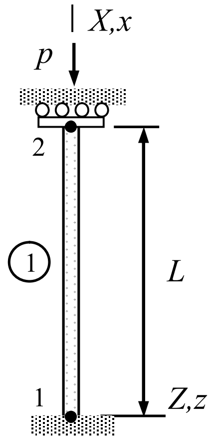
Name________________________ Student number________________

Home assignment 1



Determine the buckling force p_{cr} of the beam shown by using one beam element. Displacements are confined to the xz–plane. The cross-section properties A, I and Young's modulus of the material E are constants.

Solution template

The virtual work expressions for the buckling analysis in xz–plane consist of the internal parts for the beam bending and bar modes and the coupling (stability expression) between them

$$\delta W^{\mathrm{int}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}, \quad \delta W^{\mathrm{int}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix},$$

$$\delta W^{\mathrm{sta}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{N}{30h} \begin{bmatrix} 36 & -3h & -36 & -3h \\ -3h & 4h^2 & 3h & -h^2 \\ -36 & 3h & 36 & 3h \\ -3h & -h^2 & 3h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix} \quad \text{where } N = EA\left(\frac{u_{x2}-u_{x1}}{h}\right).$$

$$\delta W^{\mathrm{ext}} = \begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \\ \delta u_{Z1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix} + \begin{Bmatrix} \delta\theta_{X1} \\ \delta\theta_{Y1} \\ \delta\theta_{Z1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} M_X \\ M_Y \\ M_Z \end{Bmatrix}.$$

The non-zero displacement/rotation components of the structure are $u_{z2} = u_{Z2}$ and $u_{x2} = u_{X2}$ so that the expressions simplify to (keep the symbolic form N for the normal force in the coupling term)

Bending term: $\delta W^{\mathrm{int}} = -\delta u_{Z2} 12 \frac{EI}{L^3} u_{Z2}$,

Bar term: $\delta W^{\mathrm{int}} = -\delta u_{X2} \frac{EA}{L} u_{X2}$,

Coupling term: $\delta W^{\mathrm{sta}} = -\delta u_{Z2} \frac{6}{5} \frac{N}{L} u_{Z2}$, Normal force: $N = EA\dfrac{u_{X2}}{L}$,

Point force term: $\delta W^{\text{ext}} = -p\delta u_{X2}$.

Virtual work expression is the sum of the element contributions. When the expression is manipulated into the standard form, and the expression for the normal force N is substituted into the matrix of the representation, the outcome is

$$\delta W = -\begin{Bmatrix}\delta u_{X2}\\ \delta u_{Z2}\end{Bmatrix}^{\mathrm{T}} \left(\begin{bmatrix}\dfrac{EA}{L} & 0\\ 0 & 12\dfrac{EI}{L^3}+\dfrac{6}{5}\dfrac{EAu_{X2}}{L^2}\end{bmatrix} \begin{Bmatrix}u_{X2}\\ u_{Z2}\end{Bmatrix} - \begin{Bmatrix}-p\\ 0\end{Bmatrix}\right).$$

Principle of virtual work and the fundamental lemma of variation calculus imply the non-linear algebraic equation system

$$\begin{bmatrix}\dfrac{EA}{L} & 0\\ 0 & 12\dfrac{EI}{L^3}+\dfrac{6}{5}\dfrac{EAu_{X2}}{L^2}\end{bmatrix} \begin{Bmatrix}u_{X2}\\ u_{Z2}\end{Bmatrix} - \begin{Bmatrix}-p\\ 0\end{Bmatrix} = 0.$$

As the connection between the bar and bending modes is one-sided, the first equation associated with the bar mode can be solved for the axial displacement

$$u_{X2} = -\frac{pL}{EA}.$$

When the solution is substituted there, the bending equation simplifies to the homogeneous form

$$(12\frac{EI}{L^3} - \frac{6}{5}\frac{p}{L})u_{Z2} = 0.$$

A non-trivial (non-zero) solution $u_{Z2} \neq 0$ is possible only if the multiplier vanishes. This occurs when the external force

$$p_{\text{cr}} = 10\frac{EI}{L^2}. \quad \leftarrow$$