Aalto University
School of Science

Continuation of: Regularity of minimizers of the area functional in metric spaces

Cintia Pacchiano Camacho

Aalto University

March 17, 2019

Heikki Hakkarainen, Juha Kinnunen and Panu Lahti, *Regularity of minimizers of the area funtcional in metric spaces.*

Preliminaries

In this paper, (X, d, μ) is a complete, separable and connected metric space endowed with a Borel measure μ on X. The measure μ is assumed to fulfill a doubling property, i.e. there exist a constant $c \geq 1$ such that

$$0 < \mu\left(B_{2r}(x)\right) \leq C_D \mu\left(B_r(x)\right) < \infty$$

for all radii $r > 0$ and centres $x \in X$.
The doubling condition implies that for any ball $B(y, R)$ in X, $x \in B(y, R)$ and $0 < r \leq R < \infty$, we have

$$\frac{\mu(B(y,R))}{\mu(B(x,r))} \leq C\left(\frac{R}{r}\right)^{Q}$$

for some $Q > 1$ and $C \geq 1$ that only depends on C_μ.

Poincaré inequality

They assume that X supports a weak $(1,1)-$Poincaré inequality, in the sense that there exist a constant $c_P > 0$ and a dilation factor $\tau \geq 1$ such that for all open balls $B_\rho(x_0) \subset X$, for all L^1-functions u on X and all upper gradients $\widetilde{g_u}$ of u there holds

$$\fint_{B_\rho(x_0)} |u - u_{\rho,x_0}| \, \mathrm{d}\mu \leq c_P \rho \fint_{B_{\tau\rho}(x_0)} \widetilde{g_u} \, \mathrm{d}\mu,$$

where

$$u_{\rho,x_0} := \fint_{B_\rho(x_0)} u \, \mathrm{d}\mu := \frac{1}{\mu(B_\rho(x_0))} \int_{B(x,r)} u \, \mathrm{d}\mu.$$

denotes the mean value integral of the function u on the ball $B_\rho(x_0)$ with respect to the measure μ.
The Poincaré inequality implies the Sobolev-Poincaré inequality:

$$\left(\fint_{B(x,r)} |u - u_{B(x,r)}|^{\frac{Q}{(Q-1)}} \mathrm{d}\mu \right)^{\frac{(Q-1)}{Q}} \leq C_S r \fint_{B(x,2\tau r)} g_u \mathrm{d}\mu.$$

for every $u \in L^1_{\mathrm{loc}}(X)$ and every $1-$weak upper gradient g of u. Here the constant $C > 0$ depends only on the doubling constant and the constants in the Poincaré inequality.

Lemma

There exists a constant $C > 0$, depending only on the doubling constant and the constants in the Poincaré inequality, such that if $B(x,r)$ is a ball in X with $0 < r < diam(X)$ and $u \in L^1_{loc}(X)$ with a compact support in $B(x,r)$, then

$$\left(⨍_{B(x,r)} |u|^{\frac{Q}{(Q-1)}} \, d\mu \right)^{\frac{(Q-1)}{Q}} \leq \frac{Cr}{\mu(B(x,r))} \|Du\|(B(x,r)).$$

Definition

Let Ω and Ω^* be bounded open subsets of X such that $\Omega \Subset \Omega^*$, and assume that $f \in BV(\Omega^*)$. For every $u \in BV_f(\Omega)$, we define the generalized surface area functional by

$$\mathcal{F}(u, \Omega) = \inf\{\liminf_{i\to\infty} \int_{\Omega^*} \sqrt{1 + g_{u_i}^2}\, \mathrm{d}\mu\}$$

where g_{u_i} is a $1-$weak upper gradient of u_i, and the infimum is taken over sequences of functions $u_i \in \mathrm{Lip}_{\mathrm{loc}}(\Omega^*)$ such that $u_i \to u$ in $L^1_{\mathrm{loc}}(\Omega^*)$. A function $u \in BV_f(\Omega)$ is a minimizer of the generalized surface area functional with the boundary values f if

$$\mathcal{F}(u, \Omega) = \inf \mathcal{F}(v, \Omega)$$

where the infimum is taken over all $v \in BV_f(\Omega)$

Theorem

Let Ω and Ω^* be bounded open subsets of X such that $\Omega \Subset \Omega^*$. Then for every $f \in BV(\Omega^*)$ there exists a minimizer $u \in BV_f(\Omega)$ of the generalized surface area functional with the boundary values f.

Theorem

Let Ω and Ω^* be bounded open subsets of X such that $\Omega \Subset \Omega^*$, and assume that $f \in BV(\Omega^*)$. Let $u \in BV_f(\Omega)$ be a minimizer of the generalized surface area functional with the boundary values f. Assume that $B(x,R) \subset \Omega$, and let $0 < r < R$. Then for every $k \in \mathbb{R}$, we have

$$\|D(u-k)_+\|(B(x,r)) \leq \frac{2}{R-r}\int_\Omega (u-k)_+ \, d\mu + \mu(A_k, R)$$

where $A_k, R = B(x,R) \cap \{u > k\}$.

Theorem

Let Ω and Ω^* be bounded open subsets of X such that $\Omega \Subset \Omega^*$, and assume that $f \in BV(\Omega^*)$. Let $u \in BV_f(\Omega)$ be a minimizer of the generalized surface area functional with the boundary values f. Assume that $B(x,R) \subset \Omega$ with $R > 0$, and let $k_0 \in \mathbb{R}$. Then

$$\operatorname*{ess\,sup}_{B(x,R/2)} u \le k_0 + C \fint_{B(x,R)} (u - k_0)_+ \, d\mu + R,$$

where the constant C depends only on the doubling constant of the measure and the constants in the Poincaré inequality.

EXTRA

- If $u, v \in \mathcal{N}^{1,p}(X)$, then $g_u = g_v$ a.e. on $\{x \in X : u(x) = v(x)\}$. Moreover, if $c \in \mathbb{R}$ is a constant, then $g_u = 0$ a.e. on $\{x \in X : u(x) = c\}$.
- If $u, v \in \mathcal{N}^{1,p}(X)$, then $g_u \mathcal{X}_{\{u>v\}} + g_v \mathcal{X}_{\{v \geq u\}}$ is a minimal $p-$weak upper gradient of $\max\{u, v\}$ and $g_v \mathcal{X}_{\{u>v\}} + g_u \mathcal{X}_{\{v \geq u\}}$ is a minimal $p-$w.u.g of $\min\{u, v\}$.
- Let $u, v \in \mathcal{N}^{1,p}(X)$ and $\eta \in \mathrm{Lip}(X)$ be such that $0 \leq \eta \leq$. Set $w = u + \eta(v - u) = (1 - \eta)u + \eta v$. Then $g := (1 - \eta)g_u + \eta g_v + |v - u| g_\eta$ is a $p-$w.u.g. of w.
- If $u, v \in \mathcal{N}^{1,p}(X)$, then $|u|g_v + |v|g_u$ is a $p-$w.u.g. of uv.
- Let $\{Y_n\}$ for $n = 0, 1, \cdots$ be a sequence of positive numbers, satisfying the inequalities
$$Y_{n+1} \leq C b^n Y_n^{1+\alpha}$$
where $C, b > 1$ and $\alpha > 0$ are given numbers. If
$$Y_0 \leq C^{-1/\alpha} b^{-1/\alpha^2}$$
then $\{Y_n\} \to 0$ as $n \to \infty$.