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Curvature through ruled surfaces

Kirsi Peltonen,
Crystal Flowers, 12.2.2019

Program schedule for Feb 12th

15:15 Basic examples of ruled surface

- Basic properties
- Use in architecture
- Developable surfaces

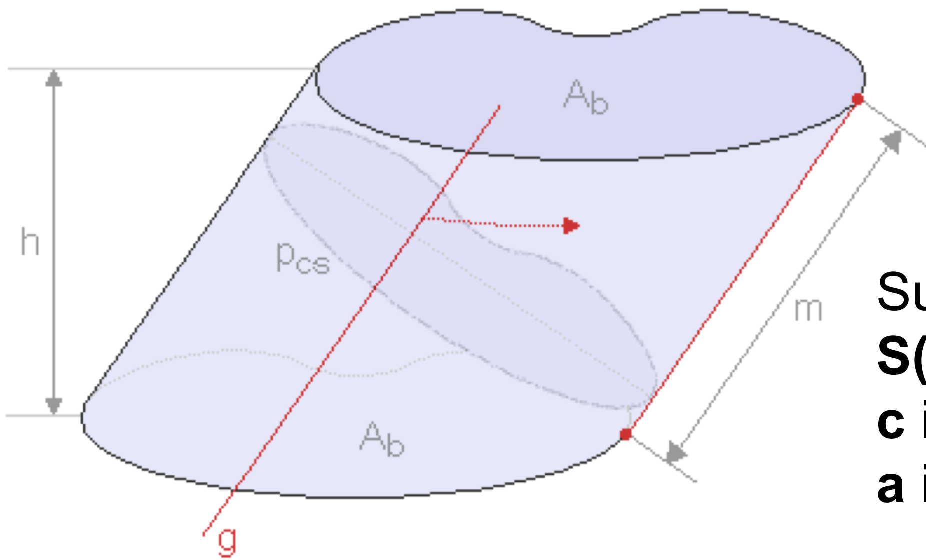
16:15 Break

16:30 What is curvature ?

- principal curvatures
- Gaussian curvature
- Mean curvature

17:00 Pablo & Markus: towards task 2 !

Generalized cylinders



Surface parametrization

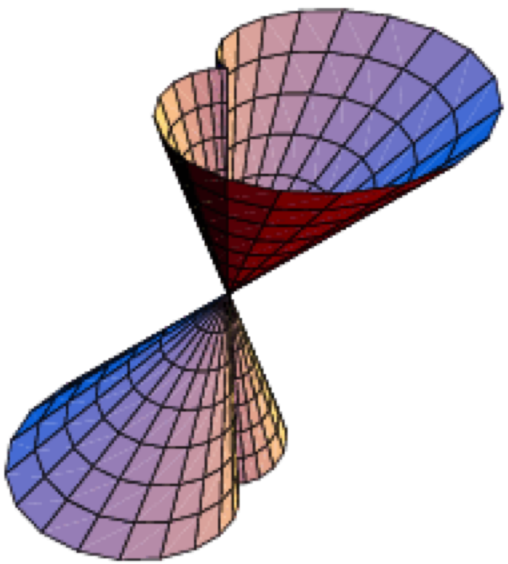
S(u,v)=c(u)+va

c is a space curve (need not be closed)

a is a fixed vector (ruler)

Note: Can be rolled from a flat piece of paper

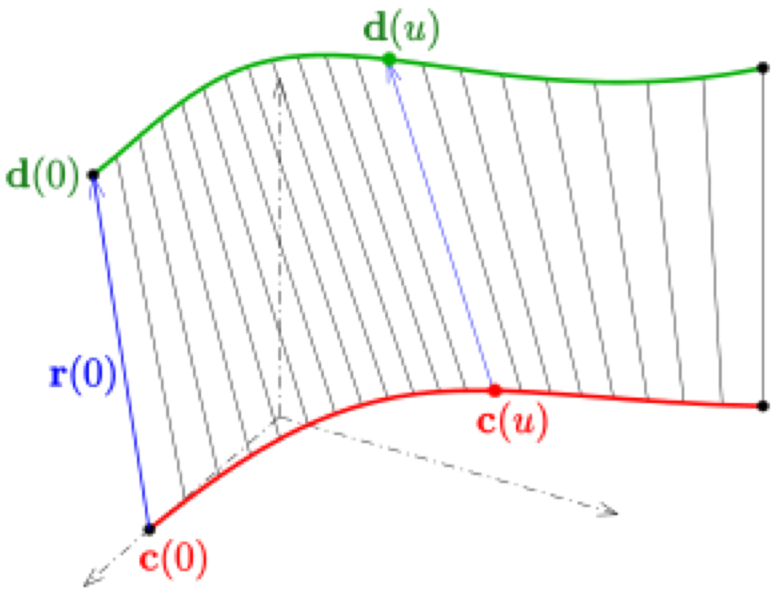
Generalized cones



Surface parametrization
S(u,v)=(1-v)p +vd(u)
p fixed point (tip of the cone)
d a space curve
Rulers: S(.,v)=(1-v)p+vd(.)

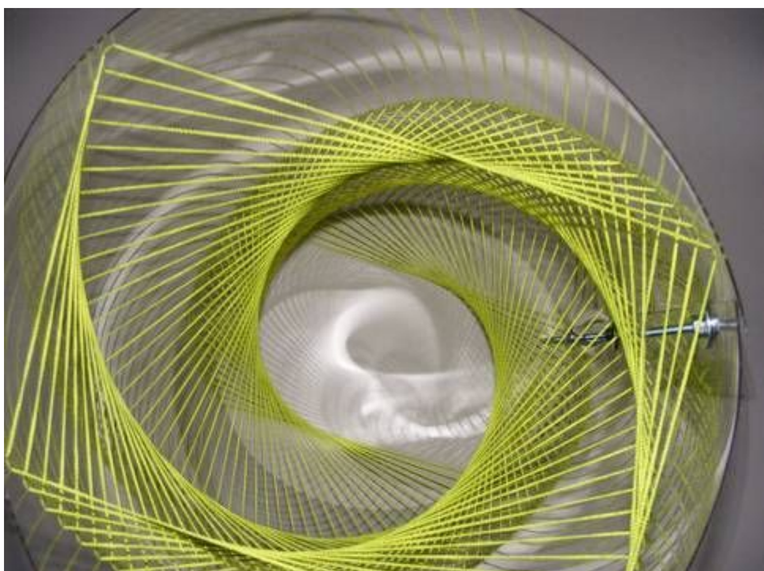
Note: Can be rolled from a flat piece of paper

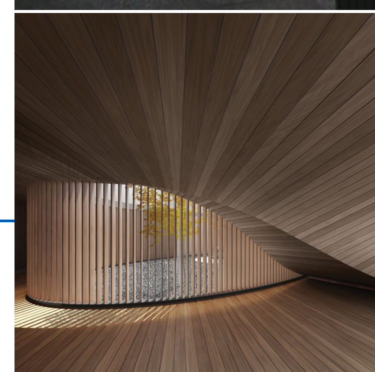
General ruled surface



S(u,v)=c(u)+vd(u)

- c,d space curves
- Ruler S(.,v)=c(.)+v(.)
- For generalized cone d(u)=a constant
- For generalized cylinder c(u)=p constant



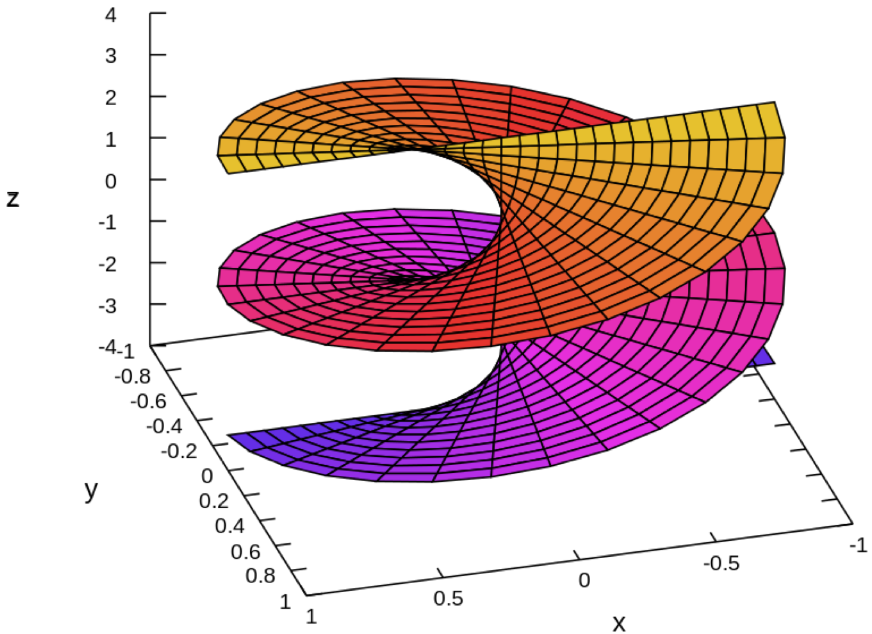



COURTESY HENNING LARSEN ARCHITECTS



Helicoid

Euler 1774, Jean Babtiste Meusnier 1776



S(u,v)=(vcos u, vsin u, ku)
=(0,0,ku) + v(cos u, sin u, 0)

- c(u)= (0,0,ku)
- d(u)=(cos u, sin u, 0)

Fixed v parametrizes a helix

Also a minimal surface !



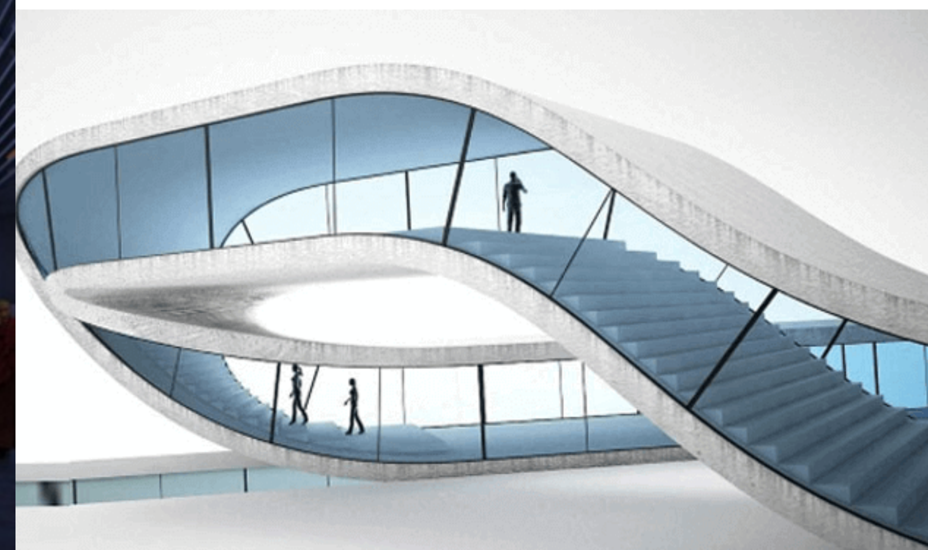
KANDINSKY



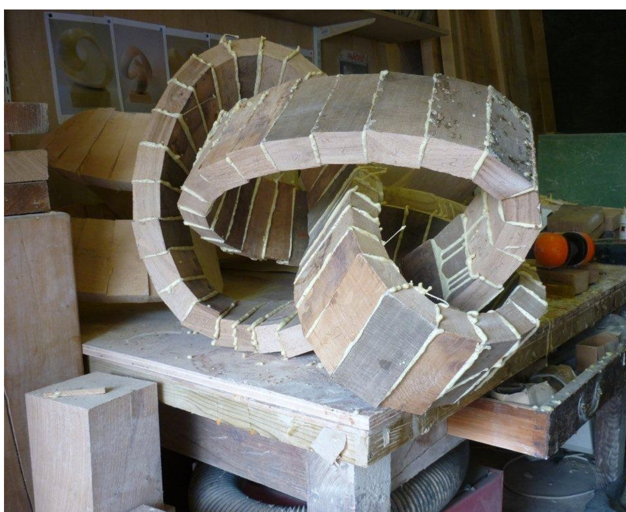
Möbius strip

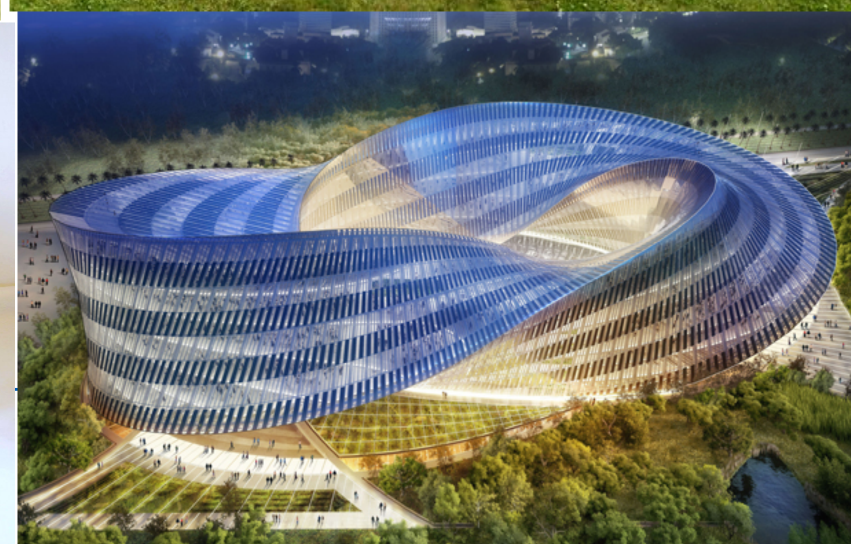
- Möbius, Listing 1858
- Roman mosaics 200-220 CE
- Nonorientable
- **S(u,v)=((1-v sin u/2)cos u,
  (1-v sin u/2)sin u, v cos u/2)
  =(cos u, sin u, 0) +
  v(-sin u/2 cos u, -sin u/2 sin u, cos u/2)**
- **c(u)=(cos u, sin u, 0)**
- **d(u)=(cos u/2 cos u, cos u/2 sin u, sin u/2)**



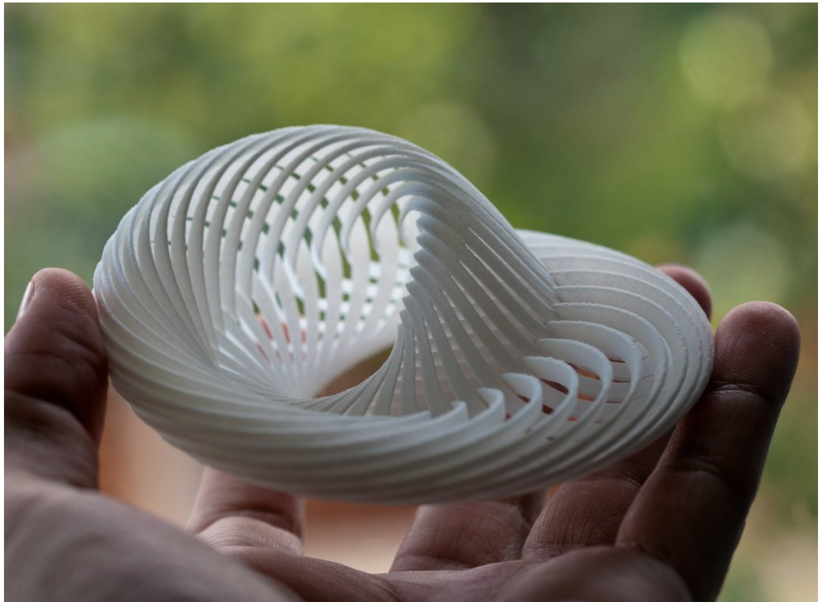












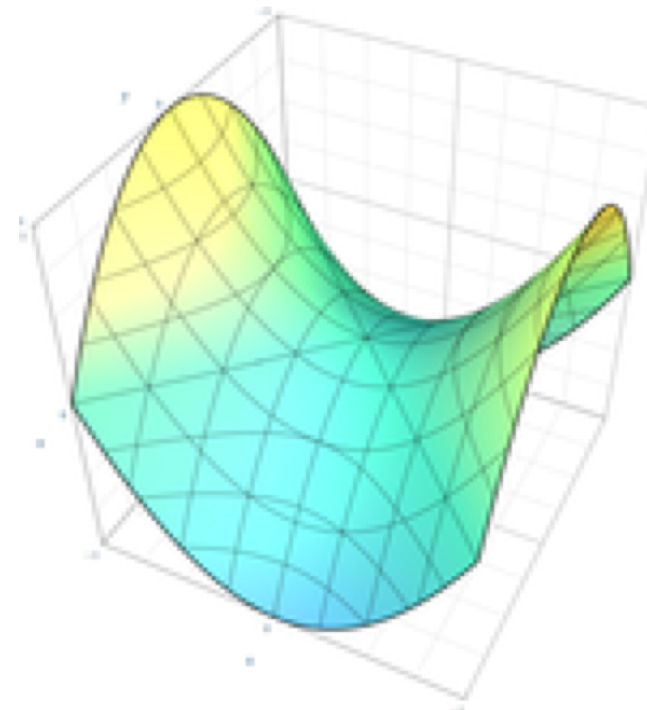


SHALL BECOME

Hyperbolic paraboloid

$z = (x/a)^2 - (y/b)^2$ (a quadric)

Where are the rulings ?

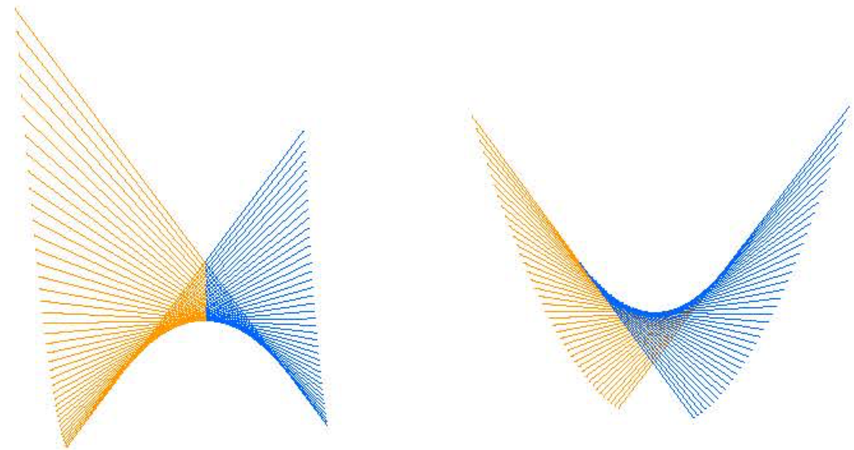


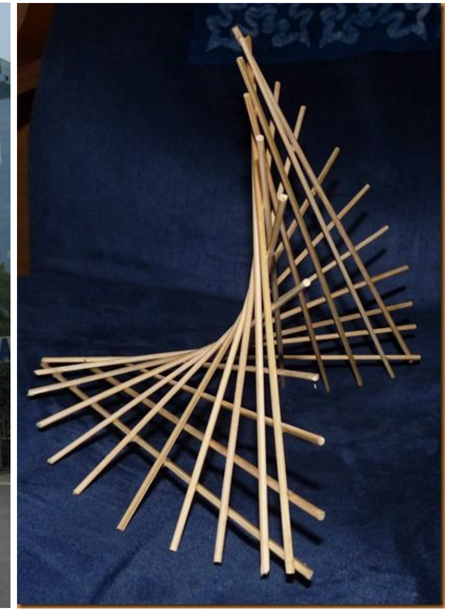
In fact it is doubly ruled (two one-parameter families of lines)

Hyperbolic paraboloid as a ruled surface

$$S^{\pm}(u,v) = (a(u+v), \pm bv, u^2+2uv)$$
$$= (au,0,u^2)+v(a, \pm b,2u)$$

- $c(u) = (au,0,u^2)$
- $d(u) = (a, \pm b,2u)$




WKD




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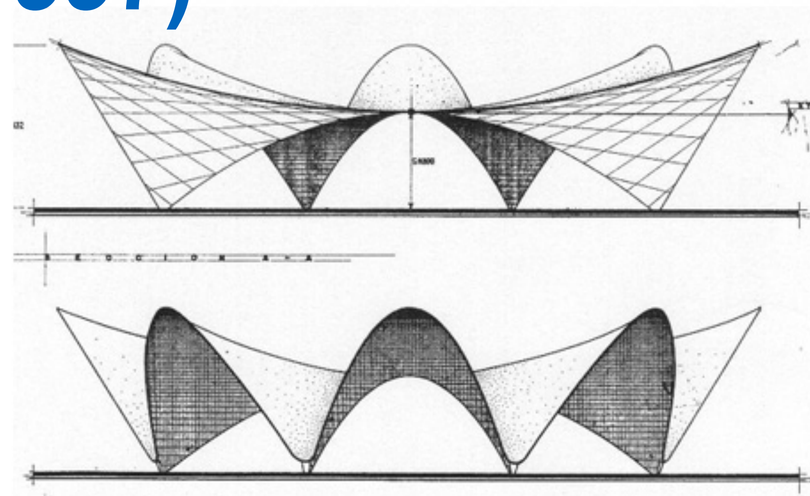
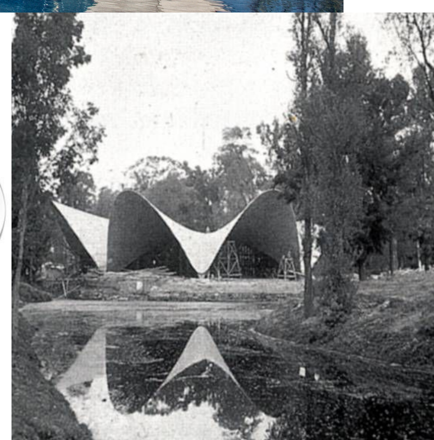
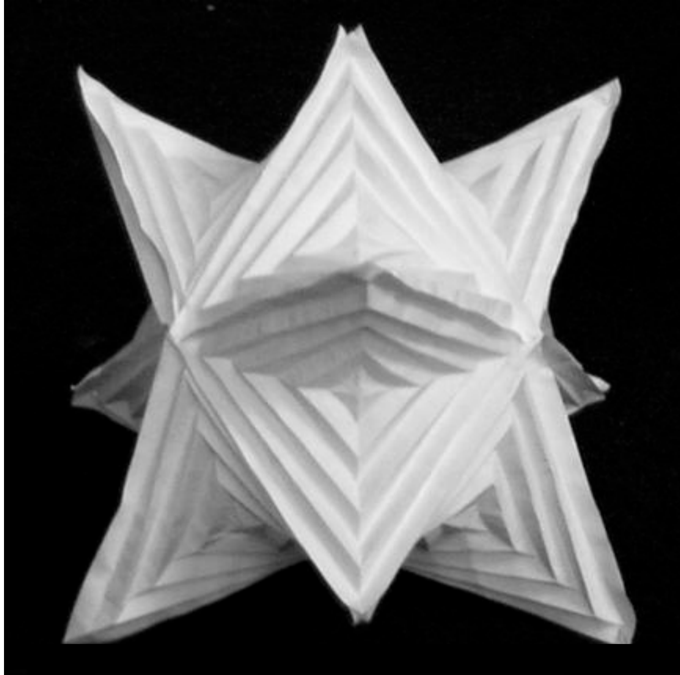
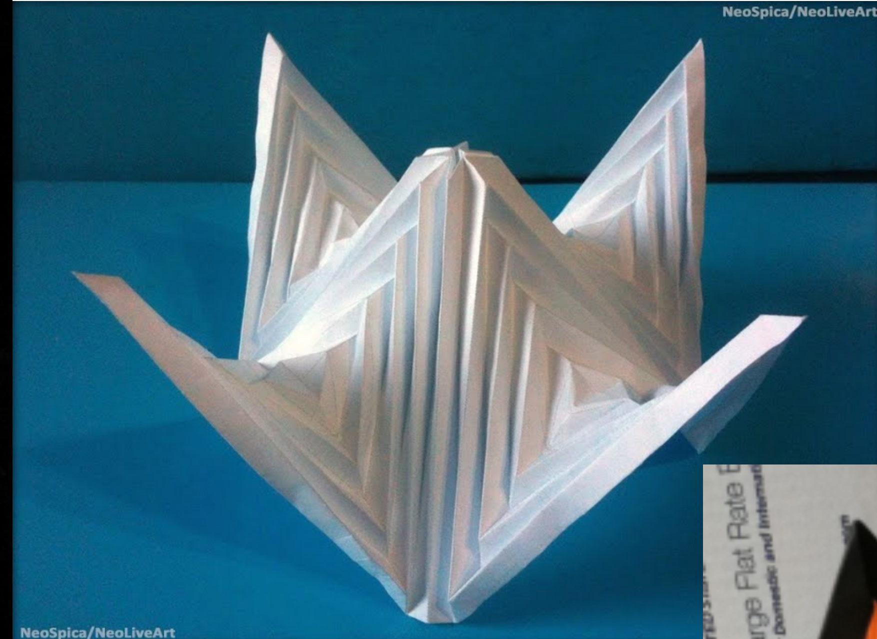
Félix Candela (1910-1997)





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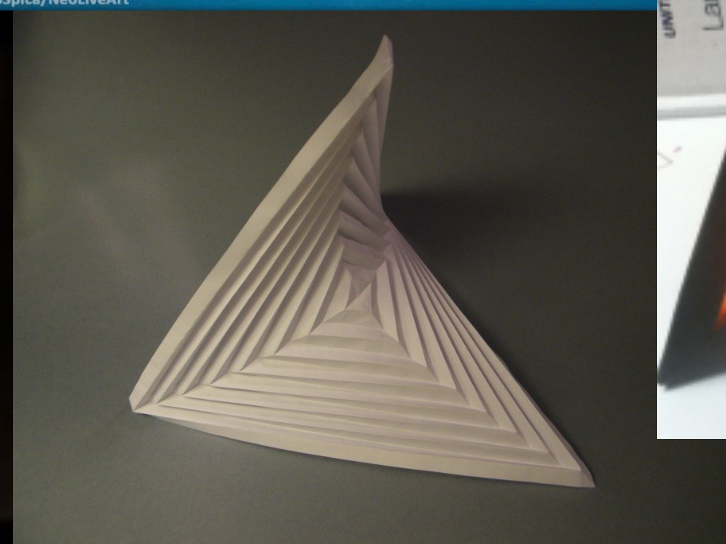
NeoSpica/NeoLiveArt




Large Flat Rate
9389774

Hyperboloid of one sheet

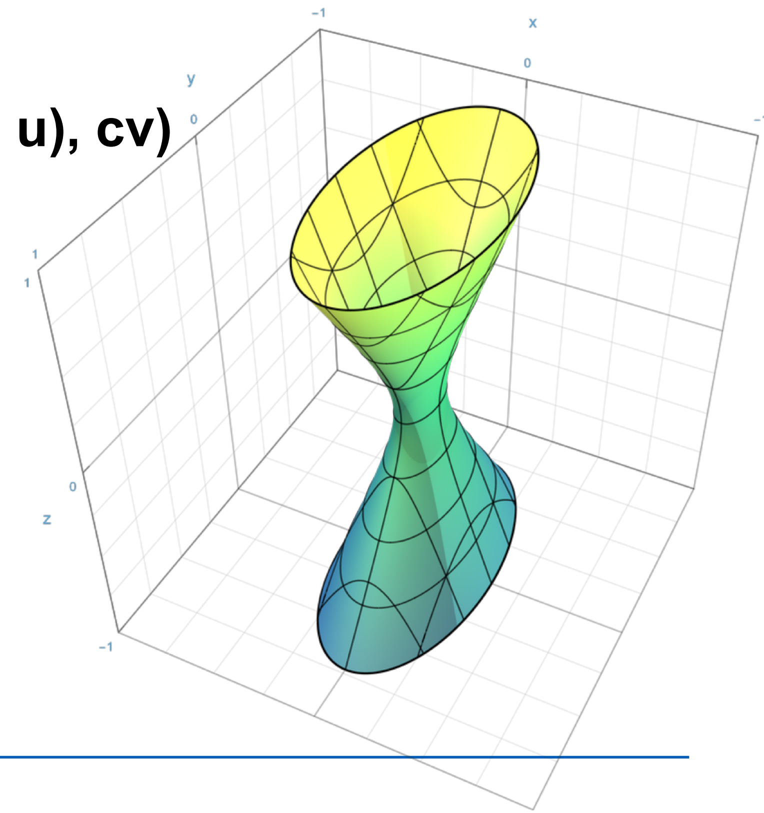
$(x/a)^2 + (y/b)^2 - (z/c)^2 = 1$ (a quadric)

$S^{\pm}(u,v) = (a(\cos u \mp v \sin u), b(\sin u \pm v \cos u), cv)$

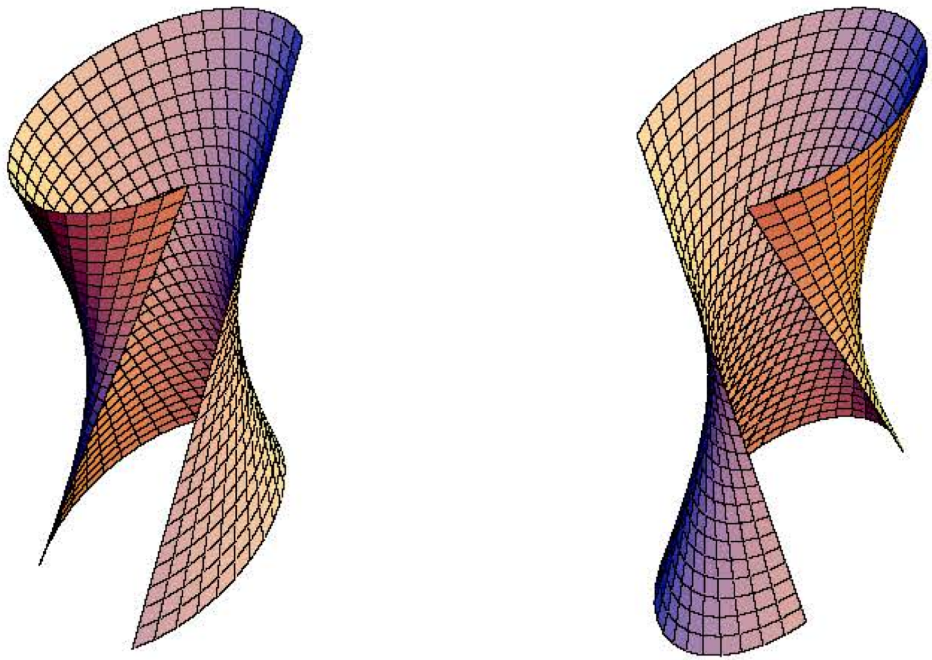
$= (a \cos u, b \sin u, 0) +$

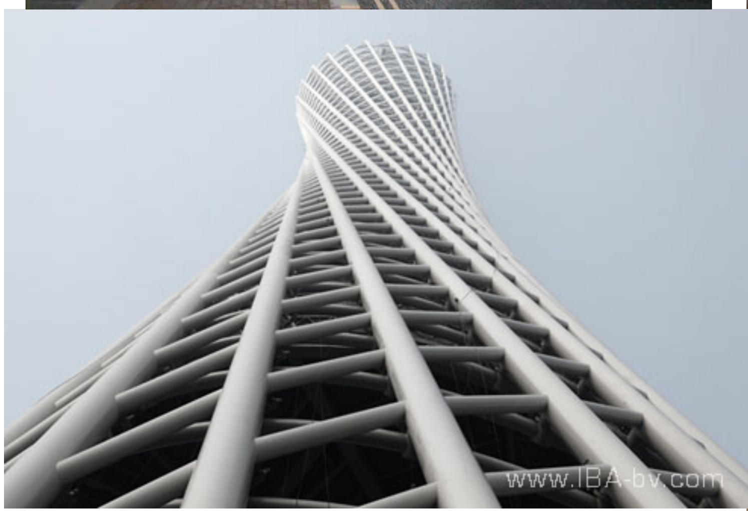
$v(\mp a \sin u, \pm b \cos u, c)$

- $c(u) = (a \cos u, b \sin u, 0)$
- $d(u) = (\mp a \sin u, \pm b \cos u, c)$



Double rulings of a hyperboloid





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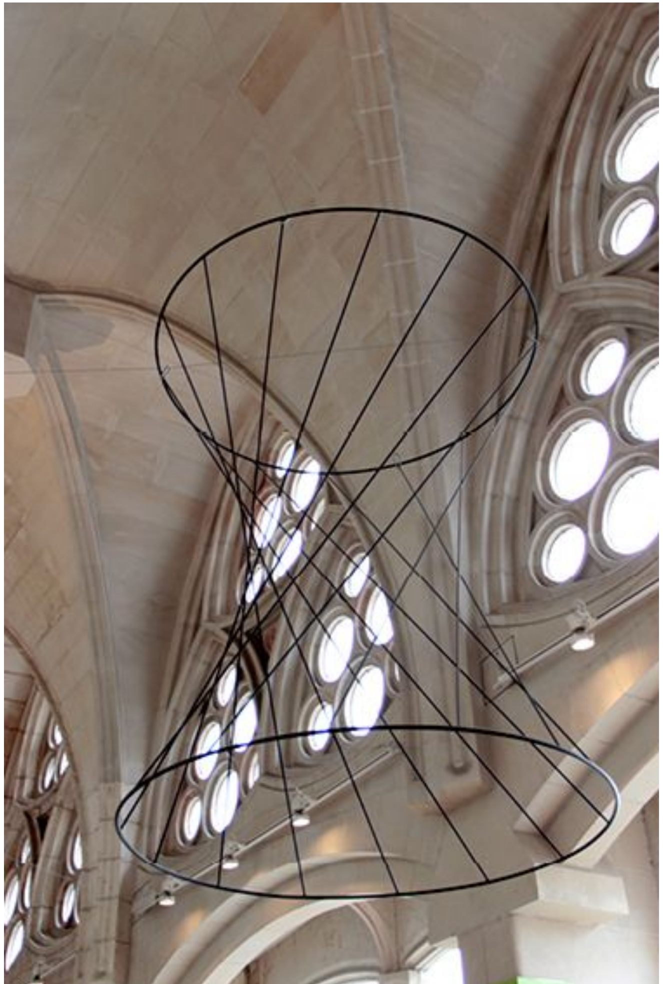

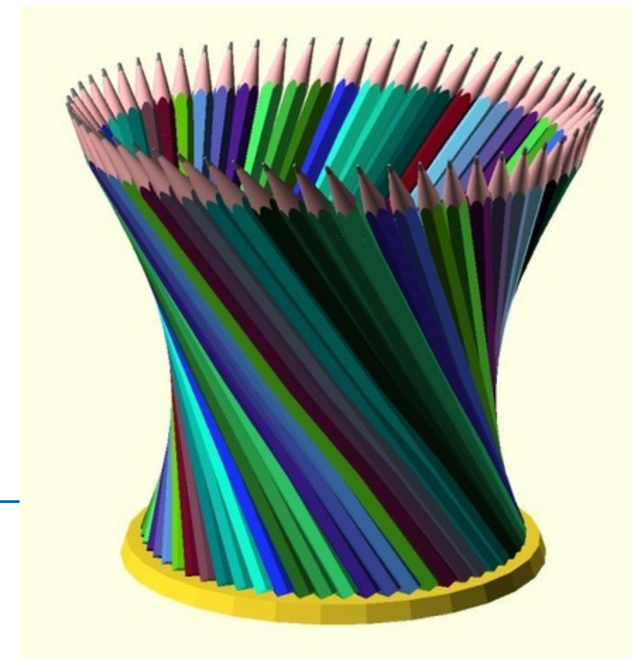

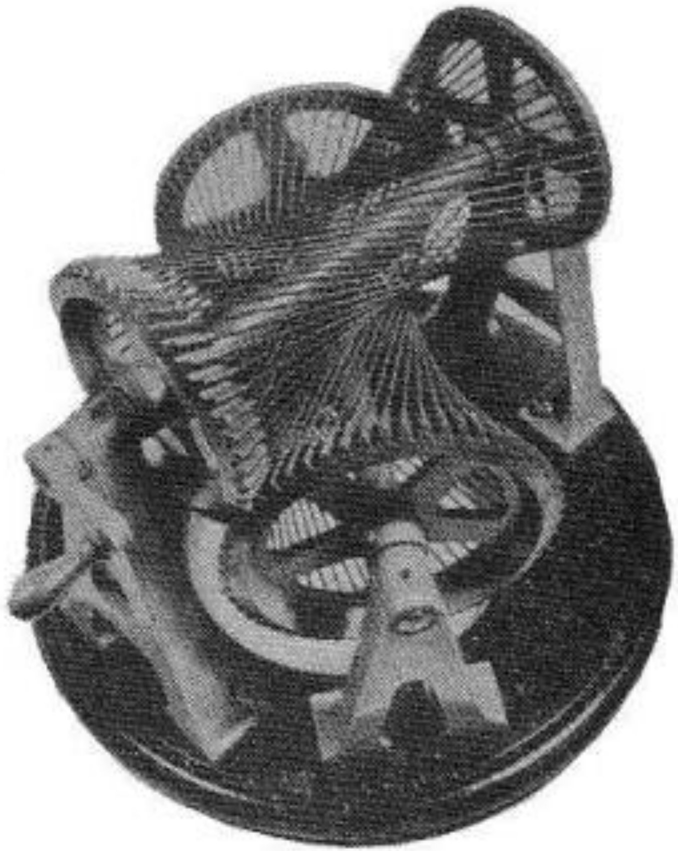

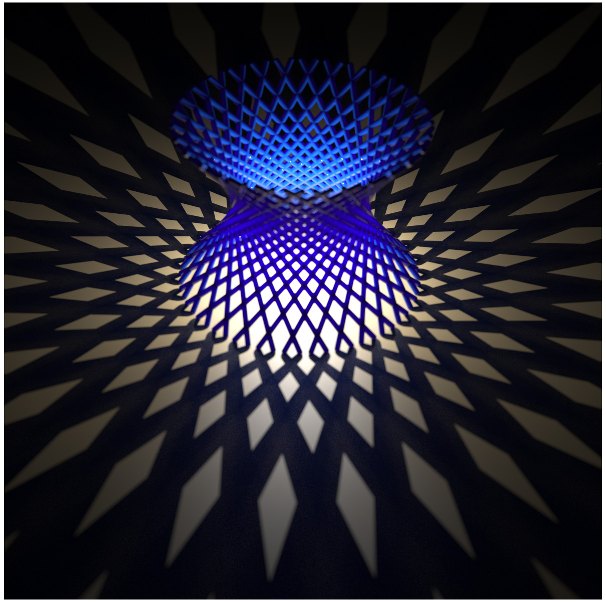

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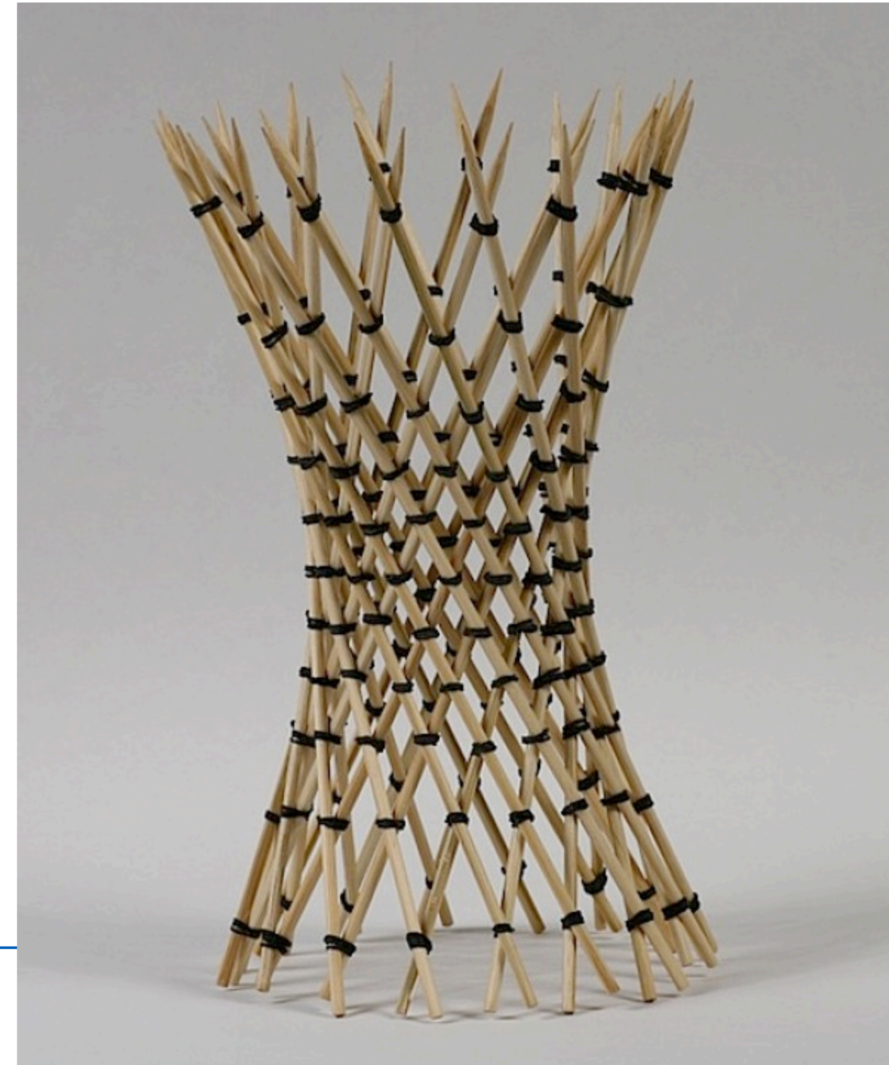
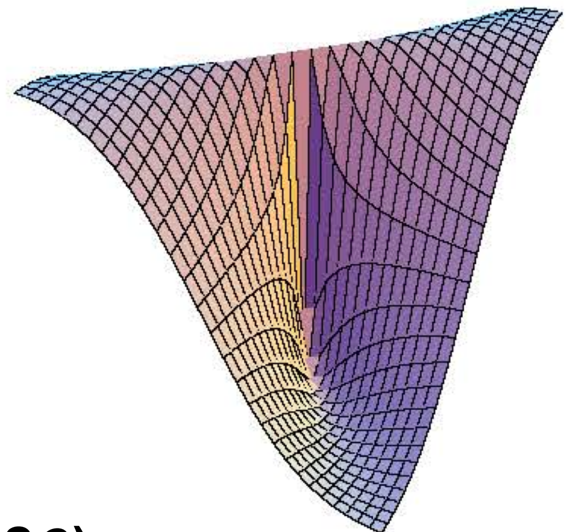
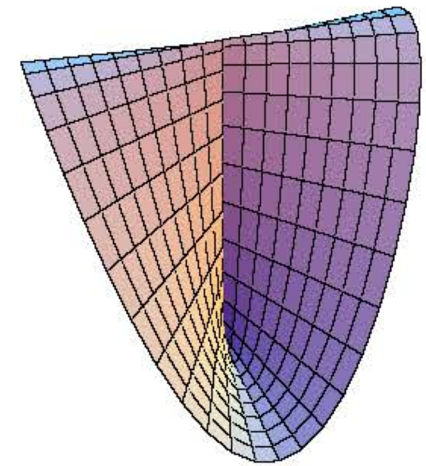
NeoSpica
NeoSpica



Plücker conoid with 2 folds





$z = 2xy/(x^2+y^2)$

In polar coordinates (r, θ)

$x = r\cos\theta$, $y = r\sin\theta$

$S(r, \theta) = (r\cos\theta, r\sin\theta, \sin 2\theta)$

$= (0, 0, \sin 2\theta) + r(\cos\theta, \sin\theta, 0)$

$c(\theta) = (0, 0, \sin 2\theta)$

$d(\theta) = (\cos\theta, \sin\theta, 0)$

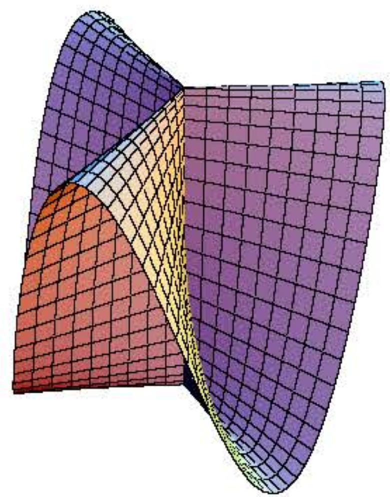
Plücker conoid with n folds

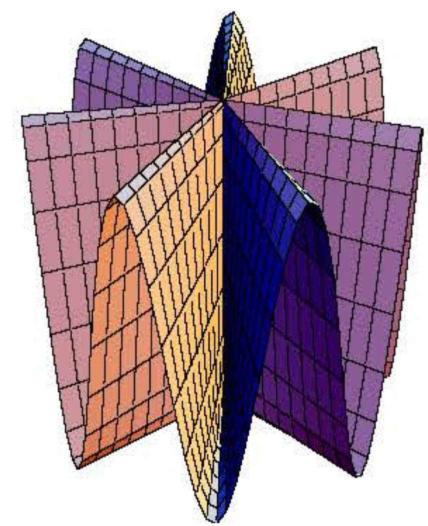
$S(r, \theta)=(r \cos\theta, r \sin\theta, \sin n\theta)$

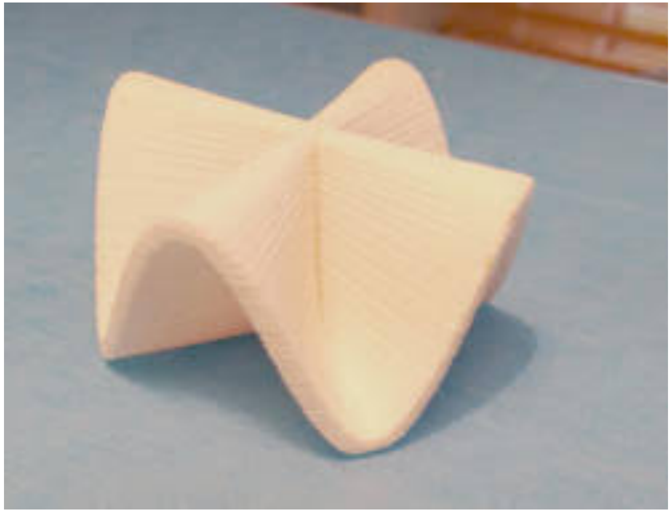
$=(0, 0, \sin n\theta)+r(\cos\theta, \sin\theta, 0)$

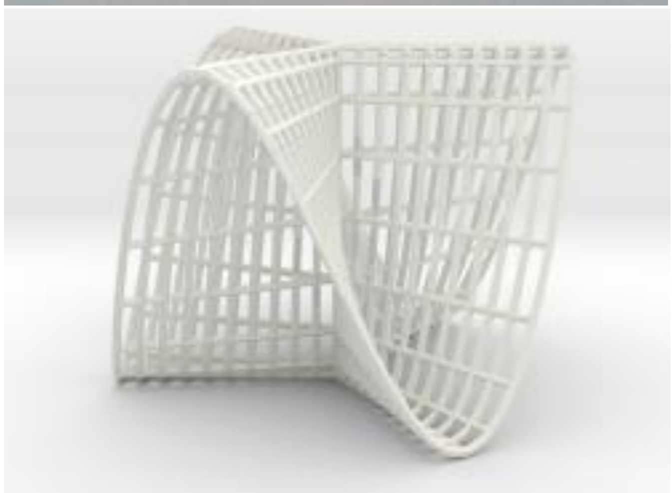
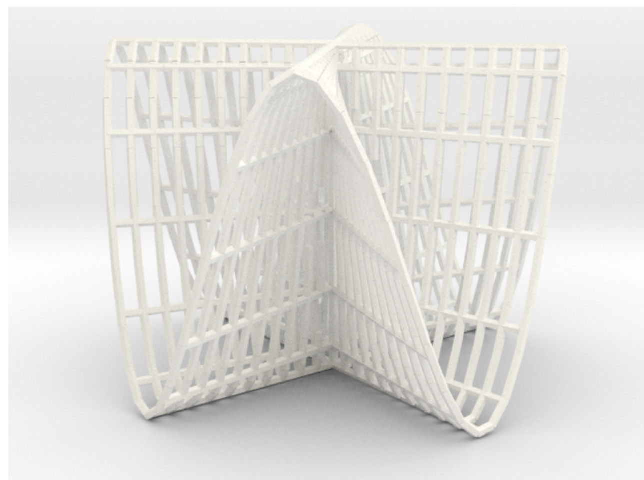
$c(\theta)=(0, 0, \sin n\theta)$

$d(\theta)=(\cos\theta, \sin\theta, 0)$







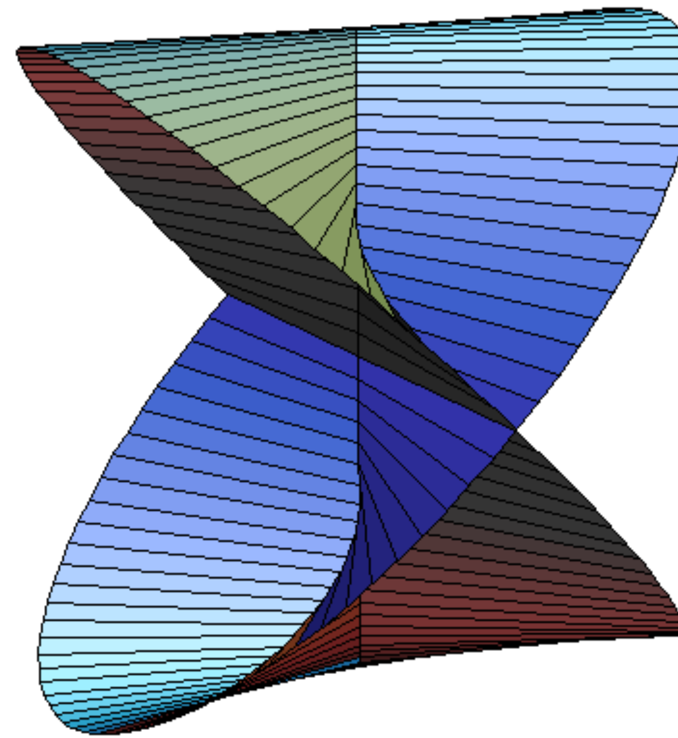



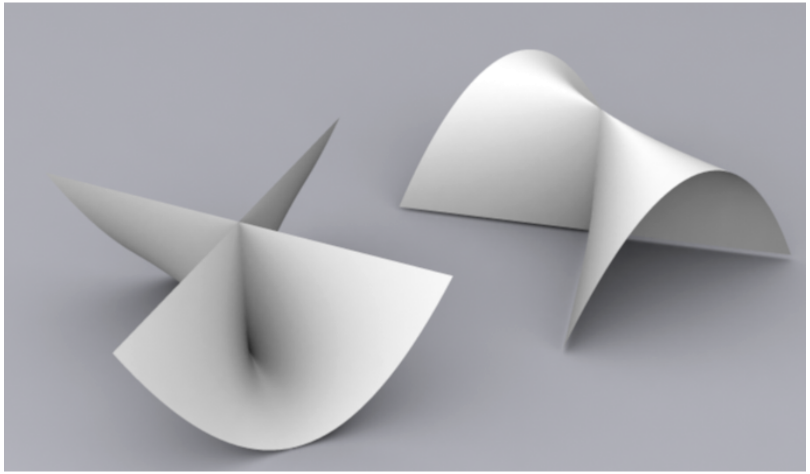
Right conoids

All ruled surfaces with rulings parallel to a plane passing through a line that is perpendicular to the plane.

Ex. Take xy-plane and z-axis, then

$$S(u,v) = (v \cos \theta(u),\ v \sin\theta(u),\ h(u))$$
$$= (0,0,\ h(u)) + v(\cos \theta(u),\ \sin\theta(u),\ 0)$$



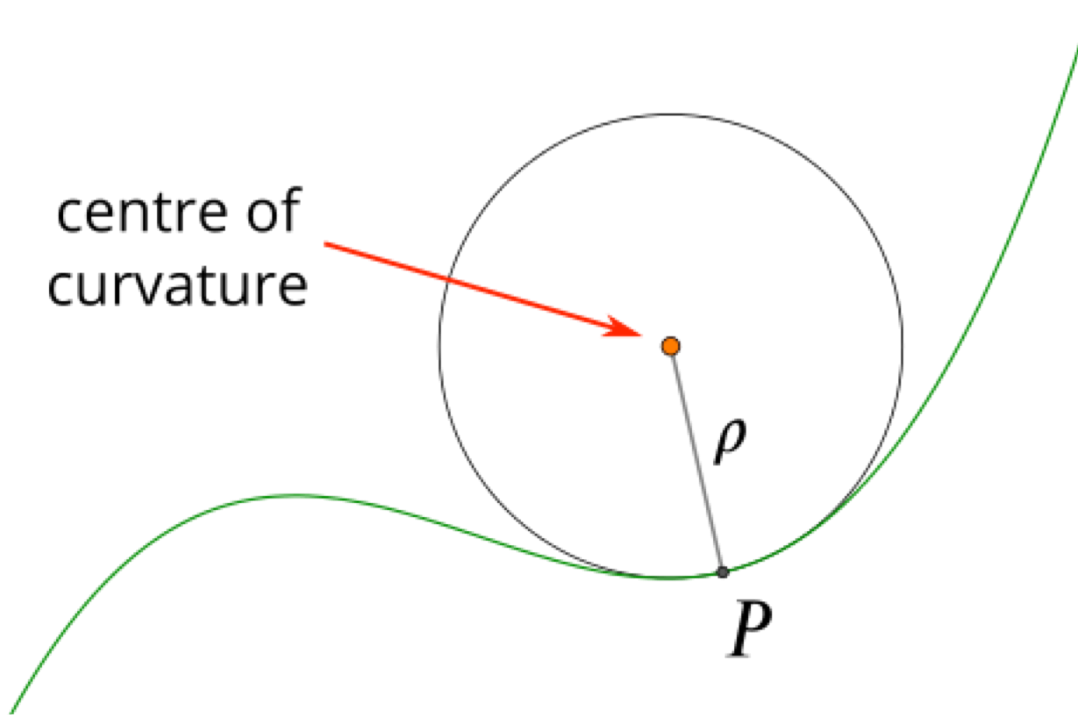



Barcelona Bus

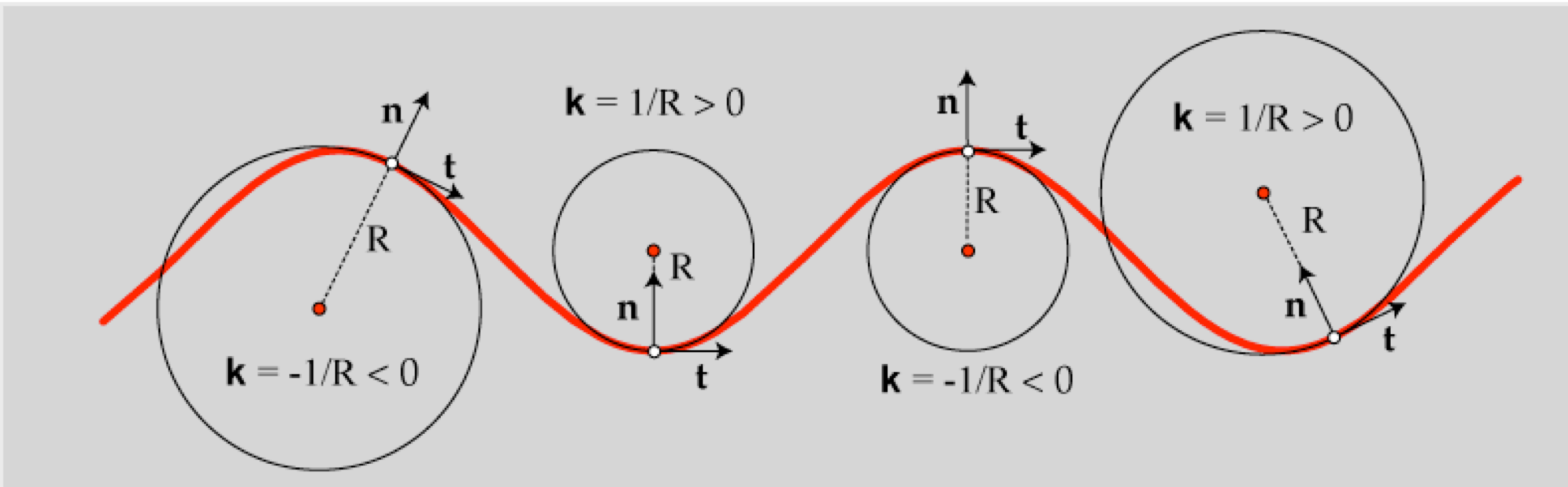
What is curvature ?

Curvature of a smooth planar ***curve*** at point **P** is $\varkappa(P)=1/\boldsymbol{\rho}$

- works also for curves in space or higher dimensions
- points should be approachable with circles
- ***extrinsic*** quantity



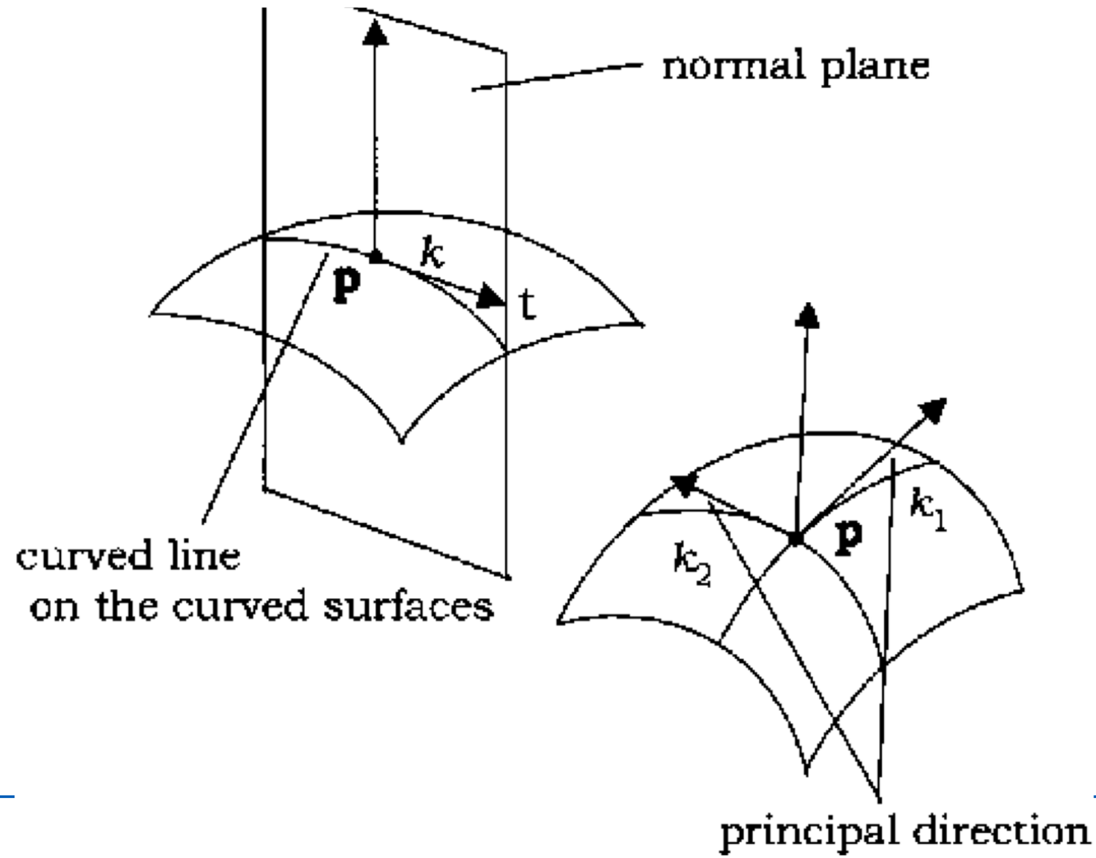
Curvature of a (parametrized planar) curve has a sign



What is curvature of a surface ?

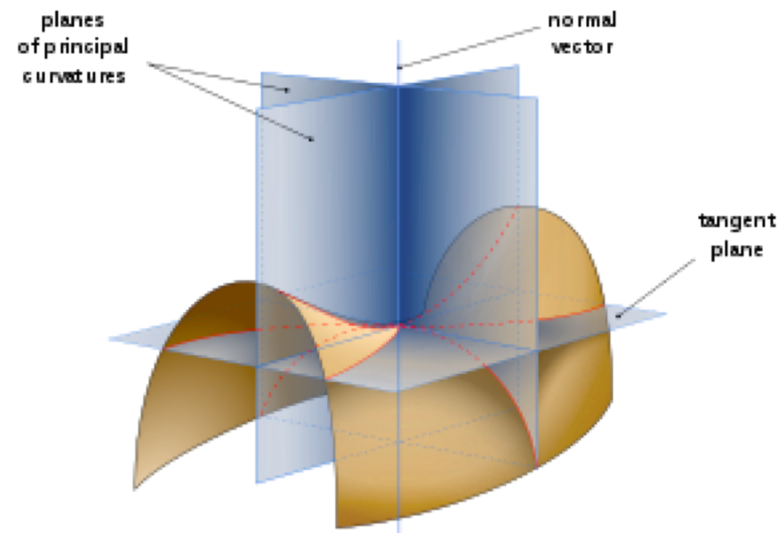
Gauss curvature

$\mathbf{K(p)} = \varkappa_1(\mathbf{p})\,\varkappa_2(\mathbf{p})$



Theorema Egregium (Gauss, 1827)

Curvature K is an *intrinsic* quantity !



- **Helicoid**
- **Hyperbolic paraboloid**
- **Hyperboloid**
- **Plücker Conoid**
- **Right conoids**

All have (varying) negative curvature !

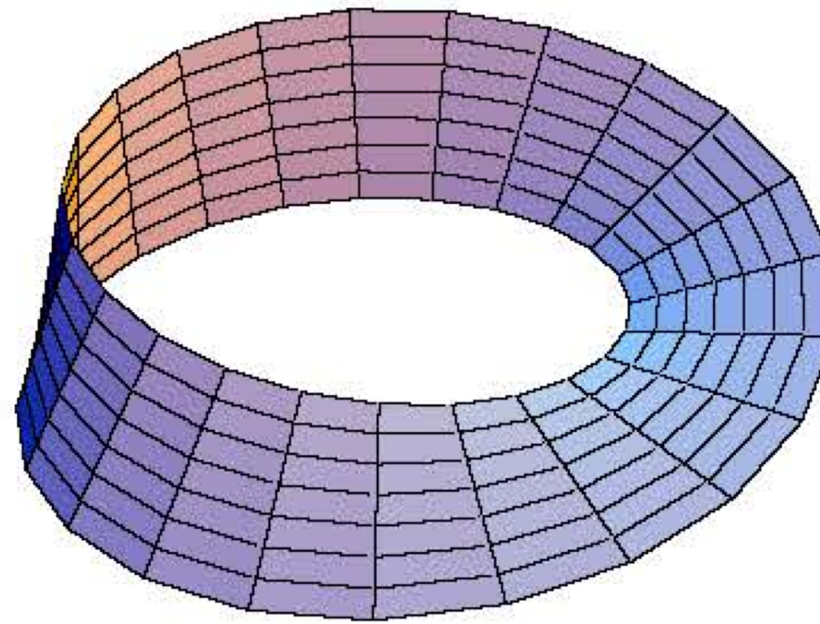
Are there other flat (K=0) ruled surfaces than plane, generalized cylinders and cones ?

Gaussian curvature of a ruled surface

$S(u,v)=c(u)+vd(u)$

$K= -(d'(u).\ N)/(EG-F^2) \leq 0$ (!)

Ex: Möbius band, with the parametrization earlier is nowhere flat !
Especially: This parametrization cannot be formed from a flat strip of paper



Mean curvature $H(p) = \frac{1}{2}(\varkappa_1(p) + \varkappa_2(p))$

- **Not an intrinsic quantity !**
- **H=0: minimal surfaces**

The only minimal ruled surfaces are plane and helicoid

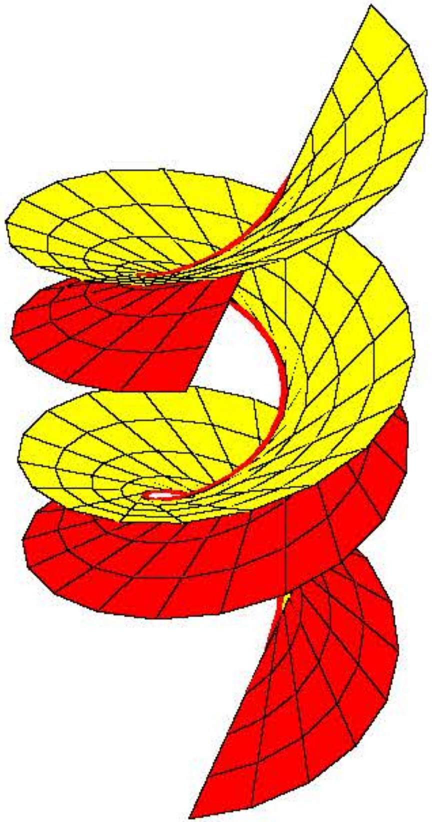
Three classes of flat ruled surfaces

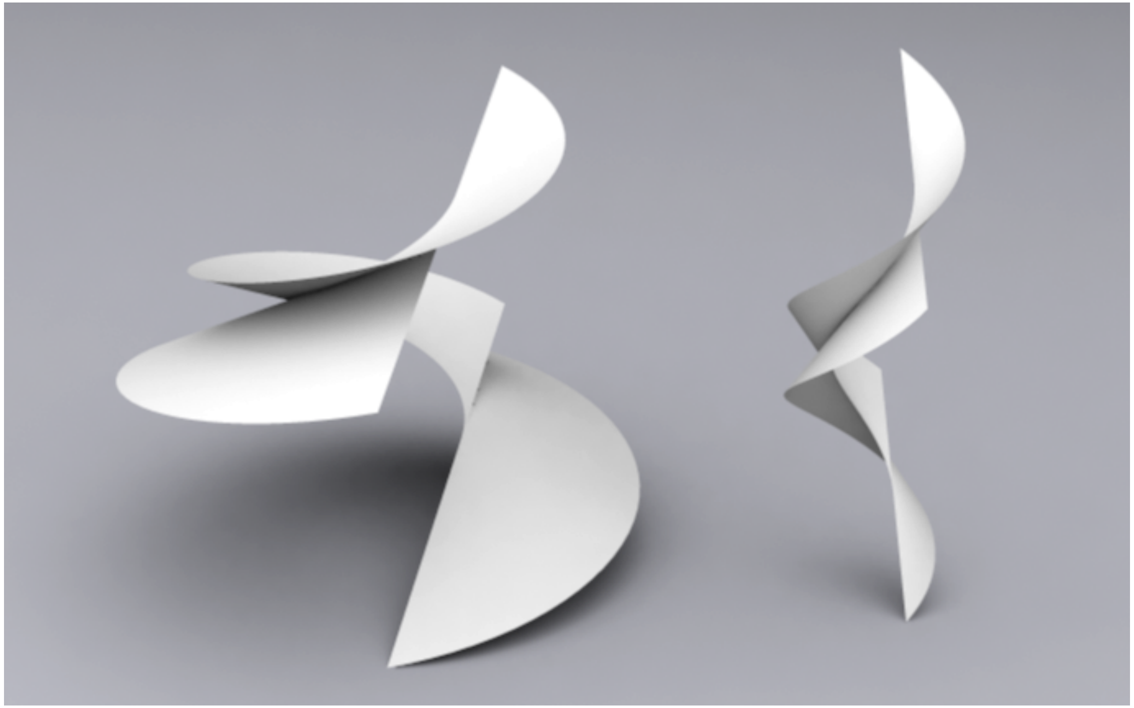
= Developable surfaces

- Generalized cones
- Generalized cylinders
- Tangent developables: S(u,v)=c(u)+vc'(u)

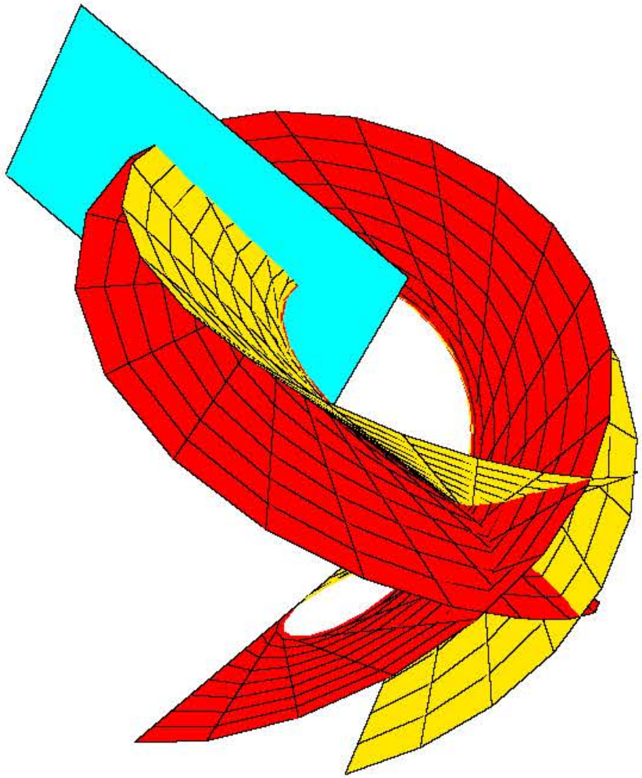
- ***Aristotle (384-322 B.C.): 'a line by its motion produces a surface'***
- ***Monge (1746-1818):*** a principle to generate surfaces => seeds to **'descriptive geometry'**

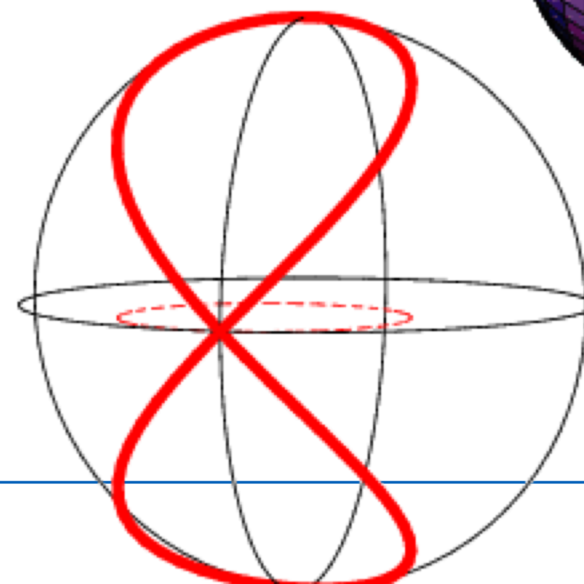
Tangent developable to a circular helix

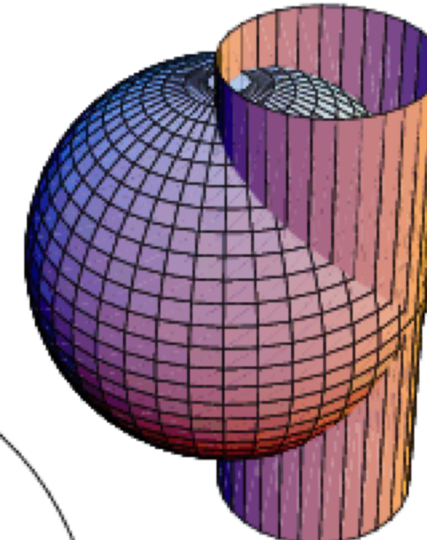




Tangent developable to a Viviani's curve







Some further history about surfaces that can be developed into plane

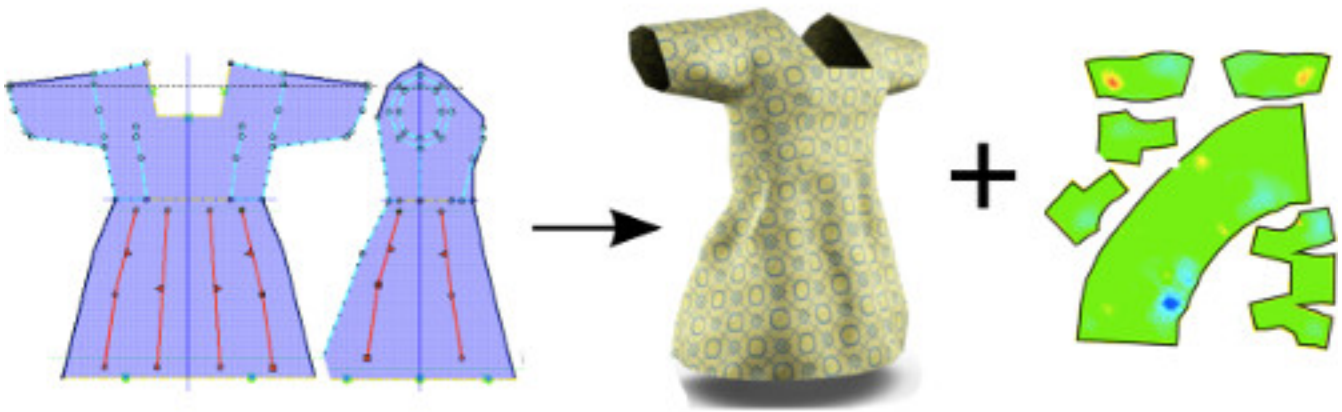
- **William Hawney** (author on surveying): 1717 described the cylinder as a surface ‘rolled over a plane so that all its points are brought into coincidence with the plane’**.**
- **1737 Amédée François Frézier** (1682-1773) also considered the rolling of the plane to form a circular cylinder and cone
- **Euler (1707-1783) & Monge** more systematic treatment of

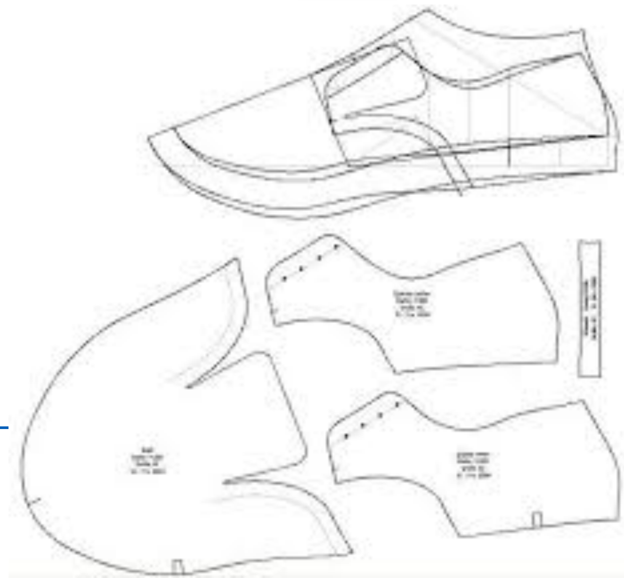
developable surfaces via differential calculus (= ’study of change’) =>

- **1886** term **“differential geometry”** was coined by **Luigi Bianchi**

Developable surfaces in ship building

….Cloth fabrication…..





….Gehry architecture

….Hans Hollein architecture

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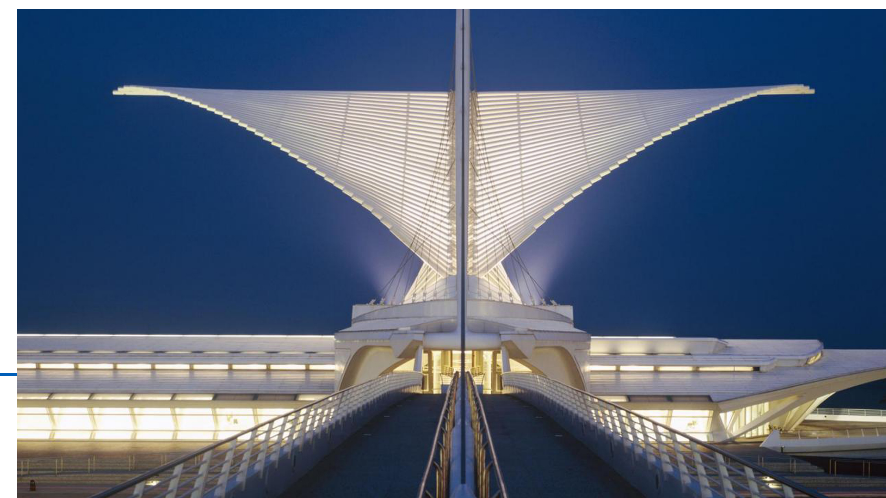
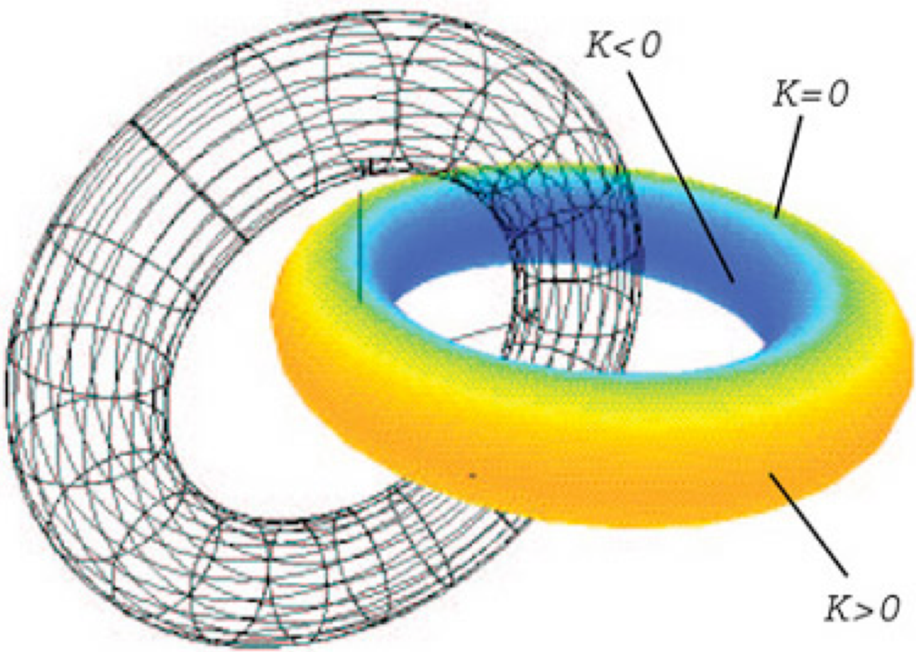
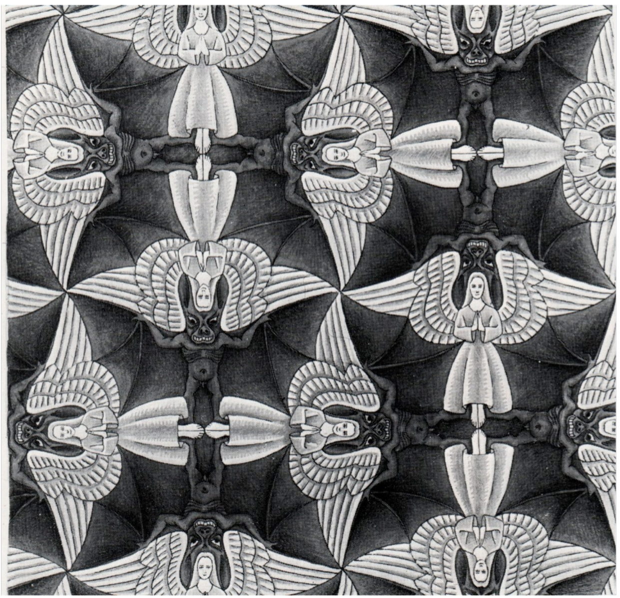
Santiago Calatrava



What are possible constant Gauss curvature geometries for smooth closed surfaces ?

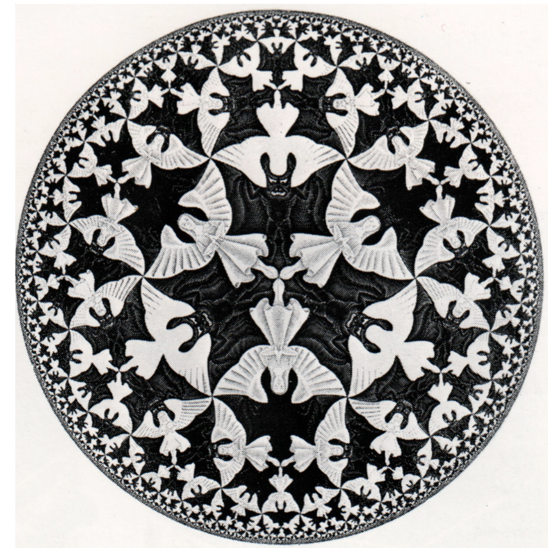


Euclidean (=flat), spherical and hyperbolic models of 2D geometry



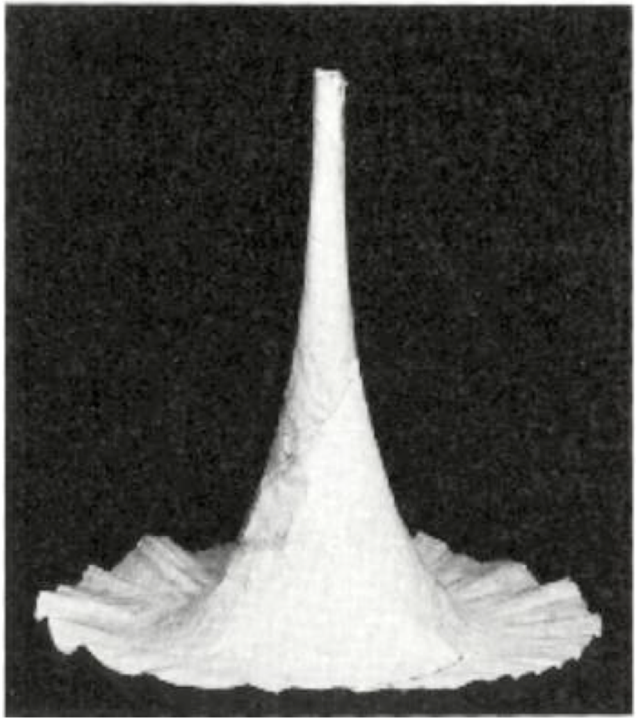
K = 0

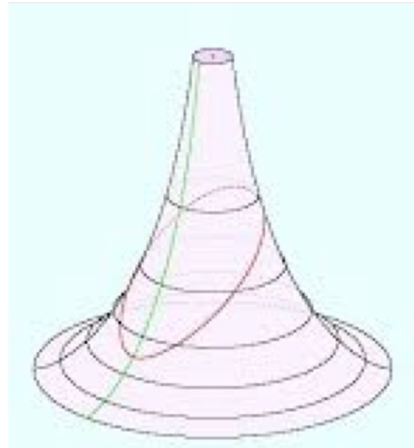
K>0



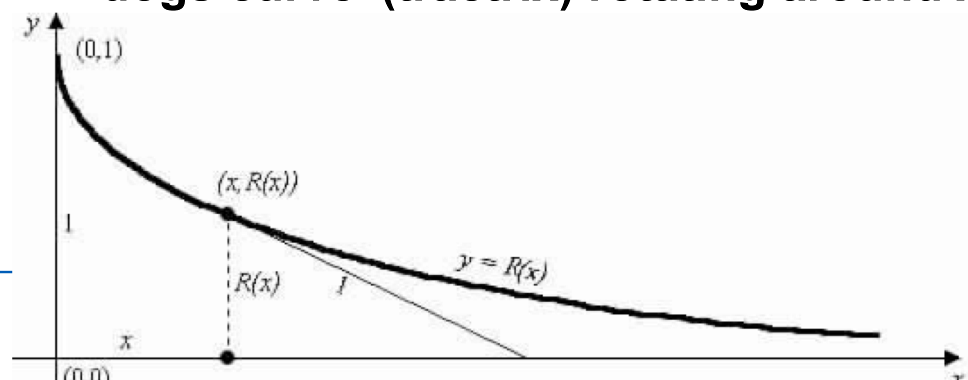
K<0

Eugenio Beltrami (1835-1900)



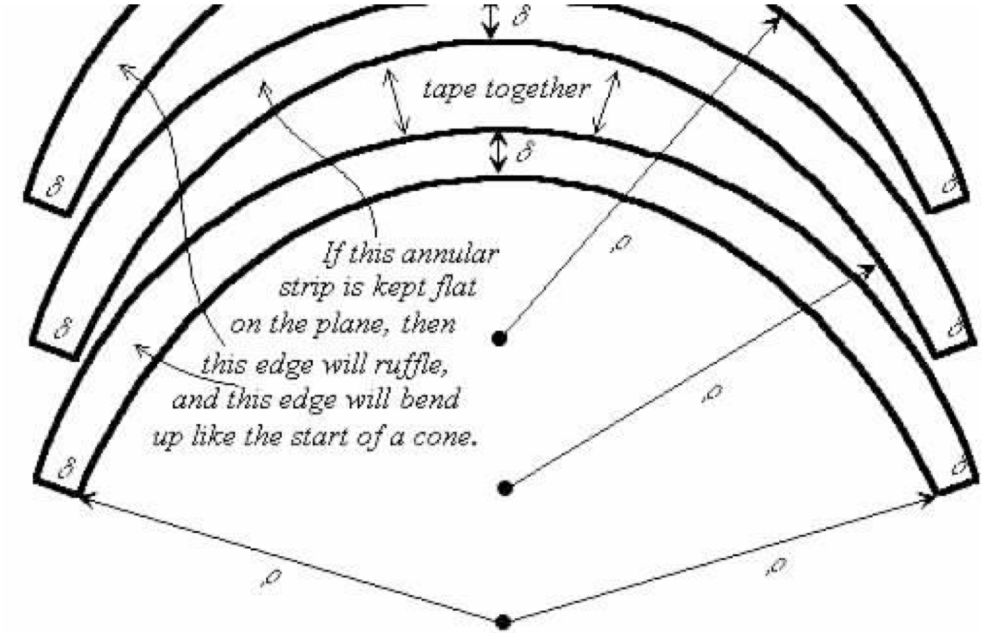


Pseudosphere (1866) : A local model via ‘lazy dogs curve’ (tractrix) rotating around x-axis



Curvature -1

Bill Thurston and his paper annuli to approximate hyperbolic surfaces

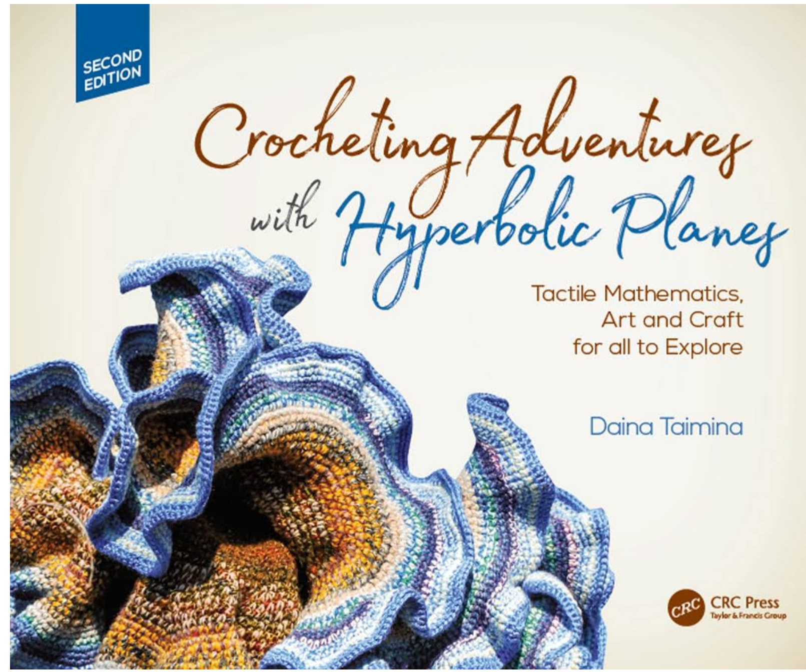


ρ = radius of the hyperbolic plane
Curvature - 1/ ρ^2

What I hear I forget,
What I see, I remember,
What I touch, I understand.
- Confius (555-479 CE)

Some outcomes from the workshop at the Institute of Figuring (theiff.org)

Crochet instructions by Daina Taimina

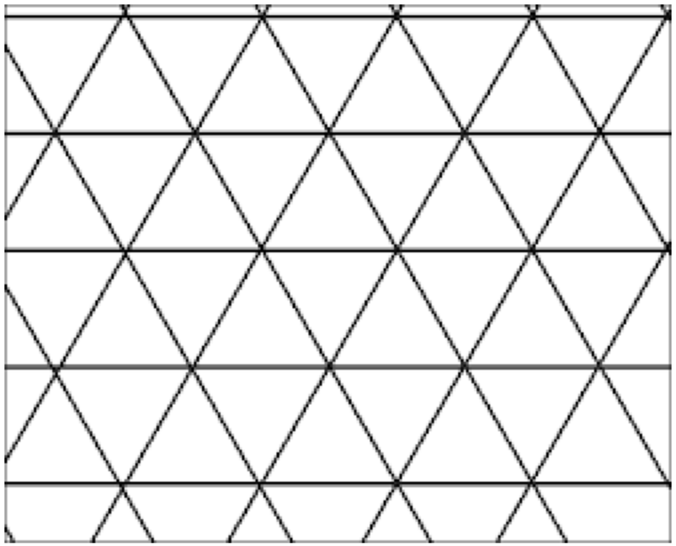


https://www.youtube.com/watch?v=rY8Uo6rSnZc

Study crocheted surfaces

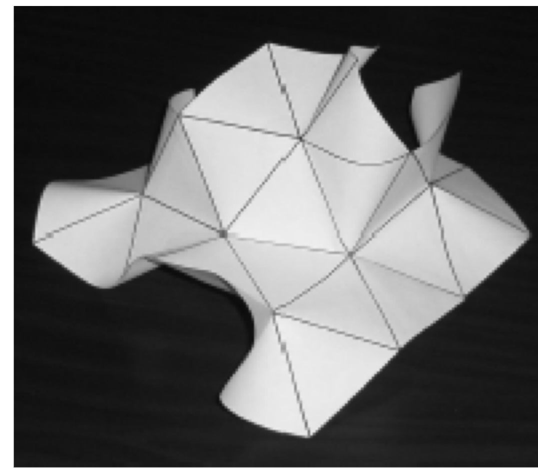
- Try to find curves that realize shortest distances between some points ie *geodesics*
- Try to convince yourself that the parallel axiom does not hold
- can you find the radius of the surface ?

Thurston model to approximate hyperbolic plane



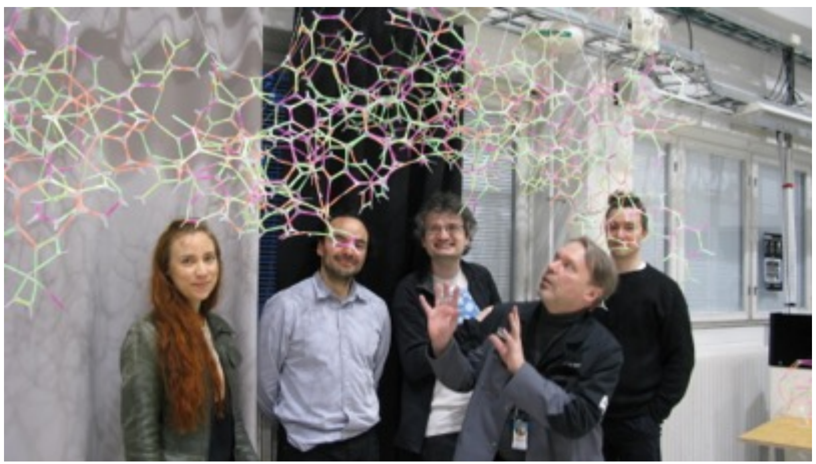
- Cut out a hexagon formed by 6 equilateral triangles
- Make a slit and tape one more triangle so that 7 triangles meet at a vertex

- Add at least two layers of triangles so that every vertex is adjacent to 7 triangles

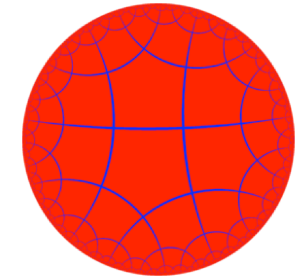


Build a hyperbolic surface

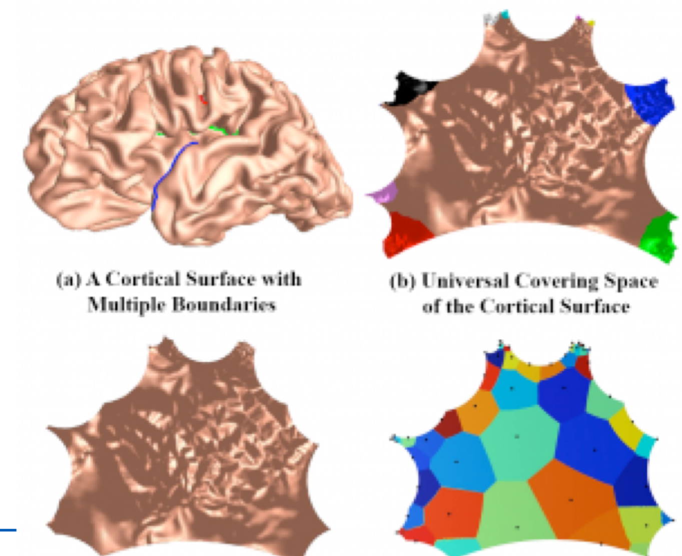
- By gluing heptagons (and hexagons)
- Compare with surfaces consisting of pentagons (and hexagons)



Why hyperbolic geometry ?



- Connections to cellular automata (Margenstern-Morita etc.)
- Visualizations of Web, Network security
- Modular functions in number theory (Fermat's last theorem)
- Algebraic geometry, differential geometry, complex variables, dynamical systems
- Biology



(a) A Cortical Surface with Multiple Boundaries

(b) Universal Covering Space of the Cortical Surface

(c) Canonical Fundamental Domain for Hyperbolic Harmonic Map

(d) Hyperbolic Power Voronoi Diagram for Optimal Mass Transport Map

Some References

Console: *Ruled surfaces*

Glaeser & Gruber: *Developable surfaces in contemporary architecture,* Journal of Mathematics and Arts, 2007

Lawrence: *Developable surfaces: Their history and applications,* Journal of Mathematics and Arts, 2010