Existence of minimizers for functionals of linear growth in metric measure spaces

Kari Väisänen
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In these notes, which are based on the research article Regularity of minimizers of the area functional in metric spaces by Heikki Hakkarainen, Juha Kinnunen and Panu Lahti from 2014, we will consider a complete metric measure space $(X, d, \mu)$, where $\mu$ is a nontrivial doubling Borel regular outer measure. We will also assume that our space is unbounded and supports a ( 1,1 )-Poincaré inequality.

The notes are divided into two parts, we will first give an introduction through an informal example in $\mathbb{R}^{n}$, and only then consider the more general metric case. For convenience we will only consider the area functional, but the same methods apply for any functional of linear growth as stated in remark 9.

Definition 1. Let $\Omega$ and $\Omega^{*}$ be open sets in $X$ such that $\Omega \Subset \Omega^{*}$ and let $f \in \mathrm{BV}\left(\Omega^{*}\right)$. We define $\operatorname{BV}_{f}(\Omega)$ as the space of functions $u \in \operatorname{BV}\left(\Omega^{*}\right)$ such that $u=f$ in $\mu$-almost everywhere in $\Omega^{*} \backslash \Omega$.

In the notation of the previous definition, we call the function $f$ boundary values of a BVfunction. Thus $\mathrm{BV}_{0}(\Omega)$ is the BV -space with zero boundary values and clearly $u \in \mathrm{BV}_{f}(\Omega)$ if and only if $u-f \in \operatorname{BV}_{0}(\Omega)$.
$\Omega^{*}$ is only a reference set and even though we integrate over it in the area functional the minimizers themselves are not dependent of it because of the predetermined boundary values $f$ outside of the subset $\Omega$.

Example 2. $\left(\mathbb{R}^{n}\right)$ By using the reference set $\Omega^{*}$ we are consistent with the Euclidean case, where we already need this kind of set, because while working with boundary values of BV-functions, we often encounter the issue that the approximation procedure leaves us with a measure that has singular part on the boundary of $\Omega$. Thus we need to take a larger reference set to take this part into account. For simplicity, it is often assumed that $\Omega$ has at least Lipschitz boundary.

We may write our minimization problem in $\mathbb{R}^{n}$ as minimize $\int_{\bar{\Omega}} \sqrt{1+|D u|^{2}}$ in the function class $\left\{u \in \operatorname{BV}\left(\Omega^{*}\right) \mid u=\phi\right.$ on $\left.\Omega^{*} \backslash \bar{\Omega}\right\}$, where $\phi \in W^{1,1}\left(\Omega^{*} \backslash \Omega\right)$ has boundary values $\varphi$ on $\partial \Omega$ in the sense of traces.

Traces are a way to make sense of boundary values of Sobolev functions. The idea is to approximate the boundary values by restrictions of smoother functions to the boundary in the sense that the partial integration formula works with the trace playing the role of the function integrated on the boundary of the set.

Furthermore we may write

$$
\int_{\Omega^{*}} \sqrt{1+|D u|^{2}}=\int_{\Omega} \sqrt{1+|D u|^{2}}+\int_{\partial \Omega}\left|(D u)^{s}\right|+\int_{\Omega^{*} \backslash \bar{\Omega}} \sqrt{1+|D \phi|^{2}} d x
$$

Note that the last term only depends on the boundary values and the boundary set is of zero Lebesgue measure and thus there can only exist a singular term of the variational measure on that set. To show that the singular term is precisely of this form is a delicate issue and would require a project of it's own. It follows from a structure theorem that gives an explicit integral representation for the minimizer. An extremely hand waving argument to see the intuition behind this fact is that since the measure is singular, the derivatives of the approximating functions have to blow up and we have $1+D u^{2} \approx D u^{2}$.

To analyze the singular part of the measure on the boundary further, we denote the set $E_{t}=\{u>t\}$ and then we have by the coarea formula and the Gauss-Green formula (technically for all Borel sets on the measure theoretic boundary, but with approximation we can neglect this technicality)

$$
\begin{aligned}
\left|(D u)^{s}\right|(\partial \Omega) & =\|D u\|(\partial \Omega) \\
& =\int_{\mathbb{R}}\left\|D \chi_{E_{t}}\right\|(\partial \Omega) d t \\
& =\int_{\mathbb{R}}\left(\int_{\partial \Omega} \chi_{\partial E_{t}} d \mathcal{H}^{n-1}\right) d t \\
& =\int_{\partial \Omega}\left(\int_{\mathbb{R}} \chi_{\partial E_{t}} d t\right) d \mathcal{H}^{n-1} \\
& =\int_{\partial \Omega}\left(u^{+}-u^{-}\right) d \mathcal{H}^{n-1},
\end{aligned}
$$

where the last equality follows from the fact that

$$
] u^{-}(x), u^{+}(x)\left[\subset\left\{t \in \mathbb{R}: x \in \partial_{M} E_{t} \cap \partial \Omega\right\} \subset\left[u^{-}(x), u^{+}(x)\right] .\right.
$$

Thefore the minimization problem can be written as minimization of

$$
\int_{\Omega} \sqrt{1+|D u|^{2}}+\int_{\partial \Omega}|u-\varphi| d \mathcal{H}^{n-1}
$$

where for the first term we have

$$
\int_{\Omega} \sqrt{1+|D u|^{2}}=\sup \left\{\int _ { \Omega } \left(\phi_{n+1}+u \sum_{i=1}^{n} D_{i} \phi_{i} d x\left|\phi \in C_{0}^{1}\left(\Omega ; \mathbb{R}^{n+1}\right),|\phi| \leq 1\right\} .\right.\right.
$$

This follows from basic calculus, as

$$
\begin{aligned}
\int_{\Omega}\left(\phi_{n+1}+u \sum_{i=1}^{n} D_{i} \phi_{i}\right) d x & =\int_{\Omega}\left(\phi_{n+1}-\sum_{i=1}^{n}\left(\phi_{i} D_{i} u\right)\right) d x \\
& =\int_{\Omega}\left(\phi_{n+1}-\nabla u \cdot\left(\phi_{1}, \ldots, \phi_{n}\right)\right) d x \\
& =\int_{\Omega} \phi \cdot\left((0, \ldots, 0,1)-\left(D_{1} u, \ldots, D_{n} u, 0\right)\right) d x \\
& \leq \int_{\Omega}|\phi|\left|(0, \ldots, 0,1)-\left(D_{1} u, \ldots, D_{n} u, 0\right)\right| d x \\
& \leq \int_{\Omega} \sqrt{|D u|^{2}+1} d x
\end{aligned}
$$

In the metric setting, which is our main topic, we are going to define the area functional by method of relaxation. This method will not be discussed in this project and we will take as granted that the following definition gives a formulation of the same problem as discussed above. The benefit is that we can consider the gradients as functions instead of measures.

Definition 3. Let $\Omega$ and $\Omega^{*}$ be bounded open sets in $X$ such that $\Omega \Subset \Omega^{*}$ and let $f \in B V\left(\Omega^{*}\right)$. We define for any $u \in \mathrm{BV}_{f}(\Omega)$, the surface area functional

$$
\mathcal{I}(u, \Omega)=\inf \left\{\liminf _{i \rightarrow \infty} \int_{\Omega^{*}} \sqrt{1+g_{u_{i}}^{2}} d \mu\right\}
$$

where $g_{u_{i}}$ is the minimal 1-weak upper gradient of $u_{i}$ and the infimum is taken over all sequences $\left(u_{i}\right)_{i}$ with $u_{i} \in \operatorname{Lip}_{\text {loc }}\left(\Omega^{*}\right)$ and $u_{i} \rightarrow u$ in $L_{l o c}^{1}\left(\Omega^{*}\right)$.

Note that $\sqrt{1+g_{u_{i}}^{2}} \leq 1+g_{u_{i}}$ and thus

$$
\begin{aligned}
\mathcal{I}(u, \Omega) & =\inf \left\{\liminf _{i \rightarrow \infty} \int_{\Omega^{*}} \sqrt{1+g_{u_{i}}^{2}} d \mu\right\} \\
& \leq \inf \left\{\liminf _{i \rightarrow \infty} \int_{\Omega^{*}} 1+g_{u_{i}} d \mu\right\} \\
& =\int_{\Omega^{*}} 1 d \mu+\inf \left\{\liminf _{i \rightarrow \infty} \int_{\Omega^{*}} g_{u_{i}} d \mu\right\} \\
& =\mu\left(\Omega^{*}\right)+\|D u\|\left(\Omega^{*}\right) \\
& <\infty
\end{aligned}
$$

as $\Omega^{*}$ is bounded and $u \in \operatorname{BV}\left(\Omega^{*}\right)$.

Remark 4. A function $u \in \operatorname{BV}_{f}(\Omega)$ is a minimizer of the area functional with boundary values $f$ if $\mathcal{I}(u, \Omega)=\inf \mathcal{I}(v, \Omega)$, where the infimum is taken over all $v \in \mathrm{BV}_{f}(\Omega)$.

Next we will show three needed Lemmas; semicontinuity for the area functional, Sobolevtype inequality for BV functions and a compactness result in BV.

Lemma 5. (Semicontinuity) Let $\Omega$ and $\Omega^{*}$ be bounded open sets in $X$ such that $\Omega \Subset \Omega^{*}$ and let $f \in B V\left(\Omega^{*}\right)$. If $u, u_{i} \in \operatorname{BV}_{f}(\Omega)$ for $i=1,2,3 \ldots$ and $u_{i} \rightarrow u$ in $L^{1}(\Omega)$ as $i \rightarrow \infty$, then

$$
\mathcal{I}(u, \Omega) \leq \liminf _{i \rightarrow \infty} \mathcal{I}\left(u_{i}, \Omega\right)
$$

Proof. In order to obtain the required $L_{l o c}^{1}\left(\Omega^{*}\right)$ convergence later, we define for all positive natural numbers $k$ the set

$$
\Omega_{k}=\left\{y \in \Omega^{*}: \operatorname{dist}\left(y, X \backslash \Omega^{*}\right)>\frac{1}{k}\right\} .
$$

Since we aim to estimate the infimum, we will choose for every index $i$ an example sequence $\left(v_{i, j}\right)_{j}$ such that as $j \rightarrow \infty$, we have $v_{i, j} \rightarrow u_{i}$ in $L_{l o c}^{1}\left(\Omega^{*}\right)$ and $\int_{\Omega^{*}} \sqrt{1+g_{v_{i, j}}^{2}} d \mu \rightarrow \mathcal{I}\left(u_{i}, \Omega\right)$. The existance of such sequence is clear from the definition of the area functional. In what follows, we will show that this particular sequence satisfies the claim and thus the infimum of all sequences does too. First we choose the indices $j(i)$ such that

$$
\int_{\Omega_{i}}\left|u_{i}-v_{i, j(i)}\right| d \mu<\frac{1}{i}
$$

and

$$
\int_{\Omega^{*}} \sqrt{1+g_{v_{i, j(i)}}^{2}} d \mu<\mathcal{I}\left(u_{i}, \Omega\right)+\frac{1}{i}
$$

Denoting $v_{i}=v_{i, j(i)}$, we obtain by using the triangle inequality and the fact that $\left|u-u_{i}\right|=0$ outside of $\Omega$

$$
\begin{aligned}
\int_{\Omega_{k}}\left|u-v_{i}\right| d \mu & \leq \int_{\Omega_{k}}\left|u-u_{i}\right| d \mu+\int_{\Omega_{k}}\left|u_{i}-v_{i}\right| d \mu \\
& =\int_{\Omega}\left|u-u_{i}\right| d \mu+\int_{\Omega_{k}}\left|u_{i}-v_{i}\right| d \mu \rightarrow 0
\end{aligned}
$$

as $i \rightarrow \infty$, as $\Omega_{k}$ is increasing. Thus $v_{i} \rightarrow u$ in $L_{l o c}^{1}\left(\Omega^{*}\right)$. Since the area functional takes the infimum of such sequences, we have

$$
\mathcal{I}(u, \Omega) \leq \liminf _{i \rightarrow \infty} \int_{\Omega^{*}} \sqrt{1+g_{v_{i}}^{2}} d \mu \leq \liminf _{i \rightarrow \infty}\left(\mathcal{I}\left(u_{i}, \Omega\right)+\frac{1}{i}\right)=\liminf _{i \rightarrow \infty} \mathcal{I}\left(u_{i}, \Omega\right) .
$$

Lemma 6. If $B(x, r) \subset \Omega^{*}$ is a ball with $0<r<\operatorname{diam}\left(\Omega^{*}\right)$ and $u \in \operatorname{BV}\left(\Omega^{*}\right)$ is compactly supported in $B(x, r)$, then there exists a constant $C>0$ depending only on the doubling constant and the constant in the Poincaré inequality such that

$$
\left(f_{B(x, r)}|u|^{\frac{Q}{Q-1}} d \mu\right)^{\frac{Q-1}{Q}} \leq \frac{C r}{\mu(B(x, r))}\|D u\|(B(x, r))
$$

Proof. Recall from Lemma 5.2 in the lecture notes that as $u \in \mathrm{BV}(B(x, r))$, there exists a sequence $\left(u_{i}\right)_{i} \subset \operatorname{Lip}_{l o c}(B(x, r))$ such that $u_{i} \rightarrow u$ in $L_{l o c}^{1}(B(x, r))$ and

$$
\|D u\|(B(x, r))=\lim _{i \rightarrow \infty} \int_{B(x, r)} g_{u_{i}} d \mu
$$

To use the Sobolev-type inequality from the lecture notes for the approximating functions $u_{i}$, we need to first consider the measure of the set

$$
A_{i}=\left\{y \in B(x, r):\left|u_{i}(y)\right|>0\right\}
$$

Since $u$ is compactly supported in the ball, we can assume that $u_{i}$ are in the Sobolev space with zero boundary values, thus by Remark 6.4 In the lecture notes, we conclude that

$$
\mu\left(A_{i}\right) \leq \gamma \mu(B(x, r))
$$

Now by Lemma 6.3 in the lecture notes we have that for $u_{i}$, the following inequality holds

$$
\begin{equation*}
\left(f_{B(x, r)}\left|u_{i}\right|^{\frac{Q}{Q-1}} d \mu\right)^{\frac{Q-1}{Q}} \leq \frac{C r}{\mu(B(x, r))} \int_{B(x, r)} g_{u_{i}} d \mu \tag{1}
\end{equation*}
$$

By Minkowski and Hölder inequalities, we have

$$
\begin{aligned}
\left(f_{B(x, r)}\left|u_{i}\right|^{\frac{Q}{Q-1}} d \mu\right)^{\frac{Q-1}{Q}} & \leq\left(f_{B(x, r)}\left|u_{i}-u\right|^{\frac{Q}{Q-1}} d \mu\right)^{\frac{Q-1}{Q}}+\left(f_{B(x, r)}|u|^{\frac{Q}{Q-1}} d \mu\right)^{\frac{Q-1}{Q}} \\
& \leq C\left(\int_{B(x, r)}\left|u_{i}-u\right| d \mu+\int_{B(x, r)}|u| d \mu\right) \\
& \rightarrow C \int_{B(x, r)}|u| d \mu<\infty
\end{aligned}
$$

as $u_{i} \rightarrow u$ in $L_{l o c}^{1}$ and $u \in L^{1}(B(x, r))$. Thus by taking limits on both sides of equation 1 . we conclude the claim.

Lemma 7. (Compactness) Let $\left(u_{i}\right)_{i}$ be a bounded sequence in $\mathrm{BV}_{\text {loc }}\left(\Omega^{*}\right)$ with respect to the norm of $L_{l o c}^{1}\left(\Omega^{*}\right)$, such that $\sup _{i}\left\|D u_{i}\right\|(\Omega)<\infty$ for all $\Omega \Subset \Omega^{*}$. Then there exists $u \in \operatorname{BV}_{l o c}\left(\Omega^{*}\right)$ and a subsequence $u_{i_{j}}$ that converges to $u$ in $L_{l o c}^{1}\left(\Omega^{*}\right)$.
This compactness result is stronger than is required in the case of sobolev spaces with $\mathrm{p}>1$, but as we cannot use reflexivity, it is what we have to use. The proof is a project of it's own and is thus omitted in these notes.

Theorem 8. Let $\Omega$ and $\Omega^{*}$ be bounded open sets in $X$ such that $\Omega \Subset \Omega^{*}$. For every $f \in \operatorname{BV}\left(\Omega^{*}\right)$ there exists a unique minimizer $u \in \operatorname{BV}_{f}(\Omega)$ of the area functional $\mathcal{I}(u, \Omega)$ with the boundary values $f$.

Proof. Denote $m=\inf \mathcal{I}(v, \Omega)$, where the infimum is taken over all $v \in \operatorname{BV}_{f}(\Omega)$. By the definition of infimum we can take a minimizing sequence $u_{i} \in \operatorname{BV}_{f}(\Omega)$ such that $\mathcal{I}\left(u_{i}, \Omega\right) \rightarrow m$.
In order to show the required convergence we will use Lemma 7 and thus we proceed by considering the total variation of $u_{i}$ in $\Omega^{*}$, we note that for positive real numbers it holds that $\lambda \leq \sqrt{1+\lambda^{2}}$ and therefore for every index $i$ we have

$$
\begin{aligned}
\left\|D u_{i}\right\|\left(\Omega^{*}\right) & =\inf \left\{\liminf _{i \rightarrow \infty} \int_{\Omega^{*}} g_{u_{i}} d \mu: u_{i} \in \operatorname{Lip}_{l o c}\left(\Omega^{*}\right), u_{i} \rightarrow u \operatorname{in} L_{l o c}^{1}\left(\Omega^{*}\right)\right\} \\
& \leq \inf \left\{\liminf _{i \rightarrow \infty} \int_{\Omega^{*}} \sqrt{1+g_{u_{i}}^{2}} d \mu: u_{i} \in \operatorname{Lip}_{l o c}\left(\Omega^{*}\right), u_{i} \rightarrow u \operatorname{in} L_{l o c}^{1}\left(\Omega^{*}\right)\right\} \\
& =\mathcal{I}\left(u_{i}, \Omega\right)<\infty
\end{aligned}
$$

and thus $\left\|D u_{i}\right\|\left(\Omega^{*}\right)$ is a bounded sequence. Now take $x \in \Omega$ and choose $r \in \mathbb{R}$ such that $\operatorname{diam}(\Omega)<r<\operatorname{diam}\left(\Omega^{*}\right)$, then we clearly have $\Omega \subset B(x, r)$. By Hölder's inequality

$$
\begin{aligned}
\int_{\Omega}\left|u_{i}-f\right| d \mu & \leq \int_{B(x, r)}\left|u_{i}-f\right| d \mu \\
& \leq\left(\int_{B(x, r)}\left|u_{i}-f\right|^{\frac{Q}{Q-1}} d \mu\right)^{\frac{Q-1}{Q}} \mu(B(x, r))^{\frac{1}{Q}}
\end{aligned}
$$

because

$$
\begin{aligned}
\mu(B(x, r))^{\frac{1}{Q}} & =\frac{1}{\mu(B(x, r))^{\frac{-1}{Q}}} \\
& =\frac{1}{\mu(B(x, r))^{\frac{Q-1}{Q}}} \mu(B(x, r))
\end{aligned}
$$

as $u_{i}-f \in \mathrm{BV}_{0}(\Omega)$, we have by Lemma 6

$$
\begin{aligned}
\int_{\Omega}\left|u_{i}-f\right| d \mu & \leq\left(f_{B(x, r)}\left|u_{i}-f\right|^{\frac{Q}{Q-1}} d \mu\right)^{\frac{Q-1}{Q}} \mu(B(x, r)) \\
& \leq C \operatorname{diam}(\Omega)\left\|D\left(u_{i}-f\right)\right\|(B(x, r)) \\
& =C \operatorname{diam}(\Omega)\left\|D\left(u_{i}-f\right)\right\|(\bar{\Omega}),
\end{aligned}
$$

where the last equality follows from the fact that $u_{i}-f=0$ almost everywhere outside $\Omega$. Therefore we have

$$
\begin{aligned}
\int_{\Omega^{*}}\left|u_{i}\right| d \mu & \leq \int_{\Omega^{*}}|f| d \mu+\int_{\Omega^{*}}\left|u_{i}-f\right| d \mu \\
& =\int_{\Omega^{*}}|f| d \mu+\int_{\Omega}\left|u_{i}-f\right| d \mu \\
& \leq \int_{\Omega^{*}}|f| d \mu+C \operatorname{diam}(\Omega) \| D\left(u_{i}-f\right)| |(\bar{\Omega}) \\
& \leq \int_{\Omega^{*}}|f| d \mu+C \operatorname{diam}(\Omega)\left(\left\|D u_{i}| |(\bar{\Omega})+\right\| D f| |(\bar{\Omega})\right) \\
& <\infty
\end{aligned}
$$

Thus by lemma 7 there exists a function $u \in \operatorname{BV}_{l o c}\left(\Omega^{*}\right)$ (in particular in $\operatorname{BV}(\Omega)$ ) and a subsequence, which we will denote by $u_{i}$, such that $u_{i} \rightarrow u$ as $i \rightarrow \infty$ in $L_{l o c}^{1}\left(\Omega^{*}\right)$. Thus along a subsequence, again denoted by $u_{i}$, we have pointwise convergence $\mu$-almost everywhere in $\Omega^{*}$. Thus we have

$$
|u(x)-f(x)| \leq\left|u(x)-u_{i}(x)\right|+\left|u_{i}(x)-f(x)\right| \rightarrow 0
$$

$\mu$-almost everywhere in $\Omega^{*} \backslash \Omega$, because the latter term is zero in that set. Therefore we conclude that $u=f$ for $\mu$-almost every point $x \in \Omega^{*} \backslash \Omega$. This, together with the fact that $u \in \operatorname{BV}(\Omega)$, implies that $u \in \operatorname{BV}\left(\Omega^{*}\right)$ and $u_{i} \rightarrow u$ in $L^{1}\left(\Omega^{*}\right)$. Therefore $u \in \operatorname{BV}_{f}(\Omega)$, and by lemma 5 we conclude that

$$
m \leq \mathcal{I}(u, \Omega) \leq \liminf _{i \rightarrow \infty} \mathcal{I}\left(u_{i}, \Omega\right)=m
$$

This proves the existence, for uniqueness we will argue by contradiction and assume that there exists two minimizers, namely $u_{1}$ and $u_{2}$, then for the function $\tilde{u}=\frac{u_{1}+u_{2}}{2}$ we have

$$
\begin{aligned}
\mathcal{I} & =\int_{\Omega} \sqrt{1+\left|D u_{1}\right|^{2}} \\
& \leq \int_{\Omega} \sqrt{1+|D \tilde{u}|^{2}} \\
= & \int_{\Omega \cap\left\{u_{1}=u_{2}\right\}} \sqrt{1+|D \tilde{u}|^{2}}+\int_{\Omega \cap\left\{u_{1} \neq u_{2}\right\}} \sqrt{1+|D \tilde{u}|^{2}} \\
& <\frac{1}{2} \int_{\Omega \cap\left\{u_{1}=u_{2}\right\}} \sqrt{1+\left|D u_{1}\right|^{2}}+\frac{1}{2} \int_{\Omega \cap\left\{u_{1}=u_{2}\right\}} \sqrt{1+\left|D u_{2}\right|^{2}} \\
& +\frac{1}{2} \int_{\Omega \cap\left\{u_{1} \neq u_{2}\right\}} \sqrt{1+\left|D u_{1}\right|^{2}}+\frac{1}{2} \int_{\Omega \cap\left\{u_{1} \neq u_{2}\right\}} \sqrt{1+\left|D u_{2}\right|^{2}} \\
= & \frac{1}{2} \int_{\Omega} \sqrt{1+\left|D u_{1}\right|^{2}}+\frac{1}{2} \int_{\Omega} \sqrt{1+\left|D u_{2}\right|^{2}} \\
= & \frac{1}{2} \mathcal{I}+\frac{1}{2} \mathcal{I}=\mathcal{I},
\end{aligned}
$$

where the strict inequality, obtained by using strict convexity of the integrand on the latter part of the sum, gives us a contradiction. Thus we conclude that $\left|\Omega \cap\left\{u_{1} \neq u_{2}\right\}\right|=0$, which proves uniqueness.

Remark 9. With the same methods we could obtain the same results for more general variational integrals of the form

$$
\mathcal{I}(u, \Omega)=\inf \left\{\liminf _{i \rightarrow \infty} \int_{\Omega^{*}} h\left(g_{u_{i}}\right) d \mu\right\}
$$

where $h$ is convex, continous and satisfies the following linear growth condition

$$
\alpha|x| \leq h(x) \leq \beta(1+|x|), \quad 0<\alpha \leq \beta<\infty .
$$

For the uniqueness of the minimizer we need strict convexity of $h$.

