7, p. 132].

Example 5.2. Let us consider the variational problem in the metric space $\mathbb{R}$ equipped with the Euclidean distance and the measure defined by $d\mu = w\,dx$, where

$$w(x) = \min\left\{\sqrt{2}, \sqrt{1+x^{4/3}}\right\}.$$

Note that w is continuously differentiable in $(-1,1)$.

In this case, we can equivalently look for the minimizers from the unweighted space. Indeed, since w is continuous and $1 \le w \le \sqrt{2}$, the space of BV-functions obtained via the metric measure space definition coincides with the weighted space $\mathrm{BV}(\Omega^*;\mu)$ for any open $\Omega^* \subset \mathbb{R}$, see e.g. [8, Theorem 3.2.3]. Furthermore, there is a one-to-one correspondence between the functions of the weighted space $\mathrm{BV}(\Omega^*;\mu)$ and the functions of the unweighted space $\mathrm{BV}(\Omega^*)$, see [5, Proposition 3.5].

Let $\Omega = (-1,1)$, and let $\Omega^* \Supset \Omega$ be an open and bounded set. Let $u \in \mathrm{BV}(\Omega^*)$ be a generalized minimizer of the problem

$$\mathcal{F}(u,\Omega) = \int_{-1}^{1} \sqrt{1+(u')^2}\, w\, dx,$$

with boundary values $u(-1) = -a$ and $u(1) = a$, where the constant $a > 0$ will be chosen later.

As in Example 3.1, the function u minimizes the functional

$$\widetilde{\mathcal{F}}(u) = \int_{-1}^{1} \sqrt{1+(u')^2}\, w\, dx + \int_{-1}^{1} w\, d|(Du)^s| + w(-1)|u(-1)+a| + w(1)|u(1)-a|,$$

where the boundary values are interpreted in the sense of traces and $(Du)^a = u'\,dx$ denotes the absolutely continuous part and $(Du)^s$ the singular part of the total variation measure Du, see the structure theorem and the further discussion in [15, pp. 583–585].

First we conclude that u attains the correct boundary values $u(-1) = -a$ and $u(1) = a$. If this were not the case, we could consider the function v defined as

$$v(x) = \begin{cases} u(x) - u(-1) - a, & -1 \le x < 0, \\ u(x) - u(1) + a, & 0 \le x \le 1, \end{cases}$$

and obtain that

$$\begin{aligned} \widetilde{\mathcal{F}}(v) &= \int_{(-1,1)\setminus\{0\}} \sqrt{1+(u')^2}\, w\, dx + \int_{(-1,1)\setminus\{0\}} w\, d|(Du)^s| + w(0)|u^-(0) - u^+(0) - u(-1) + u(1) - 2a| \\ &< \int_{(-1,1)\setminus\{0\}} \sqrt{1+(u')^2}\, w\, dx + \int_{(-1,1)\setminus\{0\}} w\, d|(Du)^s| + w(0)|u^-(0) - u^+(0)| + w(-1)|u(-1)+a| + w(1)|u(1)-a| \\ &= \widetilde{\mathcal{F}}(u), \end{aligned}$$

which contradicts the fact that u is a minimizer. Thus u attains the correct boundary values.

On the other hand, any minimizer in $W^{1,1}((-1,1))$ would have to satisfy the weak form of the corresponding Euler–Lagrange equation, see [10, Theorem 4.12] for instance. This together with the DuBois–Reymond's lemma, see [7, Lemma 1.8], then implies that

$$|u'(x)| = \left(\frac{w(x)^2}{C^2} - 1\right)^{-1/2}$$

almost everywhere for some constant C with $0 < |C| \le 1$. Now, choosing $a > 3$, we conclude that

$$\begin{aligned} a = \frac{1}{2}|u(1) - u(-1)| &\le \frac{1}{2}\int_{-1}^{1} |u'(x)|\, dx \\ &\le \frac{1}{2}\int_{-1}^{1} \left(\frac{w(x)^2}{C^2} - 1\right)^{-1/2} dx \\ &\le \frac{1}{2}\int_{-1}^{1} (w(x)^2 - 1)^{-1/2}\, dx \\ &= \frac{1}{2}\int_{-1}^{1} x^{-2/3}\, dx = 3 < a, \end{aligned}$$

which is a contradiction. Thus the minimizer does not belong to Sobolev space, since we saw earlier that any minimizer will attain the correct boundary values.

Let us conclude this example by showing that u has a jump discontinuity at the point $x = 0$, i.e. the singular part of the measure Du has a nontrivial point mass at $x = 0$. To see this, let $u'\,dx + (Du)^s$ be the Lebesgue decomposition of the total variation measure Du. Note that the derivative u' exists almost everywhere. Let us consider the function v defined by $v(-1) = -a$, $v' = u'$ and

$$(Dv)^s = \Big(\int_{-1}^{1} d(Du)^s\Big)\delta_0,$$

where δ_0 is the Dirac delta at $x = 0$. By applying the fact that any $h \in \mathrm{BV}(-1,1)$ has a representative of the form $h(x) = c + Dh((-1,x))$, see [3, Theorem 3.28], it is then straightforward to verify that $v(1) = a$. Furthermore, we have the estimate

$$\begin{aligned}\widetilde{\mathcal{F}}(v) &= \int_{(-1,1)\setminus\{0\}} \sqrt{1+(u')^2}\,w\,dx + w(0)\Big|\int_{-1}^{1} d(Du)^s\Big| \\ &\le \int_{(-1,1)\setminus\{0\}} \sqrt{1+(u')^2}\,w\,dx + \int_{-1}^{1} d|(Du)^s|.\end{aligned}$$

The fact that u is a minimizer implies $\widetilde{\mathcal{F}}(u) \le \widetilde{\mathcal{F}}(v)$, and since the absolutely continuous parts of Du and Dv are the same, the previous estimate gives that

$$\int_{-1}^{1} w\,d|(Du)^s| \le \int_{-1}^{1} d|(Du)^s|.$$

This is true only if $\operatorname{supp}(Du)^s \subset \{0\}$, since $w(x) > 1$ for $x \neq 0$. Since we know that the minimizer is not an absolutely continuous function, we conclude that $(Du)^s$ is not a null measure. Thus u has a jump at point $x = 0$.

Remark 5.3. The question whether the one-dimensional weighted area functional has absolutely continuous minimizers with given boundary values has been studied in [18], where the necessary and sufficient condition for the existence of such minimizers corresponds exactly to the calculation presented in the previous example, see also [9, p. 440].

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