6 Vibrations of membranes and plates
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Learning outcome
A. Understanding the basics of structural vibrations
B. Ability to solve basics vibration problems for plates and membranes

References
Text book: Chapter 9
6.1 Vibrations of plates

**Introduction.** In civil and structural engineering, vibrations or vibrational loads are present in many real-life situations:

- wind load in high rise buildings and bridges
- earthquake, blast
- human movement (walking, sports), machine (rotator) or traffic excitation (cars, trains, metros).

Lightweight and large-span structures, especially, are affected by excitation affecting on *vibration comfort*. Structural vibrations can be reduced by the techniques of

- *passive control* (time-independent masses, springs and dampers) and/or
- *active control* (time-dependent forces).

Design codes such as *Eurocode* give some limits to the fundamental frequencies floor slabs, for instance, and some formula for designing certain types of structures.
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Warm-up exercise

Solve the free oscillations of a discrete mass-spring-damper \((m,k,c)\) system modelled by a linear second-order ordinary differential equation of constant coefficients in the form

\[
\ddot{u}(t) + c \dot{u}(t) + ku(t) = f(t) = 0.
\]

**Hint:** Use the trial function \(u(t) = e^{\lambda t}\) for deriving the characteristic equation, and distinguish the cases

1. \(c^2 > 4mk\),
   two distinct real roots (overdamping)
2. \(c^2 = 4mk\),
   a real double root (critical damping)
3. \(c^2 < 4mk\),
   two complex conjugate roots (underdamping)

(damping coefficient \(\gamma = c/(2m)\).
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**Equation of motion for 3D bodies.** Let us recall that the static equilibrium (or force balance) for any subset \( \Omega \subset \mathbb{R}^3 \) of a three-dimensional body follows from the *Euler’s laws* (or *momentum principles*, equivalent generalizations of *Newton’s laws*):

*Principle of linear momentum* can be written in the form

\[
\int_\Omega b \, d\Omega + \int_{S_t} t \, dS = \int_\Omega \rho \dot{v} \, d\Omega
\]

with *spatial* (current configuration) *body load* \( b = b(x,t) \), *surface traction* \( t = t(x,t) \), *mass density* \( \rho = \rho(x) \) and *spatial velocity field* \( v = v(x,t) \) at time \( t \).

*Cauchy’s law* and *Gauss divergence theorem* imply the form

\[
\int_\Omega \rho \dot{v} \, d\Omega = \int_\Omega b \, d\Omega + \int_{S_t} t \, dS = \int_\Omega b \, d\Omega + \int_{S_t} \sigma n \, dS
\]

\[
= \int_\Omega b \, d\Omega + \int_\Omega \text{div} \sigma \, d\Omega
\]

from which one obtains the *equation of motion*:

\[
-\text{div} \sigma(x,t) + \rho(x) \frac{dv(x,t)}{dt} = b(x,t) \quad \forall (x,t) \in \Omega \times (0,t_1).
\]
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Equation of motion for structural models. Let us next write this equation of motion of forced vibrations for different structural models with certain kinematical and constitutive assumptions (leading to hyperbolic differential equations):

For an axially loaded engineering bar, the kinematical assumptions are of the form
\[ u = (u_x(x,y,z,t), u_y(x,y,z,t), u_z(x,y,z,t)) = (u(x,t), 0, 0) \]
implying the balance equation
\[ \rho(x)A(x)\partial_t^2 u(x,t) - \left( E(x)A(x)u'(x,t) \right)'(x,t) = b(x,t). \]

For an Euler-Bernoulli beam, kinematical assumptions are of the form
\[ u = (-zw'(x,t), 0, w(x,t)) \]
implying the balance equation
\[ \rho(x)A(x)\partial_t^2 w(x,t) + \left( E(x)I(x)w''(x,t) \right)''(x,t) = f(x,t). \]

For a Timoshenko beam, kinematical assumptions are of the form
\[ u = (-z\beta(x,t), 0, w(x,t)) \]
implying the balance equation
\[ \rho(x)A(x)\partial_t^2 w(x,t) - \left( (G(x)A(x)(w' - \beta)(x,t))' \right)' = f(x,t) \]
\[ \rho(x)I(x)\partial_t^2 \beta(x,t) - \left( E(x)I(x)\beta'(x,t) \right)' - G(x)A(x)(w' - \beta)(x,t) = 0. \]
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For a Kirchhoff plate, kinematical assumptions are of the form

\[ u = (-z \hat{\partial}_x w(x, y, t), -z \hat{\partial}_y w(x, y, t), w(x, y, t)) \]

implying, in case of constant thickness and constant material parameters, the balance equation

\[ \rho t \partial_i^2 w(x, y, t) + D \Delta^2 w(x, y, t) = f(x, y, t) \]

or equivalently

\[ D \Delta^2 w(x, y, t) = f(x, y, t) - \rho t \partial_i^2 w(x, y, t). \]

Remark. The so-called d'Alemberts principle is explicitly visible: the external loading (cf. statics) is augmented by a term of inertial forces and "everything" depends on time.

This principle enables us to use the principle of virtual work in a form analogous to the case of statics.
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**Initial-boundary value problem.** The principle of virtual work can be written as

\[ 0 = \delta W_{\text{int}} + \delta W_{\text{ext}} - \delta W_{\text{acc}} \]

\[ = -\int_V \sigma : \delta \varepsilon \, dV + \int_V b \cdot \delta u \, dV + \int_{S_t} t \cdot \delta u \, dS - \int_V \rho \partial_t^2 u \cdot \delta u \, dV \]

\[ = -\int_V \sigma : \delta \varepsilon \, dV + \int_V (b - \rho \partial_t^2 u) \cdot \delta u \, dV + \int_{S_t} t \cdot \delta u \, dS \]

which implies the governing equations and boundary conditions of the problem in a usual manner (kinematical and constitutive assumptions, integration by parts etc.).

**Remark.** Time-dependency additionally requires that intial values are given for both displacement and velocity (implied by Hamilton’s principle which is an implication of the principle of virtual work). The final *initial value problem*, or better *initial-boundary value problem*, consists of the governing balance equation together with boundary and initial conditions.

For the simplest possible structural model, the engineering bar, the initial-boundary value problem is of the following form:
### 6.1 Vibrations of plates

**Elastodynamics of an axially loaded rod.** Find displacement \( u = u(x,t) \) such that

\[
(1) - \left( E(x)A(x)u'(x,t) \right)'(x,t) = b(x,t) - \rho(x)A(x)\partial_t^2 u(x,t), \quad (x,t) \in \Omega \times (0,t_1)
\]

(2a) \[ u(0,t) = u_0(t), \quad t \in (0,t_1) \] (2) boundary conditions

(2b) \[ N(L,t) = N_L(t), \quad t \in (0,t_1) \]

(3a) \[ u(x,0) = \bar{u}(x), \quad x \in \Omega \] (3) initial conditions

(3b) \[ \partial_t u(x,0) = \bar{v}(x), \quad x \in \Omega. \]

\( u \) axial displacement (unknown function of \( x \) and \( t \))
\( E \) Young’s modulus; \( \rho \) mass density
\( A \) cross-sectional area; \( L \) length; \( \Omega = (0,L) \)
\( b \) axial body force
\( u_0 \) axial end point/facet displacement
\( N_L \) axial end point/facet force
\( \bar{u}, \bar{v} \) initial displacement and initial velocity;
\( (0,t_1) \) time domain
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**Free vibrations.** Assuming that there is no loading exiting the body leads to the so-called *problem of free vibrations*:

\[
-\mathbf{div} \sigma(x,t) + \rho(x) \frac{dv(x,t)}{dt} = 0 \quad \forall (x,t) \in \Omega \times (0,t_1).
\]

These types of problems can be solved by searching for the solution in a product form \( u(x,t) = U(x) T(t) \), where \( U \) needs to satisfy the boundary conditions and \( T \) is a *harmonic function*: \( T(t) = e^{i\omega t} \) for parameter \( \omega \) and imaginary unit \( i \). This technique is called *separation of variables*.

For Euler–Bernoulli beams, with constant bending rigidity, the problem of free vibrations takes the form

\[
\rho A \partial_t^2 w(x,t) + EI w''''(x,t) = 0.
\]

Inserting a trial solution \( w(x,t) = W(x) T(t) \) gives equation

\[
- \frac{EI}{\rho A} \frac{W''''(x)}{W} = \frac{\partial_t^2 T(x)}{T}.
\]
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which leads (Why?) to an equation system of uncoupled equations (separation of variables)

\[- \frac{EI}{\rho A} \frac{W'''(x)}{W(x)} = -\omega^2 \quad \Rightarrow \quad EI W'''(x) - \rho A\omega^2 W(x) = 0\]

\[\frac{\partial^2 T(t)}{T(t)} = -\omega^2 \quad \Rightarrow \quad \partial^2 T(t) + \omega^2 T(t) = 0\]

These equations have general solutions (see the exercises) of the form

\[W(x) = A \sin kx + B \cos kx + C \sinh kx + D \cosh kx, \quad k = \left( \frac{\rho A\omega^2}{EI} \right)^{1/4}\]

\[T(t) = E \sin \omega t + F \cos \omega t\]

where (integration) constants \(A, B, C, D\) and \(E, F\) are determined according to the boundary and initial conditions, respectively.
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For a simply supported beam, boundary conditions lead to an equation system which results in a determinant equation (for the existence of a solution), called the \textit{frequency equation}, or \textit{characteristic equation}:

\[
\sin kL \sinh kL = 0 \implies kL = n\pi, \quad n = 1, 2, \ldots
\]

leading to \textit{natural angular frequencies and frequencies}

\[
\omega_n = \left( \frac{n\pi}{L} \right)^2 \left( \frac{EI}{\rho A} \right)^{1/2}, \quad f_n = \frac{\omega_n}{2\pi}, \quad n = 1, 2, \ldots
\]

as well as \textit{eigensolutions} and \textit{time-harmonic solution}

\[
W_n(x) = \sin \frac{n\pi x}{L}
\]

\[
w(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left( E_n \sin \omega_n t + F_n \cos \omega_n t \right)
\]

where constant \( E_n, F_n \) are determined by the initial conditions.
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For Kirchhoff plates, the procedure is analogous. For a simply supported rectangular plate, the solution of the spatial equation is searched for in the form

\[ W(x, y) = \sum_{m,n=1}^{\infty} C_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \]

which is inserted to the spatial equation

\[ D \Delta^2 W(x, y) - \omega^2 \rho t W(x, y) = 0 \]

leading to a characteristic equation and then to natural frequencies

\[ \omega_{mn} = \pi^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \left( \frac{D}{\rho t} \right)^{1/2}, \quad m, n = 1, 2, \ldots \]

and solutions

\[ W_{mn}(x, y) = \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \]

\[ w(x, t) = \sum_{m,n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (E_n \sin \omega_n t + F_n \cos \omega_n t). \]
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**Rayleigh’s method.** If a system is conservative (no energy added or lost), the maximums of kinetic and potential (strain) energies are equal (*principle of conservation of energy*). The kinetic energy of a Kirchhoff plate is of the form

\[
K = \frac{1}{2} \int_A \rho t (\partial_t w(x, y, t))^2 dA.
\]

Assuming that the plate vibrates in its fundamental mode $W_1$ with the fundamental angular frequency $\omega_1$ leads to writing the deflection of the plate as

\[
w(x, t) = W_1(x) \sin \omega_1 t.
\]

The kinetic energy then takes the form

\[
K = \frac{\omega_1^2}{2} \cos^2 \omega t \int_A \rho t W_1(x, y)^2 dA, \quad K_{\text{max}} = \frac{\omega_1^2}{2} \int_A \rho t W_1(x, y)^2 dA.
\]

The potential energy of a Kirchhoff plate is of the form

\[
U = \frac{1}{2} \int_\Omega D \left( \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + \left( \frac{\partial^2 w}{\partial y^2} \right)^2 + 2\nu \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + 2(1 - \nu)(\frac{\partial^2 w}{\partial x \partial y})^2 \right) dA.
\]
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The maximum value of the potential energy takes the form

\[ U_{\text{max}} = \frac{1}{2} \int_{\Omega} D \left( \frac{\partial^2 W_1}{\partial x^2} \right)^2 + \left( \frac{\partial^2 W_1}{\partial y^2} \right)^2 + 2\nu \frac{\partial^2 W_1}{\partial x^2} \frac{\partial^2 W_1}{\partial y^2} + 2(1 - \nu) \left( \frac{\partial^2 W_1}{\partial x \partial y} \right) \right) dA. \]

The conservation principle leads to the so-called Rayleigh’s quotient

\[ U_{\text{max}} = K_{\text{max}} \implies \omega_1^2 = \frac{2U_{\text{max}}}{\int_A \rho t W_1(x, y)^2 dA} \]

Any kinematically admissible approximation for \( W_1 \) gives an approximation for the lowest eigenfrequency \( \omega_1 \).

For Euler–Bernoulli beams, the same principle applies and the corresponding condition reads as

\[ \omega_1^2 = \frac{\int_L EI W_1''(x)^2 dx}{\int_L \rho A W_1(x)^2 dx}. \]
6.1 Vibrations of plates

**Numerical methods.** For problems including many structural members, numerical methods are practically the only possible way to obtain solutions for the problems of free vibrations or forced vibrations.

In numerical discretizations, the problem of free vibrations leads to a *generalized eigenvalue problem*:

\[(K - \omega^2 M) d = 0 \quad \Rightarrow \quad \omega_n = \ldots, \quad d_n = \ldots\]

where \(K\) and \(M\) denote the stiffness and mass matrix, whereas \(d\) includes the (displacement) degrees of freedom (such as nodal displacements).
6.2 Vibrations of membranes

**String vibration.** Let us first consider a homogeneous, perfectly flexible string with *linear mass density* $\mu$ (mass per length, whereas mass density is mass per volume), *length* $L$ and *tension* $T$. For small displacements in plane, the governing equation for the string problem of *free vibrations* is the one-dimensional *wave equation*

$$\frac{\partial^2 w(x,t)}{\partial t^2} - c^2 \frac{\partial^2 w(x,t)}{\partial x^2} = 0, \quad c = \sqrt{\frac{T}{\mu}}.$$ 

The solution of this equation can be written in different forms: *d’Alembert’s solution*

$$w(x,t) = w_R(x - ct) + w_L(x + ct)$$

$$w_R(x,t) = \frac{1}{2} \bar{w}(x) - \frac{1}{2c} \int_0^x \bar{v}(s) ds$$

$$w_L(x,t) = \frac{1}{2} \bar{w}(x) + \frac{1}{2c} \int_0^x \bar{v}(s) ds$$

represents two waves travelling in opposite directions (Right, Left) determined by initial (overlined) displacement $\bar{w}$ and velocity $\bar{v}$. 

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This solution can be written in the form

\[ w(x, t) = \frac{1}{2} \ddot{w}(x - ct) + \frac{1}{2} \ddot{w}(x + ct) + \frac{1}{2c} \int_{x-ct}^{x+t} \ddot{v}(s) ds. \]

Bernoulli’s solution superposes standing wave modes as products of spatial (space-harmonic) and temporal (time-harmonic) components:

\[ w(x, t) = \sum_{n=1}^{\infty} \sin \left( \frac{n\pi x}{L} \right) \left( A_n \sin \omega_n t + B_n \cos \omega_n t \right), \quad \omega_n = \frac{n\pi c}{L} = 2\pi f_n \]

with mode number \( n \), eigenmode \( \sin(n\pi x/L) \) and angular eigenfrequency \( \omega_n \), or eigenfrequency \( f_n \). The amplitude (strength, height) of each mode is determined by constants \( A_n \) and \( B_n \) given by the initial conditions (overline) for the displacement \( w \) and velocity \( v \):

\[ A_n = \frac{2}{\omega_n L} \int_0^L \ddot{v}(s) \sin \left( \frac{n\pi x}{L} \right) ds, \quad B_n = \frac{2}{L} \int_0^L \ddot{w}(s) \sin \left( \frac{n\pi x}{L} \right) ds. \]
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**Remark.** Eigenfrequencies are multiples of the fundamental (lowest) frequency, i.e., they are equally spaced in the *eigenspectrum*:

\[ f_n = n \frac{c}{2L} = nf_1. \]

**Remark.** For a *plucked spring*, the initial displacement is constant and \( A_n = 0 \).

**Remark.** For a *striked string* (*struck string*), excited at position \( x_0 \), the initial velocity is constant meaning that \( B_n = 0 \). The vibration of a striked string follows the solution

\[
 w(x,t) = \sum_{n=1}^{\infty} \sin \left( \frac{n \pi x}{L} \right) A_n \sin \left( \frac{n \pi c t}{L} \right) \\
 A_n = \frac{2}{n \pi c \rho} \sin \left( \frac{n \pi x_0}{L} \right) \xrightarrow{n \to \infty} 0
\]

showing that the motion is a sum of eigenmodes vibrating with the corresponding frequencies but the amplitudes are decreasing along the mode number as \( 1/n \). If the strike point is \( x_0 = L/2 \), every even mode in the sum is identically zero, whereas if \( x_0 = L/m \), every \( m \)th mode vanishes (used in pianos and in playing string instruments).
6.2 Vibrations of membranes

The problem of *forced vibration* of a string excited by a time-dependent force $f = f(x,t)$ is governed by equation

$$\frac{\partial^2 w(x,t)}{\partial t^2} - c^2 \frac{\partial^2 w(x,t)}{\partial x^2} = f(x,t), \quad c = \sqrt{\frac{T}{\mu}}.$$  

**Remark.** Excitation at eigenfrequency leads to an infinite amplitude, this phenomenon is called *resonance* (to be avoided in structural design).

Due to *dissipative losses* in a non-ideal vibrating string (viscosity of air, viscoelasticity of the string, energy transfer through the end point fixings), a *damping* term should be added in the governing equation when modelling non-ideal, i.e., real, strings:

$$\frac{\partial^2 w(x,t)}{\partial t^2} + C \frac{\partial w(x,t)}{\partial t} - c^2 \frac{\partial^2 w(x,t)}{\partial x^2} = f(x,t), \quad c = \sqrt{\frac{T}{\mu}}$$

where it is assumed (according to experimental evidence) that damping is proportional to velocity with *damping coefficient* $C$. 
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Since a real string cannot be perfectly flexible but has always a bending stiffness, a bending term must be added into the governing equation:

\[
\frac{\partial^2 w(x,t)}{\partial t^2} + C \frac{\partial w(x,t)}{\partial t} - c^2 \frac{\partial^2 w(x,t)}{\partial x^2} + \frac{EAR^2}{\mu} \frac{\partial^4 w(x,t)}{\partial x^4} = f(x,t), \quad c = \sqrt{\frac{T}{\mu}}
\]

with \textit{radius of gyration} \( R^2 = I/A \). This equation resembles the governing equation of the vibration of beam bending

\[
\rho A \partial_t^2 w(x,t) + (EI w'')''(x,t) = f(x,t), \quad \mu = m/L = \rho V / L = \rho A
\]

which we could have augmented by the tension and damping terms (even with nonconstant stiffness \( EI \) and nonconstant tension \( T \)):

\[
\rho A \partial_t^2 w(x,t) + C \partial_t w(x,t) - (T w')'(x,t) + (EI w'')''(x,t) = f(x,t).
\]

\textbf{Remark.} A tensioned beam structure (or a cable) is another practical example following this theory.
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For a real string, taking into account the bending stiffness (think of a thick wounded piano or bass guitar string) implies that wave propagation is frequency-dependent and overtones are not any more exact multiples of the base tone, i.e., eigenfrequencies are not any more uniformly distributed in the spectrum (loss of harmonicity):

\[ f_n = n f_1 \sqrt{1 + B n^2}, \quad B = \frac{E I \pi^2}{L^2 T}. \]
6.2 Vibrations of membranes

**Membrane vibration.** Let us consider a homogeneous, perfectly flexible membrane with *mass per unit area* $\mu$, length $L$ and tension $T$. For small displacements in plane, the governing equation for the membrane problem of *free vibrations* is the two-dimensional *wave equation*

$$\frac{\partial^2 w(x, y, t)}{\partial t^2} - c^2 \left( \frac{\partial^2 w(x, y, t)}{\partial x^2} + \frac{\partial^2 w(x, y, t)}{\partial y^2} \right) = 0$$

$c = \sqrt{T / \mu}$.

**Bernoulli’s solution** for a rectangular membrane superposes standing wave modes as products of spatial (space-harmonic) and temporal (time-harmonic) components:

$$w(x, y, t) = \sum_{m,n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (A_{mn} \sin \omega_{mn} t + B_{mn} \cos \omega_{mn} t)$$

where the spatial component of the separative product $w(x,y,t) = W(x,y) T(t)$ satisfies the so-called *Helmholtz equation*:

$$\Delta W(x, y) + k^2 W(x, y) = 0.$$
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The characteristic equation leads to eigenfrequencies corresponding to each eigenmode $W_{mn}$:

$$\omega_{mn} = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}, \quad \omega_{mn} = \frac{f_{mn}}{2\pi}$$

**Remark.** Eigenfrequencies of the membrane are not harmonically distributed, which makes drums and other instruments relying on membrane (or plate) vibration sound specific (and somewhat inharmonic). The so-called *Chladni figures* illustrate the shapes of eigenmodes distinguishing still *nodal lines* and vibrating areas.
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For circular membranes, the spatial eigenequation reads as

\[ W''(r) + \frac{1}{r} W'(r) + k^2 W(r) = 0 \]

which can be written \((s = kr)\) as a **Bessel’s equation**

\[ W''(s) + \frac{1}{s} W'(s) + W(s) = 0 \]

with solutions given in terms of **Bessel functions** (omitted).