A very short introduction to Γ-convergence

Throughout, $X = (X, d)$ will be a metric space as in the course. In applications, this could be a function space such as $W^{k,p}(\Omega)$, $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$.

Definition

Problem: Suppose that you are given a sequence of functions $(F_n)_n$ on X, as well as a sequence $(x_n)_n$ such that x_n minimizes F_n. Does $\lim_n x_n$ exist and, if yes, does it minimize anything?
In what sense does F_n have to converge to some F to ensure that the minimizers converge as well?

Γ-convergence is a natural concept of convergence for such a sequence of variational problems.

De Giorgi, Franzoni 1975: *Su un tipo de convergenza variazionale*, Atti Accademia Nazionale de Lincei, Rendiconti della Classe di Scienze Fisiche, Matematiche e Naturali 58

Definition 0.1 (Γ-convergence)**.** Γ-*lower limit* and Γ-*upper limit* of the sequence $(F_n)_n$ at a point $x \in X$ are defined as

$$\Gamma\text{-}\liminf_{n\to\infty} F_n(x) = \inf\left\{\liminf_{n\to\infty} F_n(x_n) : x_n \to x\right\},$$
$$\Gamma\text{-}\limsup_{n\to\infty} F_n(x) = \inf\left\{\limsup_{n\to\infty} F_n(x_n) : x_n \to x\right\}.$$

The infimum is taken over the sequences $(x_n)_n$ converging to x. We say that $(F_n)_n$ Γ-converges to F, denoted $F = \Gamma\text{-}\lim_{n\to\infty} F_n$, if

$$\Gamma\text{-}\limsup_{n\to\infty} F_n(x) \leq \Gamma\text{-}\liminf_{n\to\infty} F_n(x) \quad \forall x \in X.$$

One sometimes sees the shorthand notation $F_n \xrightarrow{\Gamma} F$.

There are several equivalent ways of stating these conditions that may or may not be more appropriate for proving certain things. The following is one workable version.

Definition 0.2 (Testing Γ-convergence)**.** Let X be a metric space, $F_n : X \to [-\infty, \infty]$ functions. We say that $(F_n)_n$ Γ-converges to F if

(i) for every sequence $(x_n)_n$ converging to some $x \in X$,
$$F(x) \leq \liminf_{n\to\infty} F_n(x_n);$$

(ii) for every $x \in X$, there exists a sequence $(x_n)_n$ converging to x with
$$\limsup_{n\to\infty} F_n(x_n) \leq F(x).$$
This sequence (x_n) is sometimes called a *recovery sequence*.

We immediately note that Γ-limits are well-defined in the sense that if $F_n \xrightarrow{\Gamma} F$, then every subsequence $(F_{n_k})_k$ also Γ-converges to F.

In applications, it is often useful to note that Γ-convergence is *stable under continuous perturbations*: Let G be a continuous function. Then $\Gamma\text{-}\lim_n (F_n + G) = \Gamma\text{-}\lim_n F + G$.

In general, Γ-convergence does not have anything to do with pointwise convergence. One does not imply another, and even if a sequence of functions converges both Γ and pointwise, the two limits may not be the same (it is possible to construct examples to this effect).

A simple example on the real line. Consider $F_n(x) = \sin(nx)$. It is well known that this sequence does not have a pointwise limit. How about Γ-convergence? We have $-1 \leq \liminf_n \sin(nx_n)$; let $f(x) = -1$ be our candidate for the Γ-limit. For any $x \in \mathbb{R}$, let x_n be the nearest point such that $\sin(nx_n) = -1$;

$$x_n = -\frac{\pi}{2n} + \frac{2\,[nx/2]\,\pi}{n}$$

should work if you want a formula. Because $\sin(nx)$ is $2\pi/n$-periodic, we have $|x_n - x| \leq \pi/n$ and $x_n \to x$ as $n \to \infty$. So $\Gamma\text{-}\lim_n \sin(nx) = -1$.

Some fundamental properties

One important structural property of Γ-limits is that they are lower semicontinuous. This guarantees that the limit function attains its minimum on a compact set.

Definition 0.3 (Lower semicontinuity). $F : X \to [-\infty, \infty]$ is *lower semicontinuous* (lsc) if

$$x_n \to x \quad \Rightarrow \quad F(x) \leq \liminf_{n\to\infty} F(x_n).$$

Equivalently, $\{x \in X : F(x) \leq t\}$ is closed in X for each t.

Theorem 0.4. *Let $F_n \xrightarrow{\Gamma} F$. Then F is lsc.*

Proof. By contradiction: assume not, in which case there exist a $x \in X$ and a sequence $(x_m)_m$ with

$$\lim_{m\to\infty} x_m = x, \qquad \lim_{m\to\infty} F(x_m) < F(x).$$

By Γ-convergence, for every m there exists a sequence $(x_{m,n})_n$ with

$$\lim_{n\to\infty} x_{m,n} = x_m, \qquad \lim_{n\to\infty} F_n(x_{m,n}) = F(x_m).$$

For simplicity, assume $\lim F(x_m), F(x)$ are finite. Let

$$\delta = \frac{1}{4}\Big(F(x) - \lim_{m\to\infty} F(x_m)\Big) > 0.$$

For every m, there is an index $N_m \in \mathbb{N}$ such that

$$\begin{gathered} F_{N_m}(x_{m,N_m}) - F(x_m) < \delta, \qquad (1) \\ \lim_{m\to\infty} x_{m,N_m} = x, \qquad \lim_{m\to\infty} N_m = \infty. \end{gathered}$$

Choose m so large that

$$F(x_m) < F(x) - 3\delta, \qquad (2)$$

which is possible by the choice of δ on (1), as well as

$$F(x) < F_{N_m}(x_{m,N_m}) + \delta. \qquad (3)$$

Combining (2) and (3), we have $F(x_m) + 3\delta < F(x) < F_{N_m}(x_{m,N_m}) + \delta$, that is

$$F_{N_m}(x_{m,N_m}) - F(x_m) > 2\delta,$$

which contradicts the choice of N_m on (1). $\square$

The Γ-limit of the constant sequence. Let's take a look at the Γ-limit of the constant sequence $F_n(x) = F\ \forall n \in \mathbb{N}$. By Definition 0.2 (i), the Γ-limit $\widetilde{F}(x)$ has to satisfy

$$\widetilde{F}(x) \leq \liminf_n F(x_n)$$

for all x and sequences $(x_n)_n$ converging to x. If F is not lsc, then there exists a $\widetilde{x}$ and a sequence $(\widetilde{x}_n)_n$ converging to a $\widetilde{x} \in X$ such that

$$\liminf_{n\to\infty} F(\widetilde{x}_n) < F(\widetilde{x});$$

in particular, $\widetilde{F}(\widetilde{x}) \neq F(\widetilde{x})$. In this case, the Γ-limit is the *lower semicontinuous envelope* a.k.a. *relaxed function* of F, that is, the supremum of semicontinuous functions below or equal to F.

Finally, the fundamental theorem of Γ-convergence, namely that the minimizers converge under an equicoercivity assumption.

Lemma 0.5. *Let $F_n \xrightarrow{\Gamma} F$. We have*

(i) *for $K \subset X$ compact, $\inf_{x\in K} F(x) \leq \liminf_n \inf_{x\in K} F_n(x)$,*

(ii) *for $U \subset X$ open, $\inf_{x\in U} F(x) \geq \limsup_n \inf_{x\in U} F_n(x)$.*

Proof.

(i) Let $(\widetilde{x}_n)_n \subset K$ be a sequence such that $\liminf_n \inf_{x\in K} F_n(x) = \liminf_n F_n(\widetilde{x}_n)$. Extract a subsequence $(\widetilde{x}_{n_k})_k$ such that

$$\lim_{k\to\infty} F_{n_k}(\widetilde{x}_{n_k}) = \liminf_{n\to\infty} \inf_{x\in K} F_n(x),$$

and $\widetilde{x}_{n_k} \to \widetilde{x} \in K$. Define the sequence

$$x_n = \begin{cases} \widetilde{x}_{n_k} & \text{if } n = n_k, \\ \widetilde{x} & \text{otherwise.} \end{cases}$$

Then, we have

$$\inf_{x\in K} F(x) \leq F(\widetilde{x}) \leq \liminf_{n\to\infty} F_n(x_n) \leq \liminf_{k\to\infty} F_{n_k}(x_{n_k}) = \lim_{k\to\infty} F_{n_k}(\widetilde{x}_{n_k}) = \liminf_{n\to\infty} \inf_{x\in K} F_n(x)$$

as desired. The second inequality follows from the fact that $F_n \xrightarrow{\Gamma} F$.

(ii) Fix a $\delta > 0$ and let $y \in U$ be such that $F(y) \leq \inf_{x\in U} F(x) + \delta$. Then, if $(y_n)_n$ is a recovery sequence for y, we have

$$\inf_{x\in U} F(x) + \delta \geq F_n(y) \geq \limsup_{n\to\infty} F_n(y_n) \geq \limsup_{n\to\infty} \inf_{x\in U} F_n(x),$$

and the result follows because δ was arbitrary.

$\square$

Definition 0.6 (Equi-coercivity)**.** A sequence $(F_n)_n$ is said to be *equi-mildly coercive* if there exists a nonempty compact set $K \subset X$ such that $\inf_{x \in X} F_n(x) = \inf_{x \in K} F_n(x)$ for all n.

Theorem 0.7 (Convergence of minimizers)**.** *Let $(F_n)_n$ be a sequence of equi-mildly coercive functions on X, and let $F_n \xrightarrow{\Gamma} F$. Then $\min F$ exists and amounts to*

$$\min_{x \in X} F(x) = \lim_{n \to \infty} \inf_{x \in K} F_n(x).$$

Proof. Using Lemma 0.5 with K as the equi-coercivity set related to the sequence $(F_n)_n$ and $U = X$:

$$\inf_{x \in X} F(x) \leq \min_{x \in K} F(x) \leq \liminf_{n \to \infty} \inf_{x \in K} F_n(x) = \liminf_{n \to \infty} \inf_{x \in X} F_n(x) \leq \limsup_{n \to \infty} \inf_{x \in X} F_n(x) \leq \inf_{x \in X} F(x).$$

□