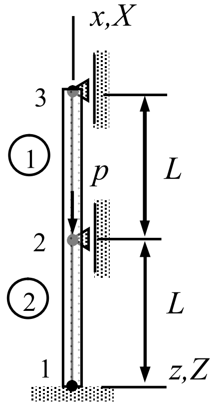
Name________________________ Student number________________

Home assignment 2



Beam structure of the figure is loaded by force p acting on node 2. Determine the buckling force p_{cr} of the structure using two beam elements. Displacements are confined to the xz-plane. Cross-sectional properties of the beam structure A and I and Young's modulus of the material E are constants.

Solution template

The normal forces of the beams can be deduced without any calculations: $N = 0$ for beam 1 and $N = -p$ for beam 2 (negative value means compression). Therefore, it is enough to consider only the bending and coupling terms for the structure. The non-zero displacement/rotation components are θ_{Y2} and θ_{Y3}. Element contributions, taking into account the beam bending mode and the interaction of the bar and beam bending modes, are given by

$$\delta W^1 = -\begin{Bmatrix} 0 \\ \delta\theta_{Y2} \\ 0 \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ \theta_{Y2} \\ 0 \\ \theta_{Y3} \end{Bmatrix} \Rightarrow$$

$$\delta W^1 = -\begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 4\dfrac{EI}{L} & 2\dfrac{EI}{L} \\ 2\dfrac{EI}{L} & 4\dfrac{EI}{L} \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix},$$

$$\delta W^2 = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} + \frac{-p}{30L} \begin{bmatrix} 36 & -3L & -36 & -3L \\ -3L & 4L^2 & 3L & -L^2 \\ -36 & 3L & 36 & 3L \\ -3L & -L^2 & 3L & 4L^2 \end{bmatrix} \right) \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_{Y2} \end{Bmatrix} \Rightarrow$$

$$\delta W^2 = -\delta\theta_{Y2}\left(4\frac{EI}{L} - \frac{p}{30}4L\right)\theta_{Y2} = -\begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 4\dfrac{EI}{L} - \dfrac{p}{30}4L & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix}.$$

Virtual work expression is sum of the element contributions

$$\delta W = \delta W^1 + \delta W^2 = -\begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 8\frac{EI}{L} - \frac{p}{30}4L & 2\frac{EI}{L} \\ 2\frac{EI}{L} & 4\frac{EI}{L} \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix}.$$

Principle of virtual work and the fundamental lemma of variation calculus imply the equation system

$$\begin{bmatrix} 8\frac{EI}{L} - \frac{p}{30}4L & 2\frac{EI}{L} \\ 2\frac{EI}{L} & 4\frac{EI}{L} \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix} = 0 .$$

A linear homogeneous equation system may have a non-trivial solution (something that is non-zero) only if the matrix is singular. The critical value of the loading parameter p, making the solution non-unique, is given by the condition

$$\det \begin{bmatrix} 8\frac{EI}{L} - 4\frac{Lp}{30} & 2\frac{EI}{L} \\ 2\frac{EI}{L} & 4\frac{EI}{L} \end{bmatrix} = (8\frac{EI}{L} - 4\frac{Lp}{30})4\frac{EI}{L} - (2\frac{EI}{L})^2 = 0 \quad \Leftrightarrow \quad p_{\mathrm{cr}} = \frac{105}{2}\frac{EI}{L^2} . \quad \leftarrow$$