Comparison principle for parabolic super- and subminimizers on space-time cylinders and a uniqueness result for minimizers of a boundary value problem

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1 Euclidian case

Let $\Omega \subset \mathbb{R}^n$ an open set, $0 < T < \infty$ and $1 < p < \infty$. In the product space $\Omega \times]0, T[$, which we denote Ω_T, we are interested in solving the evolution p-Laplacian equation

$$-\frac{du}{dt} + \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0, \tag{1}$$

which has solutions $u(x, t)$. Notice here that the gradient ∇u is only taken in the space variable $x \in \Omega$, not in t.

Lemma 1

It turns out that u being a weak solution to equation (1) is actually equivalent with u satisfying the inequality

$$p \int_{\{\phi \neq 0\}} u \frac{d\phi}{dt} dxdt + \int_{\{\phi \neq 0\}} |\nabla u|^p dxdt \leq \int_{\{\phi \neq 0\}} |\nabla u + \nabla \phi|^p dxdt \tag{2}$$

for all $\phi \in C_0^\infty(\Omega_T)$. This is called the variational approach to the p-Laplacian equation. If u satisfies (2) then u is called a parabolic minimizer.

Proof

First assume that u is a weak solution to (1). This means that

$$\int_{\Omega_T} (-u \frac{d\phi}{dt} + |\nabla u|^{p-2} \nabla u \cdot \nabla \phi) dxdt = 0$$

or in other words

$$\int_{\{\phi \neq 0\}} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi dxdt = \int_{\{\phi \neq 0\}} u \frac{d\phi}{dt} dxdt$$

for all functions $\phi \in C_0^\infty(\Omega_T)$. From this it follows that

$$\begin{aligned}
\int_{\{\phi\neq 0\}} |\nabla u|^p dxdt &= \int_{\{\phi\neq 0\}} |\nabla u|^{p-2}(\nabla u \cdot \nabla u) dxdt \\
&= \int_{\{\phi\neq 0\}} |\nabla u|^{p-2}\nabla u \cdot (\nabla u + \nabla \phi) dxdt - \int_{\{\phi\neq 0\}} |\nabla u|^{p-2}\nabla u \cdot \nabla \phi dxdt \\
&= \int_{\{\phi\neq 0\}} |\nabla u|^{p-2}\nabla u \cdot (\nabla u + \nabla \phi) dxdt - \int_{\{\phi\neq 0\}} u\frac{d\phi}{dt} dxdt.
\end{aligned}$$

Moving the last integral to the left-hand side and then using Cauchy-Schwarz inequality as well as Young's inequality we get

$$\begin{aligned}
\int_{\{\phi\neq 0\}} |\nabla u|^p dxdt + \int_{\{\phi\neq 0\}} u\frac{d\phi}{dt} dxdt &= \int_{\{\phi\neq 0\}} |\nabla u|^{p-2}\nabla u \cdot (\nabla u + \nabla \phi) dxdt \\
&\leq \int_{\{\phi\neq 0\}} ||\nabla u|^{p-2}\nabla u \cdot (\nabla u + \nabla \phi)| dxdt \\
&\leq \int_{\{\phi\neq 0\}} |\nabla u|^{p-1}|\nabla u + \nabla \phi| dxdt \\
&\leq \int_{\{\phi\neq 0\}} (1-\frac{1}{p})|\nabla u|^p + \frac{1}{p}|\nabla u + \nabla \phi|^p dxdt.
\end{aligned}$$

Reminder of Young's inequality: for $a, b \geq 0$ and $1 < p < \infty$ we have

$$ab \leq \frac{1}{p}a^p + (1-\frac{1}{p})b^{\frac{p}{p-1}}. \tag{3}$$

Moving the first term to the left-hand side and multiplying with p gives us the result

$$p\int_{\{\phi\neq 0\}} u\frac{d\phi}{dt} dxdt + \int_{\{\phi\neq 0\}} |\nabla u|^p dxdt \leq \int_{\{\phi\neq 0\}} |\nabla u + \nabla \phi|^p dxdt.$$

Now let's assume that u satisfies (2) for all functions $\phi \in C_0^\infty(\Omega_T)$. Take an arbitrary $\phi \in C_0^\infty(\Omega_T)$ and let $\epsilon > 0$. Clearly then $\epsilon\phi \in C_0^\infty(\Omega_T)$ and the set where $\phi = 0$ is the same as the set where $\epsilon\phi = 0$. Now using $\epsilon\phi$ as a test function we get

$$p\int_{\{\phi\epsilon\neq 0\}} u\frac{d\epsilon\phi}{dt} dxdt + \int_{\{\phi\epsilon\neq 0\}} |\nabla u|^p dxdt \leq \int_{\{\phi\epsilon\neq 0\}} |\nabla u + \nabla(\epsilon\phi)|^p dxdt.$$

Moving the second integral to the right-hand side and dividing by ϵ we get

$$\begin{aligned}
p\epsilon\int_{\{\phi\neq 0\}} u\frac{d\phi}{dt} dxdt &\leq \int_{\{\phi\neq 0\}} |\nabla u + \nabla(\epsilon\phi)|^p dxdt - \int_{\{\phi\neq 0\}} |\nabla u|^p dxdt \\
p\int_{\{\phi\neq 0\}} u\frac{d\phi}{dt} dxdt &\leq \int_{\{\phi\neq 0\}} \frac{|\nabla u + \epsilon\nabla\phi|^p - |\nabla u|^p}{\epsilon} dxdt.
\end{aligned}$$

Notice that what we have on the right-hand side is a difference quotient and the definition of a $\nabla\phi$-directional derivative of the function $|(x,t)|^p$. Thus by letting $\epsilon \to 0$ we get

$$\frac{|\nabla u + \epsilon\nabla\phi|^p - |\nabla u|^p}{\epsilon} \to p|\nabla u|^{p-1}\frac{\nabla u}{|\nabla u|} \cdot \nabla\phi.$$

Using this the inequality becomes

$$\begin{aligned}
p\int_{\{\phi\neq 0\}} u\frac{d\phi}{dt} dxdt &\leq \int_{\{\phi\neq 0\}} p|\nabla u|^{p-1}\frac{\nabla u}{|\nabla u|} \cdot \nabla\phi dxdt \\
p\int_{\{\phi\neq 0\}} u\frac{d\phi}{dt} dxdt - \int_{\{\phi\neq 0\}} p|\nabla u|^{p-2}(\nabla u \cdot \nabla\phi) dxdt &\leq 0.
\end{aligned}$$

By choosing $-\epsilon\phi$ as a test function and using the same logic we get instead that

$$p\int_{\{\phi\neq 0\}} u\frac{d\phi}{dt}dxdt - \int_{\{\phi\neq 0\}} p|\nabla u|^{p-2}(\nabla u\cdot\nabla\phi)dxdt \geq 0.$$

Finally by combining these results and dividing by p we have

$$\int_{\{\phi\neq 0\}} (u\frac{d\phi}{dt} - |\nabla u|^{p-2}\nabla u\cdot\nabla\phi)dxdt = 0$$

for all functions $\phi \in C_0^\infty(\Omega_T)$. This is exactly the definition of u being a weak solution to (1). So this completes the proof.

2 More general metric space

From now on let (X, d, μ) be a complete metric space with a doubling Borel measure μ supporting a weak $(1,p)$-Poincaré inequality. Similarly to the euclidian case we have an open set $\Omega \subset X$, $1 < p < \infty$ and $0 < T < \infty$. Also let $\alpha > 0$. In the spirit of inequality (2) we define that $u \in L^p_{loc}(0, T; N^{1,p}_{loc}(\Omega)) \cap L^2_{loc}(\Omega_T)$ is a parabolic minimizer in Ω_T if

$$\alpha\int_{\{\phi\neq 0\}} u\frac{d\phi}{dt}d\nu + \int_{\{\phi\neq 0\}} g_u^p d\nu \leq \int_{\{\phi\neq 0\}} g_{u+\phi}^p d\nu \tag{4}$$

for all compactly supported Lipschitz continuous functions $\phi \in \mathrm{Lip}_0(\Omega_T)$. Here g_u is, as usual, the minimal p-weak upper gradient of u. As in the euclidean case, here the upper gradient is only taken in the space X, not in time t. ν is a product measure, such that $d\nu = d\mu dt$.
We say that u is a parabolic superminimizer if it satisfies (4) for all nonnegative compactly supported Lipschitz continuous functions $\phi \geq 0$. We say that u is a parabolic subminimizer if $-u$ is a parabolic superminimizer. The notation that $u \in L^p(0, T; N^{1,p}(\Omega))$ means that for almost every $0 < t < T$, $u(x, t) \in N^{1,p}(\Omega)$ where u is considered as a function in Ω because t is fixed. The L^p at the beginning means that the integral

$$\int_0^T ||u||^p_{N^{1,p}(\Omega)}dt$$

is finite.

From now on the subscript ϵ refers to the standard mollification in the time variable, ie.

$$u_\epsilon(x, t) = \int_{-\epsilon}^{\epsilon} \eta_\epsilon(s)u(x, t-s)ds \tag{5}$$

where η_ϵ is the standard mollifier. First some lemmas.

Lemma 2

Let $g_u \in L^p_{loc}(\Omega_T)$. Then

$$\text{a) } g_{u(x,t-s)-u(x,t)} \to 0 \text{ in } L^p_{loc}(\Omega_T) \text{ as s } \to 0 \tag{6}$$

$$\text{b) } g_{u_\epsilon - u} \to 0 \text{ in } L^p_{loc}(\Omega_T) \text{ as } \epsilon \to 0. \tag{7}$$

The proof is beyond the scope of this presentation so we need to accept this as a given. The same applies for the next lemma.

Lemma 3

If $u \in L^p_{loc}(0,T;N^{1,p}(\Omega)) \cap L^2_{loc}(\Omega_T)$ is a parabolic superminimizer, then

$$-\alpha \int_{\{\phi \neq 0\}} \phi \frac{du_\epsilon}{dt} d\nu + \int_{\{\phi \neq 0\}} (g_u^p)_\epsilon d\nu \leq \int_{\{\phi \neq 0\}} (g^p_{u(x,t-s)+\phi})_\epsilon d\nu \tag{8}$$

for all nonnegative functions $\phi \in L^p_c(0,T;N^{1,p}_0(\Omega)) \cap L^2_{loc}(\Omega_T)$. Here the c means that there are some $0 < t_1 < t_2 < T$ such that $\phi = 0$ when $t < t_1$ or $t > t_2$.

The theorem

Let $u, v \in L^p(0,T;N^{1,p}(\Omega)) \cap L^2(\Omega_T)$ be a parabolic superminimizer and a parabolic subminimizer respectively. Suppose $u \geq v$ in the boundary of Ω_T in the sense that for almost every $0 < t < T$ we have $(v-u)_+ \in N^{1,p}_0(\Omega)$ and we have the initial condition

$$\frac{1}{h}\int_0^h \int_\Omega (v-u)_+^2 d\mu dt \to 0 \text{ as } h \to 0.$$

Then $u \geq v$ ν-almost everywhere in Ω_T.
This proof will be on the blackboard.

The theorem provides also a corollary about uniqueness. This is because in the definition (4) if u is a parabolic minimizer, then it is also a superminimizer and a subminimizer.

Corollary

Let $u, v \in L^p(0,T;N^{1,p}(\Omega)) \cap L^2(\Omega_T)$ be parabolic minimizers. Suppose $u = v$ in the boundary of Ω_T in the sense that for almost every $0 < t < T$ we have $v - u \in N^{1,p}_0(\Omega)$ and we have the initial condition

$$\frac{1}{h}\int_0^h \int_\Omega |v-u|^2 d\mu dt \to 0 \text{ as } h \to 0.$$

Then $u = v$ ν-almost everywhere in Ω_T.
The proof for this will be on the blackboard.