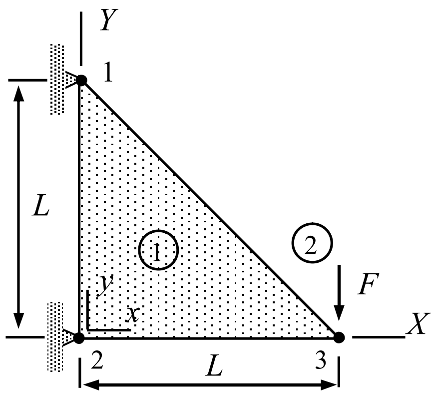
Name____________________________ Student number___________________

Home assignment 3

A thin triangular slab is loaded by a point force at node 3. Nodes 1 and 2 are fixed. Derive the equilibrium equations of the structure according to the large displacement theory in terms of the dimensionless displacement components $a_1 = u_{X3} / L$ and $a_2 = u_{Y3} / L$. Approximation is linear and material parameters C and ν are constants. Assume plane-stress conditions. When $F = 0$, side length and thickness of the slab are L and t, respectively. Find also the solution to a small displacement problem by simplifying the equilibrium equations with the assumptions $|a_1| \ll 1$ and $|a_2| \ll 1$.



Solution

Virtual work density of internal forces, when modified for large displacement analysis with the same constitutive equation as in the linear case of plane stress, is given by

$$\delta w_{\Omega^\circ}^{\text{int}} = -\begin{Bmatrix} \delta E_{xx} \\ \delta E_{yy} \\ 2\delta E_{xy} \end{Bmatrix}^{\mathrm{T}} \frac{tC}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} E_{xx} \\ E_{yy} \\ 2E_{xy} \end{Bmatrix}, \quad \begin{Bmatrix} E_{xx} \\ E_{yy} \\ 2E_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} + \dfrac{1}{2}(\dfrac{\partial u}{\partial x})^2 + \dfrac{1}{2}(\dfrac{\partial v}{\partial x})^2 \\ \dfrac{\partial v}{\partial y} + \dfrac{1}{2}(\dfrac{\partial u}{\partial y})^2 + \dfrac{1}{2}(\dfrac{\partial v}{\partial y})^2 \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\dfrac{\partial v}{\partial y} \end{Bmatrix}.$$

Let us start with the approximations and the corresponding components of the Green-Lagrange strain. Linear shape functions can be deduced from the figure. Only the shape function $N_3 = x / L$ of node 3 is needed. Displacement components and their non-zero derivatives are

$$u = \frac{x}{L} u_{X3} \text{ and } v = \frac{x}{L} u_{Y3} \quad \Rightarrow \quad \frac{\partial u}{\partial x} = \frac{u_{X3}}{L} = a_1 \text{ and } \frac{\partial v}{\partial x} = \frac{u_{Y3}}{L} = a_2$$

Green-Lagrange strain measures and their variations

$$\begin{Bmatrix} E_{xx} \\ E_{yy} \\ 2E_{xy} \end{Bmatrix} = \begin{Bmatrix} a_1 + a_1^2/2 + a_2^2/2 \\ 0 \\ a_2 \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} \delta E_{xx} \\ \delta E_{yy} \\ 2\delta E_{xy} \end{Bmatrix} = \begin{Bmatrix} \delta a_1 + a_1 \delta a_1 + a_2 \delta a_2 \\ 0 \\ \delta a_2 \end{Bmatrix}.$$

When the strain component expressions are substituted there, virtual work density simplifies to

$$\delta w_{\Omega^\circ}^{\text{int}} = -\frac{tC}{1-\nu^2}\begin{Bmatrix} \delta a_1 + a_1\delta a_1 + a_2\delta a_2 \\ 0 \\ \delta a_2 \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} a_1 + a_1^2/2 + a_2^2/2 \\ 0 \\ a_2 \end{Bmatrix} \Rightarrow$$

$$\delta w_{\Omega^\circ}^{\text{int}} = -\frac{tC}{1-\nu^2}[(\delta a_1 + a_1\delta a_1 + a_2\delta a_2)(a_1 + a_1^2/2 + a_2^2/2) + \frac{1-\nu}{2}a_2\delta a_2]$$

Integration over the (initial) domain gives the virtual work expression. As the integrand is constant

$$\delta W^1 = \frac{L^2}{2}\delta w_{\Omega^\circ}^{\text{int}} = -\frac{L^2}{2}\frac{tC}{1-\nu^2}[(\delta a_1 + a_1\delta a_1 + a_2\delta a_2)(a_1 + a_1^2/2 + a_2^2/2) + \frac{1-\nu}{2}a_2\delta a_2].$$

Virtual work expression of the external point force components

$$\delta W^2 = -F\delta u_{Y3} = -FL\delta a_2 .$$

Virtual work expression of the structure is obtained as sum over the element contributions. In terms of the dimensionless displacement

$$\delta W = -\frac{L^2}{2}\frac{tC}{1-\nu^2}[(\delta a_1 + a_1\delta a_1 + a_2\delta a_2)(a_1 + a_1^2/2 + a_2^2/2) + \frac{1-\nu}{2}a_2\delta a_2] - FL\delta a_2$$

or, when written in the standard form,

$$\delta W = -\begin{Bmatrix} \delta a_1 \\ \delta a_2 \end{Bmatrix}^{\mathrm{T}} \frac{L^2}{2}\frac{tC}{1-\nu^2}\begin{Bmatrix} (1+a_1)(a_1 + \frac{1}{2}a_1^2 + \frac{1}{2}a_2^2) \\ a_2(a_1 + a_1^2/2 + a_2^2/2) + \frac{1-\nu}{2}a_2 + 2\frac{F}{tCL}(1-\nu^2) \end{Bmatrix}$$

Principle of virtual work and the basic lemma of variation calculus imply the equilibrium equations

$$(1+a_1)(a_1 + \frac{1}{2}a_1^2 + \frac{1}{2}a_2^2) = 0 , \quad \Leftarrow$$

$$a_2(a_1 + a_1^2/2 + a_2^2/2) + \frac{1-\nu}{2}a_2 + 2\frac{F}{tCL}(1-\nu^2) = 0 . \quad \Leftarrow$$

Assuming that $|a_1| \ll 1$ and $|a_2| \ll 1$ the equilibrium equations simplify to

$$a_1 = 0 \quad \text{and} \quad \frac{1-\nu}{2}a_2 + 2\frac{F}{tCL}(1-\nu^2) = 0 \quad \Rightarrow a_1 = 0 \quad \text{and} \quad a_2 = -4\frac{F}{tCL}(1+\nu). \quad \Leftarrow$$