Existence of Minimizers in the Calculus of Variations

Nicolas Dietrich

March 7, 2019

1 Existence and uniqueness of Minimizers

1.1 What we want

In the following $U \subset \mathbb{R}^n$ is a bounded open set with smooth boundary. The Lagrangian L is a smooth function

$$L : \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R}$$

with

$$(p, z, x) \mapsto L(p, z, x).$$

Further we have Sobolev-functions $w \in W^{1,q}(U)$ with $1 < q < \infty$, where $u = g$ on ∂U. We consider a functional of the form

$$I[w] := \int_U L(Dw(x), w(x), x)\ dx.$$

We will now identify some conditions on the Lagrangian L which ensure that the functional $I[.]$ does have a minimizer within an appropriate Sobolev space.

1.2 Coercivity

Example 1.1. Even a smooth function $f : \mathbb{R} \to \mathbb{R}$ which is bounded from below does not need to attain its minimum. Therefore consider the function $f(x) := \frac{1}{1+x^2}$ with $\frac{1}{1+x^2} > 0$ and $\inf_{x\in\mathbb{R}} \frac{1}{1+x^2} = 0$ but f does not attain its minimum.

The upper example suggests that in general we need some hypothesis controlling $I[w]$ for large functions w. The most effective way to ensure this would be to introduce a hypothesis like:
$I[w]$ grows rapidly as $|w| \to \infty$.

Definition 1.2. Suppose there exist constants $\alpha > 0, \beta > 0$ such that

$$L(p, z, x) \geq \alpha|p|^q - \beta \tag{1.1}$$

for all $p \in \mathbb{R}^n$, $z \in \mathbb{R}$ and $x \in U$. Then we have

$$I[w] \geq \delta\|Dw\|^q_{L^q(U)} - \gamma \tag{1.2}$$

for $\gamma := \beta|U|$ and some constant $\delta > 0$ and we call (1.2) a coercivity condition on $I[.]$.

We interpret (1.2) as:
If $\|Dw\|_{L^q} \to \infty$ then $I[w] \to \infty$.

Example 1.3. Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ as $f(x, y) := x^2 + y^2$. Then we have

$$\lim_{|(x,y)|\to\infty} f(x, y) = \lim_{|(x,y)|\to\infty} x^2 + y^2 = \lim_{|(x,y)|\to\infty} |(x, y)|^2 = \infty.$$

Therefore f is coercive.

1.3 Problem with the boundary condition

We want to find minimizers in $W^{1,q}(U)$ which satisfy $u = g$ on ∂U.
We have a problem in making sense of $u = g$ on ∂U if $u \in W^{1,q}(U)$. In general $u \in W^{1,q}(U)$ is only defined almost everywhere in U. Since $|\partial U| = 0$ there is no meaning we can give to the expression $u\,|_{\partial U}$. The solution to this problem is that we interpret $u = g$ on ∂U as

$$(u - g) \in W_0^{1,q}(U).$$

This also gives us $g = u - (u - g) \in W^{1,q}(U)$. Now we can define the space in which we search for minimizers of the functional $I[.]$ in a proper way. We define the space

$$\mathcal{A} := \{w \in W^{1,q}(U) : (w - g) \in W_0^{1,q}(U)\}.$$

1.4 Weak compactness and lower semicontinuity

Remark 1.4. If $f : \mathbb{R} \to \mathbb{R}$ is continuous satisfying a coercivity condition, i.e. there exist constants $\alpha > 0$ and $\beta > 0$ such that

$$f(x) \geq \alpha|x|^q - \beta$$

then f attains its minimum.

Proof. Because of the coercivity we can find a minimizing sequence $(x_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} f(x_k) = \inf_{x\in\mathbb{R}} f(x) := m.$$

Since the sequence $(f(x_k))_{k\in\mathbb{N}}$ converges, it is bounded. So there exists a constant $M > 0$ such that for all $n \in \mathbb{N}$ we have

$$M \geq f(x_n) \geq \alpha|x_n|^q - \beta$$

and therefore we have

$$\left(\frac{M + \beta}{\alpha}\right)^{\frac{1}{q}} \geq |x_n| \text{ for all } n \in \mathbb{N}.$$

This gives us that the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$. Therefore by Bolzano-Weierstraß-Theorem we can find a subsequence $(x_{n_k})_{k\in\mathbb{N}} \subset (x_n)_{n\in\mathbb{N}}$ and $x \in \mathbb{R}$ such that $x_{n_k} \to x$ as $k \to \infty$. Because of the continuity of f we can interchange limit and function and we get

$$f(x) = f(\lim_{k\to\infty} x_{n_k}) = \lim_{k\to\infty} f(x_{n_k}) = m.$$

Therefore x is a minimizer of f. □

The problem is that our functional $I[.]$ in general will not have a minimizer as the following example illustrates

Example 1.5. We define the function $f : \mathbb{R} \to \mathbb{R}$ for $q > 1$ as

$$f(x) := \begin{cases} x^q & , \text{for } x > 0 \\ 1 & , \text{for } x = 0 \\ (-x)^q & , \text{for } x < 0 \end{cases}$$

Then obviously f is coercive, but it does not attend its minimum.

Unfortunately we can not use the above proof for general functionals $I[.]$ since our space is not finite dimensional and $I[.]$ is in general not continuous. But we will use the idea of the proof and introduce some additional conditions on L. To understand the problem we set

$$m := \inf_{w\in\mathcal{A}} I[w]$$

and choose a minimizing sequence $(u_k)_{k\in\mathbb{N}} \subset \mathcal{A}$ such that

$$\lim_{k\to\infty} I[u_k] = m.$$

Now we would like to show that some sequence converges to an actual minimizer. For this we need some kind of compactness argument as in the proof for continuous functions, but the problem is that the space in which we search for the minimizer is infinitely dimensional. If we utilize the corecivity argument we only get that the minimizing sequence is bounded, but nothing more. So it is not a good idea to work with strong compactness.

Let us work with weak compactness instead. Since we are assuming that $1 < q < \infty$ we have that $W^{1,q}(U)$ is reflexive. Because of the coercivity condition on $I[.]$ we know that the sequence $(Du_k)_{k\in\mathbb{N}}$ is bounded in $L^q(U)$ and because of the Poincare-inequality we further know that $(u_k)_{k\in\mathbb{N}}$ is bounded in $L^q(U)$. Therefore $(u_k)_{k\in\mathbb{N}}$ is bounded in $W^{1,q}(U)$. Because of **Theorem** 2.2 we conclude that there exists a subsequence $(u_{k_j})_{j\in\mathbb{N}} \subset (u_k)_{k\in\mathbb{N}}$ and $u \in W^{1,q}(U)$ such that $u_{k_j} \rightharpoonup u$ weakly in $W^{1,q}(U)$, i.e.

$$\begin{cases} u_{k_j} \rightharpoonup u & \text{weakly in } L^q(U) \\ Du_{k_j} \rightharpoonup Du & \text{weakly in } L^q(U;\mathbb{R}^n) \end{cases}$$

Further it is true that $u = g$ on ∂U. We have that $(u_{k_j} - g) \in W_0^{1,q}(U)$ and $u_{k_j} - g \to u - g$, as $j \to \infty$. Since $W_0^{1,q}(U)$ is closed we have $u - g \in W_0^{1,q}(U)$. Therefore we have $u \in \mathcal{A}$.

The last problem is that the functional $I[.]$ in general is not continuous. So we do not have $I[u] = \lim_{j\to\infty} I[u_j]$ if $u_j \to u$. In general weak convergence does not imply convergence almost everywhere.
The good thing is that we do not really need the full strength of continuity. It is sufficient to assume lower semicontinuity.

Definition 1.6. We say that a function $I[.]$ is sequentially weakly lower semicontinuous on $W^{1,q}(U)$, provided

$$I[u] \leq \liminf_{k\to\infty} I[u_k]$$

whenever $u_k \rightharpoonup u$ weakly in $W^{1,q}(U)$ as $k \to \infty$.

In the next chapter we will find a property of the Lagrangian L which together with the coercivity will give us lower semicontinuity.

1.5 Convexity and lower semicontinuity

Theorem 1.7. *Assume that L is smooth, bounded below and in addition the mapping $p \mapsto L(p,z,x)$ is convex or each $z \in \mathbb{R}$ and $x \in U$. Then $I[.]$ is weakly lower semicontinuous on $W^{1,q}(U)$.*

Proof. **1.** We choose any sequence $(u_k)_{k\in\mathbb{N}}$ with $u_k \rightharpoonup u$ weakly in $W^{1,q}(U)$ and set

$$l := \liminf_{k\to\infty} I[u_k]. \tag{1.3}$$

We want to show that $I[u] \leq l$. If $l = \infty$, then we are done. So we assume that $l < \infty$.
2. Note first that because $u_k \rightharpoonup u$ weakly in $W^{1,q}(U)$ we have that the sequence $(u_k)_{k\in\mathbb{N}}$ is bounded in $W^{1,q}(U)$. So we have $\sup_{k\in\mathbb{N}} \|u_k\|_{W^{1,q}(U)} < \infty$. Because of (1.3) we can find a subsequence $(u_{k_\ell})_{\ell\in\mathbb{N}} \subset (u_k)_{k\in\mathbb{N}}$ such that

$$l = \lim_{\ell\to\infty} I[u_{k_\ell}].$$

To simplify the notation we will denote the subsequence also by $(u_k)_{k\in\mathbb{N}}$. By the compactness Theorem 2.3 we also know that $u_k \to u$ strongly in $L^q(U)$, as $k \to \infty$. Further we have by **Theorem** 2.4 that there exists a subsequence $(u_{k_r})_{r\in\mathbb{N}} \subset (u_k)_{k\in\mathbb{N}}$ such that

$$u_{k_r} \to u \tag{1.4}$$

almost everywhere in U as $r \to \infty$. Again we call the subsequence $(u_k)_{k\in\mathbb{N}}$.
3. We fix $\varepsilon > 0$. Then (1.4) and **Theorem** 2.1 give us

$$u_k \overset{k\to\infty}{\longrightarrow} u \text{ uniformly on } E_\varepsilon, \tag{1.5}$$

where E_ε is a measurable set with

$$|U \setminus E_\varepsilon| \leq \varepsilon. \tag{1.6}$$

We may assume $E_\varepsilon \subset E_{\varepsilon'}$ for $0 < \varepsilon' < \varepsilon$. We define the set

$$F_\varepsilon := \left\{ x \in U : |u(x)| + |Du(x)| \leq \frac{1}{\varepsilon} \right\}. \tag{1.7}$$

For this set we also have

$$\lim_{\varepsilon \downarrow 0} |U \setminus F_\varepsilon| = 0. \tag{1.8}$$

We eventually set

$$G_\varepsilon := E_\varepsilon \cap F_\varepsilon. \tag{1.9}$$

Because of (1.6) and (1.8) we further have

$$\begin{aligned} |U \setminus G_\varepsilon| &= |U \setminus (E_\varepsilon \cap F_\varepsilon)| = |(U \setminus E_\varepsilon) \cup (U \setminus F_\varepsilon)| \\ &\leq |U \setminus E_\varepsilon| + |U \setminus F_\varepsilon| \leq \varepsilon + |U \setminus F_\varepsilon| \xrightarrow{\varepsilon \downarrow 0} 0. \end{aligned}$$

4. Because L is bounded from below we can assume without loss of generality that $L \geq 0$ (otherwise apply the following argument to $\tilde{L} := L + \beta \geq 0$ for some constant β). By using the gradient- inequality for differentiable, convex functions we get

$$\begin{aligned} I[u_k] &= \int_U L(Du_k, u_k, x)\ dx \geq \int_{G_\varepsilon} L(Du_k, u_k, x)\ dx \\ &\geq \int_{G_\varepsilon} L(Du, u_k, x)\ dx + \int_{G_\varepsilon} \langle D_p L(Du, u_k, x), Du_k - Du \rangle\ dx. \end{aligned}$$

Because of (1.5), (1.7) and (1.9) we have by using the dominated convergence theorem and the continuity of $z \mapsto L(p, z, x)$:

$$\begin{aligned} \lim_{k \to \infty} \int_{G_\varepsilon} L(Du, u_k, x)\ dx &\overset{DCT}{=} \int_{G_\varepsilon} \lim_{k \to \infty} L(Du, u_k, x)\ dx \\ &= \int_{G_\varepsilon} L(Du, \lim_{k \to \infty} u_k, x)\ dx \\ &= \int_{G_\varepsilon} L(Du, u, x)\ dx. \end{aligned}$$

Now we argue why we can use the dominated convergence theorem. Because $z \mapsto L(p, z, x)$ is smooth, it is Lipschitz-continuous. Therefore there exists a constant $C > 0$ such that for every $u_k \in W^{1,q}(U)$ and $u \in W^{1,q}(U)$ we have

$$|L(Du, u_k, x) - L(Du, u, x)| \leq C_1 |u_k(x) - u(x)|$$

for all $x \in G_\varepsilon$. Further we have because $(u_k)_{k \in \mathbb{N}}$ is bounded in $W^{1,q}(U)$ and (1.7)

$$\begin{aligned} \left| \int_{G_\varepsilon} L(Du, u_k, x) - L(Du, u, x)\ dx \right| &\leq \int_{G_\varepsilon} |L(Du, u_k, x) - L(Du, u, x)|\ dx \\ &\leq \int_{G_\varepsilon} C_1 |u_k(x) - u(x)|\ dx \end{aligned}$$

$$
\begin{aligned}
&\leq C_1|U|^{1-\frac{1}{q}}\left(\int_U |u_k|^q\ dx\right)^{\frac{1}{q}} + \int_{G_\varepsilon} |u(x)|\ dx \\
&\leq C_1|U|^{1-\frac{1}{q}} \underbrace{\sup_{k\in\mathbb{N}} \|u_k\|_{W^{1,q}(U)}}_{=:C_2} + C_1 \int_{G_\varepsilon} \frac{1}{\varepsilon}\ dx \\
&\leq C_1|U|^{1-\frac{1}{q}} C_2 + \frac{C_1|U|}{\varepsilon}.
\end{aligned}
$$

Therefore we found a uniform bound on our integral and therefore we can put the limit under the integral and get

$$
\left|\int_{G_\varepsilon} L(Du, u_k, x) - L(Du, u, x)\ dx\right| \leq \int_{G_\varepsilon} |L(Du, u_k, x) - L(Du, u, x)|\ dx \overset{\varepsilon\downarrow 0}{\longrightarrow} 0.
$$

Now we show that

$$
D_pL(Du, u_k, x) \overset{k\to\infty}{\longrightarrow} D_pL(Du, u, x) \tag{1.10}
$$

uniformly on G_ε. So we have to show that for every $\eta > 0$ there exists an $N(\varepsilon) \in \mathbb{N}$ such that

$$
|D_pL(Du, u_k, x) - D_pL(Du, u, x)| < \eta.
$$

for all $k \geq N$ and all $x \in G_\varepsilon$.
Since $z \mapsto D_pL(p, z, x)$ is Lipschitz continuous, there exists a constant $C > 0$ such that for every $x \in G_\varepsilon$ we have

$$
|D_pL(Du_k, u_k, x) - D_pL(Du, u, x)| \leq C|u_k(x) - u(x)|.
$$

Because of the uniform convergence of $(u_k)_{k\in\mathbb{N}}$ we have that for every $\eta > 0$ there exists an $N(\eta) \in \mathbb{N}$ such that for all $x \in G_\varepsilon$ we have

$$
|u_k - u| < \frac{\eta}{C}
$$

for all $k \geq N$. Combining those results we get that for every $\eta > 0$ there exists an $N \in \mathbb{N}$ such that for every $x \in G_\varepsilon$

$$
|D_pL(Du, u_k, x) - D_pL(Du, u, x)| \leq C|u_k(x) - u(x)| < C\frac{\eta}{C} = \eta.
$$

for all $k \geq N$.
Now we show that

$$
\lim_{k\to\infty} \int_{G_\varepsilon} \langle D_pL(Du, u_k, x), Du_k - Du\rangle\ dx = 0. \tag{1.11}
$$

We have

$$
\lim_{k\to\infty} \int_{G_\varepsilon} \langle D_pL(Du, u_k, x), Du_k - Du\rangle\ dx
$$

$$= \lim_{k\to\infty} \underbrace{\int_{G_\varepsilon} \langle D_pL(Du,u_k,x) - D_pL(Du,u,x), Du_k - Du\rangle \; dx}_{=:I_1(k)}$$

$$+ \lim_{k\to\infty} \underbrace{\int_{G_\varepsilon} \langle D_pL(Du,u,x), Du_k - Du\rangle \; dx}_{=:I_2(k)} .$$

Now we show that $D_pL(Du,u,x) \in L^{\frac{q}{q-1}}(G_\varepsilon)$. For some $y \in G_\varepsilon$ we have

$$\begin{aligned}
\int_{G_\varepsilon} |D_pL(Du,u,x)|^{\frac{q}{q-1}} \; dx &\leq 2^{\frac{1}{q-1}} \int_{G_\varepsilon} |D_pL(Du(x),u(x),x) - D_pL(Du(y),u(y),y)|^{\frac{q}{q-1}} \; dx \\
&\quad + 2^{\frac{1}{q-1}} \int_{G_\varepsilon} |D_pL(Du(y),u(y),y)|^{\frac{q}{q-1}} \; dx \\
&\leq 2^{\frac{1}{q-1}} \int_{G_\varepsilon} (|Du(x) - Du(y)| + |u(x) - u(y)| + |x-y|)^{\frac{q}{q-1}} \; dx \\
&\quad + 2^{\frac{1}{q-1}} |U||D_pL(Du(y),u(y),y)|^{\frac{q}{q-1}} \\
&\leq 2^{\frac{1}{q-1}} |U| \left(\frac{2}{\varepsilon} + \operatorname{diam}(U)\right)^{\frac{q}{q-1}} + C_3 < \infty.
\end{aligned}$$

It follows because we have $Du_k \rightharpoonup Du$ weakly in $L^q(U;\mathbb{R}^n)$

$$\lim_{k\to\infty} I_2(k) = 0$$

Further we have because of (1.10)

$$\begin{aligned}
|I_1(k)| &\leq \int_{G_\varepsilon} |D_pL(Du,u_k,x) - D_pL(Du,u,x)||Du_k - Du| \; dx \\
&\leq \underbrace{\int_{G_\varepsilon} |D_pL(Du,u,x) - D_pL(Du,u,x)||Du_k| \; dx}_{=:I_3(k)} \\
&\quad + \underbrace{\int_{G_\varepsilon} |D_pL(Du,u,x) - D_pL(Du,u,x)||Du| \; dx}_{:=I_4(k)} .
\end{aligned}$$

Moreover we apply Hölders inequality and because of the boundedness of $(u_k)_{k\in\mathbb{N}}$ in $W^{1,q}(U)$ we get

$$\begin{aligned}
|I_3(k)| &\leq \left(\int_{G_\varepsilon} |D_pL(Du,u,x) - D_pL(Du,u,x)|^{\frac{q}{q-1}} \; dx\right)^{1-\frac{1}{q}} \left(\int_{G_\varepsilon} |Du_k|^q \; dx\right)^{\frac{1}{q}} \\
&\leq C \left(\int_{G_\varepsilon} |D_pL(Du,u,x) - D_pL(Du,u,x)|^{\frac{q}{q-1}} \; dx\right)^{1-\frac{1}{q}} \to 0,
\end{aligned}$$

as $k \to \infty$. Analogous we show that $|I_4(k)| \to 0$, as $k \to \infty$. This gives us (1.11). If we look at the upper inequality we conclude

$$l := \lim_{k\to\infty} I[u_k] \geq \int_{G_\varepsilon} L(Du, u, x)\ dx.$$

This inequality holds for arbitrary $\varepsilon > 0$. So by letting $\varepsilon \downarrow 0$ and using the Monotone Convergence Theorem we get

$$\begin{aligned} l &\geq \lim_{\varepsilon\downarrow 0} \int_{G_\varepsilon} L(Du, u, x)\ dx \\ &= \lim_{\varepsilon\downarrow 0} \int_U \underbrace{L(Du, u, x)}_{\geq 0} \underbrace{\chi_{G_\varepsilon}(x)}_{\text{increasing}}\ dx \\ &\overset{\text{MTC}}{=} \int_U L(Du, u, x)\ dx = I[u]. \end{aligned}$$

This gives us the desired result. □

With the upper result we can finally prove the existence of a minimizer.

1.6 Existence of a Minimizer

Theorem 1.8 (Existence of a minimizer)**.** *Assume that L is a smooth function which satisfies the coercivity inequality (1.1) and is convex in the variable p. Then there exists at least one function $u \in \mathcal{A}$ solving*

$$I[u] = \min_{w\in\mathcal{A}} I[w].$$

Proof. **1.** Set $m := \inf_{w\in\mathcal{A}} I[w]$. If $m = +\infty$, we are done. So we now assume that $m < \infty$. Because of the coercivity we can find a minimizing sequence $(u_k)_{k\in\mathbb{N}}$ such that

$$I[u_k] \overset{k\to\infty}{\longrightarrow} m.$$

2. Without loss of generality we take $\beta = 0$ in the coercivity inequality (1.1), because we can otherwise just consider $\tilde{L} := L + \beta$. Therefore the coercivity inequality simplifies to $L \geq \alpha|p|^q$. This gives us

$$I[w] \geq \alpha \int_U |Dw|^q\ dx.$$

Because the sequence $(I[u_k])_{k\in\mathbb{N}}$ converges in $\mathbb{R}$, it is bounded in $\mathbb{R}$. Therefore there exists a constant $M > 0$ such that for every $k \in \mathbb{N}$

$$M \geq I[u_k] \geq \alpha \int_U |Du_k|^q\ dx.$$

This gives us

$$\|Du\|_{L^q(U)} \leq \left(\frac{M}{\alpha}\right)^{\frac{1}{q}}$$

for all $k \in \mathbb{N}$, which implies

$$\sup_{k\in\mathbb{N}} \|Du_k\|_{L^q(U)} < \infty. \tag{1.12}$$

3. Now we fix $w \in \mathcal{A}$. Since u_k and w are both equal to g on ∂U we have

$$u_k - w = (u_k - g) - (w - g) \in W_0^{1,q}(U).$$

The Poincare-Friedrichs inequality gives us

$$\begin{aligned}
\|u_k\|_{L^q(U)} &\leq \|u_k - w\|_{L^q(U)} + \|w\|_{L^q(U)} \\
&\leq C_1\|Du_k - Dw\|_{L^q(U)} + C_2 \\
&\leq C_1(\|Du_k\|_{L^q(U)} + \underbrace{\|Dw\|_{L^q(U)}}_{\leq C_3}) + C_2 \\
&\leq C_1(\underbrace{\sup_{k\in\mathbb{N}} \|Du_k\|_{L^q(U)}}_{\leq C_4} + C_3) + C_2 \leq C.
\end{aligned}$$

This gives us

$$\begin{aligned}
\sup_{k\in\mathbb{N}} \|u_k\|_{W^{1,q}(U)} &= \sup_{k\in\mathbb{N}}(\|u_k\|_{L^q(U)} + \|Du_k\|_{L^q(U)}) \\
&= \sup_{k\in\mathbb{N}} \|u_k\|_{L^q(U)} + \sup_{k\in\mathbb{N}} \|Du_k\|_{L^q(U)} < \infty.
\end{aligned}$$

So $(u_k)_{k\in\mathbb{N}}$ is bounded in $W^{1,q}(U)$.

4. By **Theorem** 2.2 there exists a subsequence $(u_{k_j})_{j\in\mathbb{N}} \subset (u_k)_{k\in\mathbb{N}}$ and a function $u \in W^{1,q}(U)$ such that

$$u_{k_j} \rightharpoonup u$$

weakly in $W^{1,q}(U)$. We show that $u \in \mathcal{A}$. That is to show that $u - g \in W_0^{1,q}(U)$. We know that $W_0^{1,q}(U)$ is a closed, convex linear subspace of $W^{1,q}(U)$. So we can apply Mazurs Theorem, which says that a convex, closed subset of a reflexive Banach-space is weakly closed. Now we take the sequence $(u_{k_j} - g)_{j\in\mathbb{N}} \subset W_0^{1,q}(U)$ which now, because of the weak compactness converges to $u - g \in W_0^{1,q}(U)$. Therefore we have $u \in \mathcal{A}$. Because of the convexity of $p \mapsto L(p, z, x)$ we can apply **Theorem** 1.7 and we get

$$I[u] \leq \liminf_{j\to\infty} I[u_{k_j}] = m.$$

Since $u \in \mathcal{A}$ and m is the infimum we have

$$I[u] \geq m.$$

So in total we get

$$I[u] = m = \min_{w\ in \mathcal{A}} I[w].$$

This proves the result. □

Now we will set the conditions under which our minimizer is unique:

$$L = L(p,x) \text{ does not depend on } z \tag{1.13}$$

There exists $\theta > 0$ such that for all $p, q \in \mathbb{R}^n$ and $x \in U$ and $t \in [0,1]$ we have

$$(1-t)L(q,x) + tL(p,x) \geq L((1-t)q + tp) + t(1-t)\theta|q-p|^2. \tag{1.14}$$

The upper condition says that $p \mapsto L(p,x)$ is uniformly convex for each $x \in U$. The next Theorem gives us the uniqueness of the minimizer.

Theorem 1.9 (Uniqueness of the Minimizer)**.** *Suppose* (1.13) *and* (1.14) *hold. Then a minimizer $u \in \mathcal{A}$ is unique.*

Proof. **1.** Assume $u, \tilde{u} \in \mathcal{A}$ are both minimizers of $I[.]$ over $\mathcal{A}$.
Then we set $v := \frac{u+\tilde{u}}{2} \in \mathcal{A}$. Indeed we have $v \in \mathcal{A}$, because

$$\frac{u+\tilde{u}}{2} - g = \frac{1}{2}(u-g) + \frac{1}{2}(\tilde{u}-g) \in W_0^{1,q}(U).$$

We claim that

$$I[v] \leq \frac{I[u] + I[\tilde{u}]}{2} \tag{1.15}$$

with is a strict inequality, unless $u = \tilde{u}$ almost everywhere.
2. We want to prove (1.15). The uniform convexity (1.14) together with the smoothness of L gives us

$$L(p,x) \geq L(q,x) + \langle D_pL(q,x), p-q\rangle + \frac{\theta}{2}|p-q|^2$$

for all $x \in U$ and $p, q \in \mathbb{R}^n$. Now we set $q := \frac{Du+D\tilde{u}}{2}$ and $p = Du$ and integrate over U

$$I[v] + \int_U \left\langle D_pL\left(\frac{Du+D\tilde{u}}{2}, x\right), \frac{Du-D\tilde{u}}{2}\right\rangle\, dx + \frac{\theta}{8}\int_U |Du-D\tilde{u}|^2\, dx \leq I[u].$$

Similarily we set $q = \frac{Du+D\tilde{u}}{2}$ and $p = D\tilde{u}$ and integrate over U

$$I[v] + \int_U \left\langle D_pL\left(\frac{Du+D\tilde{u}}{2}, x\right), \frac{D\tilde{u}-Du}{2}\right\rangle\, dx + \frac{\theta}{8}\int_U |Du-D\tilde{u}|^2\, dx \leq I[\tilde{u}].$$

Now we add the upper inequalities and divide by 2 and we get

$$I[v] \leq I[v] + \frac{\theta}{8}\int_U |Du-D\tilde{u}|^2\, dx \leq \frac{I[u]+I[\tilde{u}]}{2} = \min_{w\in\mathcal{A}} I[w].$$

Therefore we have

$$I[v] = \frac{I[u] + I[\tilde{u}]}{2},$$

which gives us

$$\frac{\theta}{8} \int_U |Du - D\tilde{u}|^2 \, dx = 0.$$

This implies $Du = D\tilde{u}$ almost everywhere in U. We have that

$$u - \tilde{u} = (u - g) - (\tilde{u} - g) \in W_0^{1,q}(U).$$

Therefore we can use Poincare Friedrichs inequality and get

$$\int_U |u - \tilde{u}|^q \, dx \leq C \int_U |Du - D\tilde{u}|^q \, dx = 0.$$

This finally gives us $u = \tilde{u}$ almost everywhere in U. □

2 Appendix

Theorem 2.1 (Egoroff)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space $\mu(X) < \infty$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions on X and let f be a measurable function on X. Assume that $f_n \to f$ μ-almost everywhere as $n \to \infty$. Then for every $\varepsilon > 0$ there exists a measurable subset D of X such that $\mu(D) < \varepsilon$ and $f_n \to f$ uniformly on $X \setminus D$.*

Proof. Let $\varepsilon > 0$ be arbitrary. By definition of convergence almost everywhere there exists a set $E \in \mathcal{M}$ such that $\mu(E) = 0$ and $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in A$, where $A := X \setminus E$. Now for all $n, k \in \mathbb{N}$ we define $A_{n,k}$ by

$$A_{n,k} := \left\{ x \in A : |f_n(x) - f(x)| \geq \frac{1}{k} \right\}.$$

Further we define $B_{n,k}$ by

$$B_{n,k} := \bigcup_{i=n}^{\infty} A_{i,k}.$$

Since f_n converges pointwise to f on A it follows by the definition of convergence that for each $x \in A$ and $k \in \mathbb{N}$ we have

$$|f_i(x) - f(x)| < \frac{1}{k}$$

for all $i \in \mathbb{N}$ sufficiently large.
Thus, when k is fixed, no element of A belongs to $A_{n,k}$ for infinitely many n. Therefore we have

$$\limsup_{n\to\infty} A_{n,k} = \emptyset.$$

So we have since $\mu(X) < \infty$ and using the continuity of the measure

$$\begin{aligned} 0 = \mu(\emptyset) &= \mu(\limsup_{n\to\infty} A_{n,k}) \\ &= \mu\left(\bigcap_{n=0}^{\infty} \bigcup_{i=n}^{\infty} A_{i,k} \right) = \mu\left(\bigcap_{n=0}^{\infty} B_{n,k} \right) \\ &= \mu(\lim_{n\to\infty} B_{n,k}) = \lim_{n\to\infty} \mu(B_{n,k}). \end{aligned}$$

Since $k \in \mathbb{N}$ was arbitrary we can choose $n_k \in \mathbb{N}$ such that $\mu(B_{n_k,k}) < \frac{\varepsilon}{2^{k+1}}$. We now set $B := \bigcup_{k\in\mathbb{N}} B_{n_k,k}$ and get by using the subadditivity of the measure

$$\mu(B) = \mu\left(\bigcup_{k\in\mathbb{N}} B_{n_k,k} \right) \leq \sum_{k\in\mathbb{N}} \mu(B_{n_k,k}) < \sum_{k\in\mathbb{N}} \frac{\varepsilon}{2^{k+1}} = \varepsilon.$$

Now we set $D := B \cup E$. Then we have since $\mu(E) = 0$

$$\mu(B \cup E) \leq \mu(B) + \mu(E) = \mu(B) < \varepsilon.$$

Also given any $k \in \mathbb{N}$ we have that $x \in X \setminus D$ implies $x \notin B_{n_k,k}$. So we have

$$|f_i(x) - f(x)| < \frac{1}{k}$$

or all $i \geq n_k$. Since this is true or all $x \in X \setminus D$, it follows that f_n converges uniformly to f as $n \to \infty$ on $X \setminus D$. □

Theorem 2.2. *If X is a reflexive Banach-space and $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence in X. Then there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ and $x \in X$ such that $x_{n_k} \rightharpoonup x$ weakly as $k \to \infty$.*

Theorem 2.3. *If $1 < q < \infty$ and $u_k \rightharpoonup u$ weakly in $W^{1,q}(U)$. Then we have $u_k \to u$ strongly in $L^q(U)$.*

Theorem 2.4. *If the sequence $(u_k)_{k\in\mathbb{N}} \subset L^q(U)$ converges to some $u \in L^q(U)$, i.e. $u_k \to u$ in $L^q(U)$. Then there exists a subsequence $(u_{k_j})_{j\in\mathbb{N}} \subset (u_k)_{k\in\mathbb{N}}$ such that $u_{k_j} \to u$ almost everywhere, as $k \to \infty$.*