1. *Let's consider atoms with three energy levels 0, 1, and 2. A system of such atoms is pumped with an outside energy source (e.g. electric current, flashlight or another laser) that lifts atoms from energy level 0 to level 2 with a rate P_2. The atoms relax from level 2 via level 1 to the ground state. The transition rates are governed by the respective lifetimes τ_2 and τ_1. Assume, that the population at level 0 is so large that pumping does not affect it.*
   a) Write down the rate equations $\frac{\partial N_1}{\partial t}$ and $\frac{\partial N_2}{\partial t}$ for states 1 and 2.
   b) Give the expressions for populations N_1 and N_2 as functions of time. Hint: the system of equations can be solved with trial functions $N_i(t) = a(t)\cdot\exp(-t/\tau_i)$. Assume that initially (when the pump is ignited) all the atoms are at the ground state.
   c) Calculate N_1 and N_2 at the steady state using constant values of τ_1 = 1 µs, τ_2 = 2 µs and P_2 = 1020 cm^{-3}s^{-1}.

a)
$$\frac{\partial N_2}{\partial t} = P_2 - \frac{N_2}{\tau_2} \qquad (1)$$
$$\frac{\partial N_1}{\partial t} = \frac{N_2}{\tau_2} - \frac{N_1}{\tau_1} \qquad (2)$$

b) First we solve N$_2$ using the given trial, *i.e.*, $N_2(t) = a(t)e^{-t/\tau_2}$ inserted into Eq. 1

$$\Rightarrow a(t)e^{-t/\tau_2} - \frac{a(t)}{\tau_2}e^{-t/\tau_2} = P_2 - \frac{a(t)}{\tau_2}e^{-t/\tau_2} \Leftrightarrow a(t)e^{-t/\tau_2} = P_2 \Leftrightarrow a(t) = P_2 e^{t/\tau_2}.$$

Then from initial values $N_2(t=0)=0, a(0)=0$ we get $a(t) = \tau_2 P_2 e^{t/\tau_2} + c$,

$$\Rightarrow \quad c = -\tau_2 P_2 e^0 \quad \Rightarrow \quad a(t) = \tau_2 P_2 e^{t/\tau_2} - \tau_2 P_2 = \tau_2 P_2\left(e^{t/\tau_2} - 1\right)$$

Therefore, $N_2(t) = \tau_2 P_2\left(e^{t/\tau_2} - 1\right)e^{-t/\tau_2} = \tau_2 P_2\left(1 - e^{-t/\tau_2}\right)$.

By inserting N$_2$ into Eq. 2

$$\frac{\partial N_1}{\partial t} = \frac{N_2}{\tau_2} - \frac{N_1}{\tau_1} = P_2\left(1 - e^{-t/\tau_2}\right) - \frac{N_1}{\tau_1}$$

and by using the trial $N_1(t) = a(t)e^{-t/\tau_1}$:

$$\frac{a(t)}{dt}e^{-t/\tau_1} - \frac{a(t)}{\tau_1}e^{-t/\tau_1} = P_2\left(1 - e^{-t/\tau_2}\right) - \frac{a(t)}{\tau_1}e^{-t/\tau_1} \quad \Leftrightarrow \quad \frac{a(t)}{dt}e^{-t/\tau_1} = P_2\left(1 - e^{-t/\tau_2}\right)$$

$$\Rightarrow \quad \frac{a(t)}{dt} = P_2\left(1 - e^{-t/\tau_2}\right)e^{t/\tau_1} = P_2 e^{t/\tau_1} - P_2 e^{t\left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right)}.$$

$$\Rightarrow \quad a(t) = \tau_1 P_2 e^{t/\tau_1} - \frac{1}{1/\tau_1 - 1/\tau_2} P_2 e^{t\left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right)} + c,$$

And again from initial values $N_1(0) = a(0) = 0$

$$\Rightarrow \quad c = \frac{1}{1/\tau_1 - 1/\tau_2} P_2 - \tau_1 P_2 = \frac{P_2\left(\tau_1\tau_2 + \tau_1^2 - \tau_1\tau_2\right)}{\tau_2 - \tau_1} = \frac{P_2\tau_1^2}{\tau_2 - \tau_1}$$

$$\Rightarrow \quad a(t) = \tau_1 P_2 e^{t/\tau_1} - \frac{\tau_1\tau_2}{\tau_2 - \tau_1} P_2 e^{t\left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right)} + \frac{P_2\tau_1^2}{\tau_2 - \tau_1}.$$

Then

$$N_1(t) = \left[\tau_1 P_2 e^{t/\tau_1} - \frac{\tau_1\tau_2}{\tau_2 - \tau_1} P_2 e^{t\left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right)} + \frac{P_2\tau_1^2}{\tau_2 - \tau_1}\right] e^{-t/\tau_1}$$

$$= \tau_1 P_2 \left(1 + \frac{\tau_2 e^{-t/\tau_2}}{\tau_2 - \tau_1} - \frac{\tau_1 e^{-t/\tau 1}}{\tau_2 - \tau_1}\right).$$

c) At equilibrium the derivatives are zero so that

$$P_2 = \frac{N_2}{\tau_2} \Leftrightarrow N_2 = P_2\tau_2 = 10^{20}\ \text{cm}^{-3}\text{s}^{-1} \cdot 2 \cdot 10^{-6}\ \text{s} = 2 \cdot 10^{14}\ \text{cm}^{-3}.$$

$$\frac{N_2}{\tau_2} = \frac{N_1}{\tau_1} \Leftrightarrow N_1 = \frac{N_2}{\tau_2}\tau_1 = P_2\tau_1 = 10^{14}\ \text{cm}^{-3}.$$

2. *a) The differential quantum efficiency of an InP laser is 30%. A voltage of 2.5 V is applied over the component. Calculate the external efficiency of the laser neglecting the losses inside the cavity. b) The following parameters are known about the laser: emission wavelength 850 nm, threshold current 12 mA, differential quantum efficiency below threshold 2%, and differential quantum efficiency above threshold 25%. Calculate the emitted power, when a current of 18 mA flows through the laser.*

a) Power conversion efficiency is $\eta_P = \dfrac{P_0}{V_f A J}$ and differential efficiency is $\eta_D = \dfrac{d\left(P_0/h\nu\right)}{d\left[\left(A/q\right)\left(J - J_{th}\right)\right]}$. With large current densities $(J >> J_{th})$

$$\eta_D = \frac{d\left(P_0/h\nu\right)}{d\left[\left(A/q\right)\left(J - J_{th}\right)\right]} = \frac{q}{Ah\nu}\frac{dP_0}{d\left[J - J_{th}\right]} \cong \frac{q}{Ah\nu}\frac{P_0}{J} \quad \Rightarrow \quad \frac{P_0}{J} = \frac{Ah\nu\eta_D}{q}.$$

Hence the conversion efficiency can be solved

$$\Rightarrow \quad \eta_P = \frac{1}{V_f A}\frac{P_0}{J} = \frac{1}{V_f A}\frac{Ah\nu\eta_D}{q} = \frac{h\nu}{qV_f}\eta_D = \frac{1.35}{2.5} \times 0.3 \approx 0.16.$$

b) Power is obtained by integrating

$$\eta_d = \frac{d\left(P_0/h\nu\right)}{d\left[\left(A/q\right)J\right]} = \frac{q}{h\nu}\frac{dP_0}{dI} \quad \Rightarrow \quad P_0 = \int_0^{18\ \text{mA}} \frac{h\nu}{q}\eta_d\, dI \quad \Rightarrow$$

$$P_0 = \int_0^{12\ \text{mA}} \frac{h\nu}{q}\eta_{d,1}\, dI + \int_{12\ \text{mA}}^{18\ \text{mA}} \frac{h\nu}{q}\eta_{d,2}\, dI = \frac{hc}{q\lambda}\left(\eta_{d,1} \cdot 12\ \text{mA} + \eta_{d,2} \cdot 6\ \text{mA}\right) = 2.54\ \text{mW}.$$

3. *Let's compare surface-emitting and edge-emitting lasers that are manufactured from the same active material having material parameters of $n_{trans} = 1.7\times10^{18}$ cm^{-3}, $\partial g/\partial n = 1.5\times10^{-16}$ cm^2, $\eta_i = 0.8$, $\tau_r = 6$ ns and $\gamma = 8$ cm^{-1}. The length of the side-emitting laser is 300 µm, the thickness of the active region is 0.1 µm, the reflectivity of the mirrors is 0.3 and the optical confinement factor is 0.6. The thickness of the surface-emitting laser is 1 µm and the reflectivity of the mirrors is 0.95. a) Calculate the threshold current density for both components. b) What are the threshold currents of the components if the active region width of the edge-emitting laser is 3 µm and the diameter of the surface-emitting laser is 1 µm?*

a) The threshold current density is $J_{th} = J_{th}^0 + \dfrac{qd}{\Gamma \eta_i \tau_r \dfrac{\partial g}{\partial n}}\left[\gamma + \dfrac{1}{l}\ln\dfrac{1}{R}\right]$,

where $J_{th}^0 = \dfrac{qd}{\eta_i \tau_r} n_{nom}$. By inserting the given numerical values we obtain

$J_{th,edge-em.} \approx 750$ A/cm^2 and $J_{th,surfacee-em.} \approx 17\ 300$ A/cm^2.

b) The threshold current is obtained by multiplying the current density with the active surface area of the component

$I_{th} = J_{th} A$, where $A_{edge-em.} = lw$ and $A_{surfacee-em.} = \pi r^2$.

Results: $I_{th,edge-em.} = 6.7$ mA and $I_{th,surfacee-em.} = 0.14$ mA.

4. *The grating period of a GaAs DFB-laser is 249.6 nm. The grating operates at second order diffraction wavelength. Let the refractive index be 3.59 and the cavity length 500 µm. a) What is the Bragg the wavelength? b) Calculate the two main emission wavelengths of the DFB laser.*

Bragg wavelength is obtained from the equation $d\sin\theta = l\,\lambda/2$ (light waves diffracted from the grating have to interfere constructively). In DFB-grating light is reflected back from the refractive index changes of the grating → $\sin\theta = 1$.

When $l = 2$ and $d = \Lambda$ $\Rightarrow$ $\Lambda = \lambda = \dfrac{\lambda_B}{n}$ where λ_B is the light wavelength in vacuum $\Rightarrow$

$\lambda_B = n\Lambda = 896.06$ nm.

$\lambda_m = \lambda_B \pm \dfrac{\lambda_B^2}{2nL}\left(m + \dfrac{1}{2}\right)$ Now $\dfrac{\lambda_B^2}{2nL} = 0.22$ nm, so that

$\lambda_0 = 896.06\text{ nm} \pm 0.11\text{ nm} = 895.95\text{ nm} \vee 896.17\text{ nm}$

5. *The length of a GaAs laser is 680 μm, the thickness of the active region is 1 μm and the width of the current-limiting stripe is 50 μm. The refractive index of GaAs is 3.6, recombination time is 1 ns and the attenuation of the laser cavity is 10 cm^{-1}. Assume that the emission spectrum of GaAs is triangle-shaped with a peak energy of 1.476 eV and the width of the amplification band is 43 meV. Compute a) the wavelength of the peak energy emission, b) the FWHM of the amplification band in [Hz], c) amplification at laser threshold, and d) the minimum value for the electrical power fed in the laser when the population inversion (injected carrier concentration) in the active region is 10^{16} cm^{-3}.*

a) $\lambda_{peak} = \dfrac{h \cdot c}{E_{peak}} = \dfrac{4.1357 \cdot 10^{-15}\,\mathrm{eV \cdot s} \cdot 3 \cdot 10^{8}\,\mathrm{m \cdot s^{-1}}}{1.476\mathrm{eV}} = 841\ \mathrm{nm}.$

b) $E = hf \Leftrightarrow \Delta E = h\Delta f \Leftrightarrow \Delta f = \dfrac{\Delta E}{h} = \dfrac{0.5 \cdot 0.043\ \mathrm{eV}}{4.3157 \cdot 10^{-15}\,\mathrm{eV \cdot s}} = 4.94 \cdot 10^{12}\ \mathrm{Hz}.$

c) The gain has to overcome the cavity losses

$$G_{th} = \gamma + \frac{1}{2l}\ln\left(\frac{1}{R_1 R_1}\right) = \gamma + \frac{1}{l}\ln\left(\frac{1}{R}\right) = 10\ \mathrm{cm^{-1}} + \frac{1}{0.068\ \mathrm{cm}}\ln\left(\frac{3.6+1}{3.6-1}\right)^2 =$$

$$= 10\ \mathrm{cm^{-1}} + \frac{1.14}{0.068\ \mathrm{cm}} \approx 26.8\ \mathrm{cm^{-1}}.$$

d) The amount of electrons injected to the laser has to be at least equal to the amount of recombination

$$R = \frac{N}{\tau_r} = \frac{n \cdot V}{\tau_r} = \frac{10^{16}\ \mathrm{cm^{-3}} \cdot 0.068\ \mathrm{cm} \cdot 0.005\ \mathrm{cm} \cdot 0.0001\ \mathrm{cm}}{10^{-9}\mathrm{s}} = 3.4 \cdot 10^{17}\mathrm{s^{-1}}.$$

Power is obtained by multiplying this with the minimum energy of the electron. This depends on many factors but it has to be at least as large as the energy of the recombined photon:

$$P = E_{el} \cdot R = 1.476\ \mathrm{eV} \cdot 3.4 \cdot 10^{19}\ \mathrm{s^{-1}} = 1.476 \cdot 1.6 \cdot 10^{-19}\ \mathrm{J} \cdot 3.4 \cdot 10^{17}\ \mathrm{s^{-1}} \approx 80\ \mathrm{mW}.$$