Midterm 1 – Mon 12 Feb 2017, 13.00–16.00

Write down on each answer sheet:

- Your name, degree programme, student number, and the date
- The course title: CS-E4530 Computational Complexity Theory

1. (a) Define what it means for a language $L \subseteq \{0,1\}^*$ to be *decidable*. Prove that if languages $L_1, L_2 \subseteq \{0,1\}^*$ are decidable, then the language

$$L_1 \cap L_2 = \{x \in \{0,1\}^* : \ x \in L_1 \text{ and } x \in L_2 \ \}$$

is also decidable.

(b) Define what it means for a language $L \subseteq \{0,1\}^*$ to be NP-*complete*. Prove that if $L \subseteq \{0,1\}^*$ is an NP-complete language and there is a polynomial-time Turing machine that decides L, then $\mathsf{P} = \mathsf{NP}$.

(*You may use results about Turing machines and reductions that were proved during the lectures.*)

2. The *set packing problem* is defined as follows:

- **Instance:** A family $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ of subsets of a finite set U, an integer k.
- **Question:** Is there a subfamily $\mathcal{T} \subseteq \mathcal{S}$ such that $\mathcal{T}$ contains at least k distinct sets from $\mathcal{S}$, and for all $T_1, T_2 \in \mathcal{T}$, we have $T_1 \cap T_2 = \emptyset$?

(a) Prove that the set packing problem is in NP; specify what are certificates for the set packing problem instances, prove that only yes-instances have valid certificates of polynomial size and prove that certificates can be verified in polynomial time.

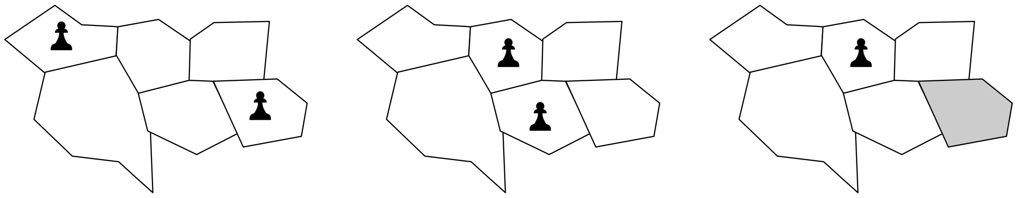
(b) Prove that the set packing problem is NP-hard via reduction from maximum independent set; prove that the reduction you give is a valid reduction, and show that it can be computed in polynomial time.

(*Hint: consider sets* $S_v = \{v\} \cup \{u \in V : \{u,v\} \in E\}$.)

3. Consider the following problem setting (if you want, you may image it as e.g. a military logistics problem, or as an artificial intelligence problem for a computer game): A king controls an area of land, divided into *provinces*, and needs to maintain a number of *armies* to protect against attacks from neighbouring kingdoms and barbarians. Any two provinces may be adjacent (i.e., sharing a border), or non-adjacent.

To ensure protection against possible attacks, each province either must have an army in that province or an army in at least one adjacent province. However, generals leading the armies are known to be scheming, and may try to overthrow the king if they are able to join forces; thus, no armies may be placed in two adjacent provinces. The task is to find a placement of armies into the provinces so that the above constraints are satisfied; to minimise the kingdom's military expenditure, the number of armies should also be as small as possible.

(*Continued on the next page.*)



(a) (b) (c)

The figure above shows an example of (a) a valid placement of armies, (b) an invalid placement of armies (two armies in adjacent provinces) and (c) an invalid placement of armies (an undefended province).

(a) Formalise the setting described above as a graph decision problem; specify a set of valid instances and the associated yes/no question.

(b) Determine if the resulting decision problem is in P or is NP-hard. Prove your claim. (*You may use any results from the lectures and the appendix below about problems being in* P *or* NP*-hard.*)

Grading: Each problem 8pt, total 24pt.
Passing grade: 12pt.

Appendix: some NP-complete problems

CNF-SAT
Instance: A CNF formula $\varphi = \bigwedge_{i=1}^{m} \bigvee_{j=1}^{k_i} \ell_{i,j}$, where $\ell_{i,j}$ is either x or $\neg x$ for some variable x.
Question: Is φ satisfiable?

***k*-SAT** ($k \geq 3$)
Instance: A CNF formula φ with at most k literals per clause.
Question: Is φ satisfiable?

Chromatic Number
Instance: Graph $G = (V, E)$, an integer $k \geq 1$.
Question: Is there a function $c \colon V \to \{1, 2, \dots, k\}$ such that $c(u) \neq c(v)$ for all $\{u, v\} \in E$?

Maximum Independent Set
Instance: Graph $G = (V, E)$, an integer $k \geq 1$.
Question: Is there a set of vertices I such that $|I| \geq k$ and for all $u, v \in I$, we have $\{u, v\} \notin E$?

Maximum Clique
Instance: Graph $G = (V, E)$, an integer $k \geq 1$.
Question: Is there a set of vertices C such that $|C| \geq k$ and for all $u, v \in C$, we have $\{u, v\} \in E$?

Minimum Vertex Cover
Instance: Graph $G = (V, E)$, an integer $k \geq 1$.
Question: Is there a set of vertices C such that $|C| \leq k$ and for all $\{u, v\} \in E$, either $v \in C$ or $u \in C$ (or both)?

Minimum Dominating Set
Instance: Graph $G = (V, E)$, an integer $k \geq 1$.
Question: Is there a set of vertices D such that $|D| \leq k$ and for all $v \in V$, either $v \in D$ or at least one of the neighbours of v is in D?

Hamiltonian Path
Instance: A graph $G = (V, E)$, vertices $s, t \in V$.
Question: Does there exist a path from s to t that visits each vertex exactly once?
(*NP-complete for directed and undirected graphs.*)

Hamiltonian Cycle
Instance: A graph $G = (V, E)$.
Question: Is there a cycle that visits each vertex exactly once?
(*NP-complete for directed and undirected graphs.*)

Travelling Salesman Problem
Instance: A graph $G = (V, E)$ with edge weights, integer W.
Question: Is there a cycle that visits vertices exactly once with weight at most W?
(*NP-complete for directed and undirected graphs.*)

Set Cover
Instance: A finite set U, a family $\mathcal{S} = \{S_1, S_2, \dots, S_m\}$ of subsets of U, an integer k.
Question: Is there a subfamily $\mathcal{T} \subseteq \mathcal{S}$ such that $\mathcal{T}$ contains at most k sets from $\mathcal{S}$, and any element $u \in U$ is contained in at least one set $T \in \mathcal{T}$?