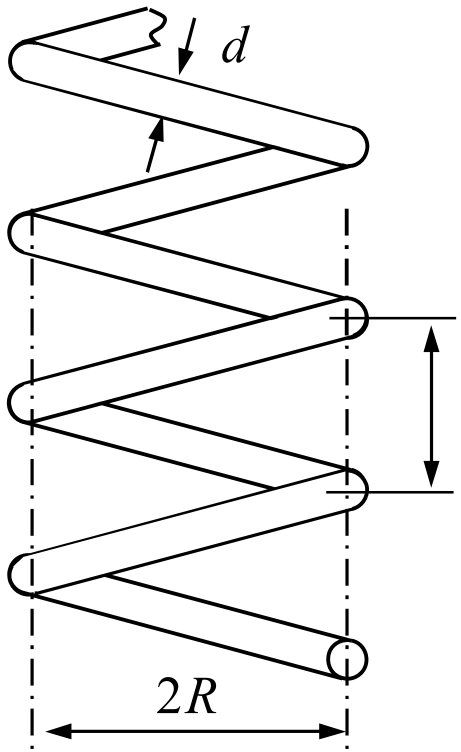
MEC-E8003 BEAM, PLATE AND SHELL MODELS (P)

Spring-2019

Week 9/1

BEAM MODEL



Beam is a thin body in two directions. Bar, torsion bar, and bending beam are used to refer to the loading modes of beams in straight geometry. In curved geometry, string denotes the loading mode of vanishing bending moments.

PLATE MODEL

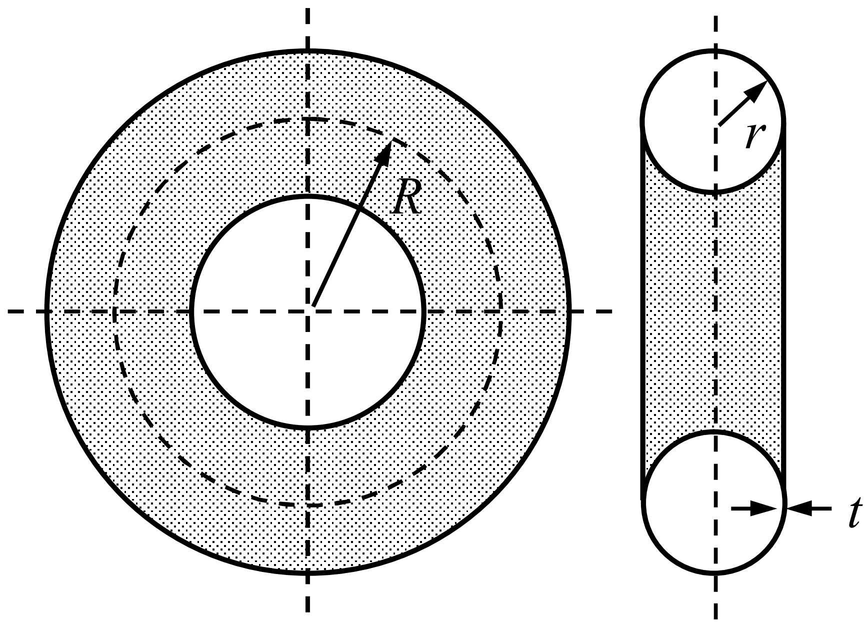


Plate is a thin body in one direction. Thin-slab and bending plate refer to the loading modes of the mode. In curved geometry, membrane denotes the mode of vanishing bending moments.

BEAM, PLATE AND SHELL MODELS 5cr (P)

- Beam and plate kinematics in flat and curved geometries.
- Beam and plate equilibrium equations, constitutive equations and boundary conditions in flat and curved geometries in tensor forms.
- Component forms of the beam and plate equations in orthonormal Cartesian and curvilinear coordinate systems (cylindrical shell equations etc.)
- Example solutions in flat and curved geometries.

BALANCE LAWS

Balance of mass (def. of a body or a material volume) Mass of a body is constant

Balance of linear momentum (Newton 2) The rate of change of linear momentum within a material volume equals the external force resultant acting on the material volume. ←

Balance of angular momentum (Cor. of Newton 2) The rate of change of angular momentum within a material volume equals the external moment resultant acting on the material volume. ←

Balance of energy (Thermodynamics 1)

Entropy growth (Thermodynamics 2)

PRINCIPLE OF VIRTUAL WORK

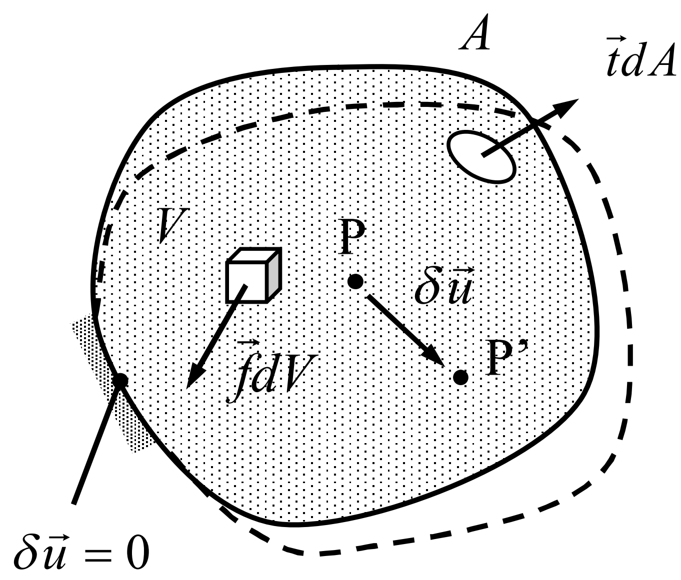
Principle of virtual work $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = 0 \quad \forall \delta \vec{u} \in U$ is just one representation of the balance laws of continuum mechanics. It is important due to its wide applicability and physical meanings of the terms.

virtual work density

$$\delta W^{\text{int}} = \int_V \delta w_V^{\text{int}} dV = -\int_V (\delta \vec{\vec{\varepsilon}}_{\text{c}} : \vec{\vec{\sigma}}) dV$$

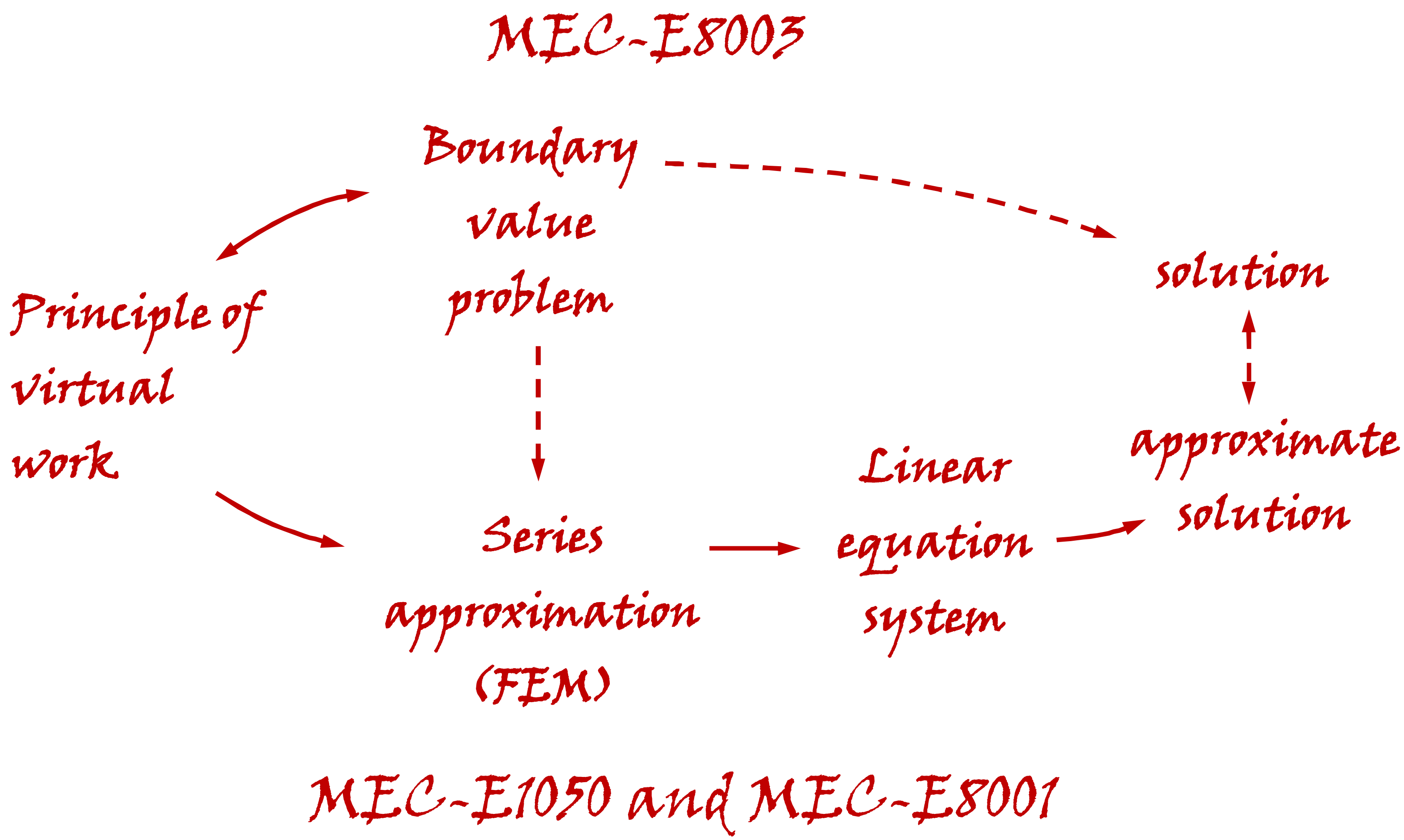
$$\delta W_V^{\text{ext}} = \int_V \delta w_V^{\text{ext}} dV = \int_V (\vec{f} \cdot \delta \vec{u}) dV$$

$$\delta W_A^{\text{ext}} = \int_A \delta w_A^{\text{ext}} dA = \int_A (\vec{t} \cdot \delta \vec{u}) dA$$



The details of the expressions vary case by case, but the principle itself does not!

THE PLAN


MEC-E8003
Boundary value problem
Principle of virtual work
solution
approximate solution
Series approximation (FEM)
Linear equation system
MEC-E1050 and MEC-E8001

1 KINEMATICS

LEARNING OUTCOMES

Students are able to solve the weekly lecture problems, home problems, and exercise problems on the topics of week 9:

- ☐ Vector and tensor quantities in mechanics. Representations in orthonormal curvilinear basis. Tensor products and expressions.

- ☐ Material coordinate system. Vectors, basis vector derivatives, and gradient operator in the polar, cylindrical, and spherical material coordinate systems.

- ☐ Basis vectors, basis vector derivatives, and gradient operator in the beam and shell material coordinate systems.

- ☐ Curvature of curves and surfaces.

1.1 QUANTITIES OF MECHANICS

The quantities in mechanics can be classified into scalars a, vectors $\vec{a}$ and multi-vectors or $\overleftrightarrow{a}$ which are vectors of vectors called also as tensors. Rank 0, 1, 2, 4 tensors are common in mechanics

Vector $$\vec{a} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix} = \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k} \quad \text{(rank 1 tensor)}$$

Tensor $$\overleftrightarrow{a} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{yx} & a_{yy} & a_{yz} \\ a_{zx} & a_{zy} & a_{zz} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + \ldots + a_{zz}\vec{k}\vec{k} \quad \text{(rank 2 tensor)}$$

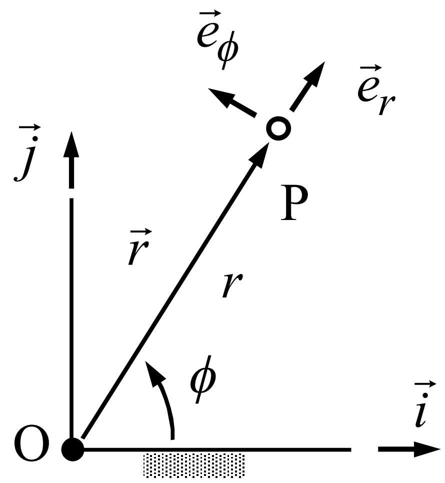
COORDINATE SYSTEM INVARIANCE

Physical tensor quantities are invariant with respect to coordinate system although components and basis vector are not. Representation change from one coordinate system to another requires the relationship between the basis vectors.

$$\vec{a} = \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} a_{xx} & a_{xy} \\ a_{yx} & a_{yy} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} \Leftrightarrow \text{(Cartesian system representation)}$$

$$\vec{a} = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} a_{xx} & a_{xy} \\ a_{yx} & a_{yy} \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} \Leftrightarrow$$

$$\vec{a} = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} a_{rr} & a_{r\phi} \\ a_{\phi r} & a_{\phi\phi} \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}. \quad \text{(Polar system representation)}$$



TENSOR PRODUCTS

In manipulation of expression containing tensors, it is important to remember that tensor ($\otimes$), cross ($\times$), inner ($\cdot$) products are non-commutative (order matters). For simplicity of presentation, outer (tensor) products like $\vec{a}\otimes\vec{b}$ are denoted by $\vec{a}\vec{b}$ in MEC-E8003. Otherwise, the usual rules of vector algebra apply:

$$\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y + a_z b_z,$$

$$\vec{a}\times\vec{b} = (a_y b_z - a_z b_y)\vec{i} + (a_z b_x - a_x b_z)\vec{j} + (a_x b_y - a_y b_x)\vec{k},$$

$$\vec{a}\vec{b} = a_x b_x \vec{i}\vec{i} + a_x b_y \vec{i}\vec{j} + a_x b_z \vec{i}\vec{k} + a_y b_x \vec{j}\vec{i} + a_y b_y \vec{j}\vec{j} + a_y b_z \vec{j}\vec{k} + a_z b_x \vec{k}\vec{i} + a_z b_y \vec{k}\vec{j} + a_z b_z \vec{k}\vec{k}.$$

Calculation with tensors is straightforward although the number of terms may make manipulations somewhat tedious.

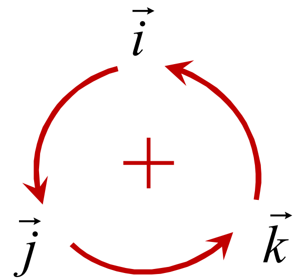
- As an example, manipulations needed to find the cross-product of two vectors in a Cartesian system (orthonormal and right-handed) consists of steps

$$\vec{a} \times \vec{b} = (a_x \vec{i} + a_y \vec{j} + a_z \vec{k}) \times (b_x \vec{i} + b_y \vec{j} + b_z \vec{k}) \quad \Leftrightarrow$$

$$\vec{a} \times \vec{b} = a_x b_x \vec{i} \times \vec{i} + a_x b_y \vec{i} \times \vec{j} + a_x b_z \vec{i} \times \vec{k} +$$

$$a_y b_x \vec{j} \times \vec{i} + a_y b_y \vec{j} \times \vec{j} + a_y b_z \vec{j} \times \vec{k} +$$

$$a_z b_x \vec{k} \times \vec{i} + a_z b_y \vec{k} \times \vec{j} + a_z b_z \vec{k} \times \vec{k} \quad \Rightarrow$$



$$\vec{a} \times \vec{b} = 0 + a_x b_y \vec{k} - a_x b_z \vec{j} - a_y b_x \vec{k} + 0 + a_y b_z \vec{i} + a_z b_x \vec{j} - a_z b_y \vec{i} + 0 \quad \Leftrightarrow$$

$$\vec{a} \times \vec{b} = (a_y b_z - a_z b_y)\vec{i} + (a_z b_x - a_x b_z)\vec{j} + (a_x b_y - a_y b_x)\vec{k}. \quad \Leftarrow$$

- The manipulations are often (but not always) easier when the components and basis vectors are arranged as matrices

$$\vec{a}=\begin{Bmatrix}a_x\\a_y\\a_z\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\vec{i}\\\vec{j}\\\vec{k}\end{Bmatrix}=\begin{Bmatrix}\vec{i}\\\vec{j}\\\vec{k}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}a_x\\a_y\\a_z\end{Bmatrix} \text{ and } \vec{b}=\begin{Bmatrix}\vec{i}\\\vec{j}\\\vec{k}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}b_x\\b_y\\b_z\end{Bmatrix}=\begin{Bmatrix}b_x\\b_y\\b_z\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\vec{i}\\\vec{j}\\\vec{k}\end{Bmatrix} \quad \Rightarrow$$

$$\vec{a}\times\vec{b}=\begin{Bmatrix}a_x\\a_y\\a_z\end{Bmatrix}^{\mathrm{T}}\left(\begin{Bmatrix}\vec{i}\\\vec{j}\\\vec{k}\end{Bmatrix}\times\begin{Bmatrix}\vec{i}\\\vec{j}\\\vec{k}\end{Bmatrix}^{\mathrm{T}}\right)\begin{Bmatrix}b_x\\b_y\\b_z\end{Bmatrix}=\begin{Bmatrix}a_x\\a_y\\a_z\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}0 & \vec{k} & -\vec{j}\\-\vec{k} & 0 & \vec{i}\\\vec{j} & -\vec{i} & 0\end{bmatrix}\begin{Bmatrix}b_x\\b_y\\b_z\end{Bmatrix} \Leftrightarrow$$

$$\vec{a}\times\vec{b}=(a_y b_z-a_z b_y)\vec{i}+(a_z b_x-a_x b_z)\vec{j}+(a_x b_y-a_y b_x)\vec{k}. \quad \leftarrow$$

EXAMPLE. The local forms of the balance laws of momentum and moment of momentum are $\nabla\cdot\vec{\vec{\sigma}}+\vec{f}=0$ and $\vec{\vec{\sigma}}=\vec{\vec{\sigma}}_{\mathrm{c}}$ (conjugate tensor). Assuming a planar case and a Cartesian coordinate system so that

$$\nabla=\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\partial/\partial x\\ \partial/\partial y\end{Bmatrix},\quad \vec{f}=\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}f_x\\ f_y\end{Bmatrix},\ \text{and}\ \vec{\vec{\sigma}}=\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\sigma_{xx} & \sigma_{xy}\\ \sigma_{yx} & \sigma_{yy}\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}$$

derive the component forms of the balance laws.

Answer $\dfrac{\partial\sigma_{xx}}{\partial x}+\dfrac{\partial\sigma_{yx}}{\partial y}+f_x=0,\quad \dfrac{\partial\sigma_{xy}}{\partial x}+\dfrac{\partial\sigma_{yy}}{\partial y}+f_y=0$ and $\sigma_{xy}=\sigma_{yx}$

- In a Cartesian system, basis vectors are constants and one may transpose the gradient operator to get (transposing cannot be used with non-constant basis vectors)

$$\nabla\cdot\vec{\sigma}+\vec{f}=\begin{Bmatrix}\partial/\partial x\\ \partial/\partial y\end{Bmatrix}^{\mathrm{T}}(\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}\cdot\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}^{\mathrm{T}})\begin{bmatrix}\sigma_{xx} & \sigma_{xy}\\ \sigma_{yx} & \sigma_{yy}\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}+\begin{Bmatrix}f_x\\ f_y\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}=0$$

$$\nabla\cdot\vec{\sigma}+\vec{f}=(\begin{Bmatrix}\partial/\partial x\\ \partial/\partial y\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\sigma_{xx} & \sigma_{xy}\\ \sigma_{yx} & \sigma_{yy}\end{bmatrix}+\begin{Bmatrix}f_x\\ f_y\end{Bmatrix}^{\mathrm{T}})\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}=0. \quad \Leftarrow$$

$$\vec{\vec{\sigma}}-\vec{\vec{\sigma}}_{\mathrm{c}}=\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\sigma_{xx} & \sigma_{xy}\\ \sigma_{yx} & \sigma_{yy}\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}-\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\sigma_{xx} & \sigma_{xy}\\ \sigma_{yx} & \sigma_{yy}\end{bmatrix}^{\mathrm{T}}\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}=0 \quad \Leftrightarrow$$

$$\vec{\vec{\sigma}}-\vec{\vec{\sigma}}_{\mathrm{c}}=\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}0 & \sigma_{xy}-\sigma_{yx}\\ \sigma_{yx}-\sigma_{xy} & 0\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}=0. \quad \Leftarrow$$

SOME DEFINITIONS AND IDENTITIES

Conjugate tensor $\overleftrightarrow{a}_{\mathrm{c}}$: $\overleftrightarrow{a}\cdot\vec{b}=\vec{b}\cdot\overleftrightarrow{a}_{\mathrm{c}} \quad \forall\vec{b}$

Second order identity tensor $\overleftrightarrow{I}$: $\overleftrightarrow{I}\cdot\vec{a}=\vec{a}\cdot\overleftrightarrow{I}=\vec{a} \quad \forall\vec{a}$

Fourth order identity tensor $\overset{\leftrightarrow}{\overleftrightarrow{I}}$: $\overset{\leftrightarrow}{\overleftrightarrow{I}}:\overleftrightarrow{a}=\overleftrightarrow{a}:\overset{\leftrightarrow}{\overleftrightarrow{I}}=\overleftrightarrow{a} \quad \forall\overleftrightarrow{a}$

Associated vector $\vec{a}$ **of an antisymmetric tensor** $\overleftrightarrow{a}$: $\vec{b}\cdot\overleftrightarrow{a}=\vec{a}\times\vec{b}$, when $\overleftrightarrow{a}=-\overleftrightarrow{a}_{\mathrm{c}}$

Scalar triple product $\vec{a}\cdot(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\cdot\vec{c}$

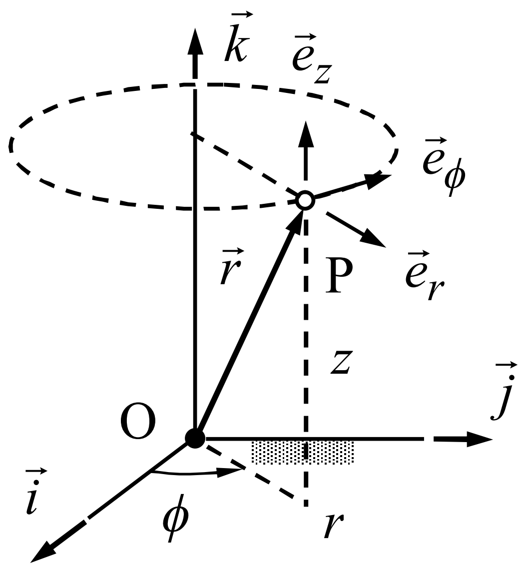
Vector triple product $\vec{a}\times(\vec{b}\times\vec{c})=\vec{b}(\vec{a}\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b})$

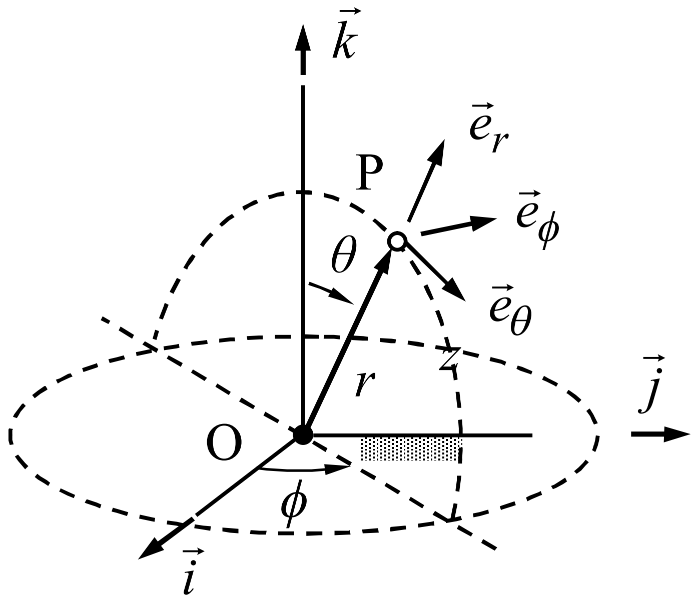
Symmetric-antisymmetric double product $\overleftrightarrow{a}=-\overleftrightarrow{a}_{\mathrm{c}}$ ja $\overleftrightarrow{b}=\overleftrightarrow{b}_{\mathrm{c}} \quad \Rightarrow \quad \overleftrightarrow{a}:\overleftrightarrow{b}=0$

Symmetric-antisymmetric division $\overleftrightarrow{a}=\overleftrightarrow{a}_s+\overleftrightarrow{a}_u=\frac{1}{2}(\overleftrightarrow{a}+\overleftrightarrow{a}_{\mathrm{c}})+\frac{1}{2}(\overleftrightarrow{a}-\overleftrightarrow{a}_{\mathrm{c}})$

1.2 COORDINATE SYSTEMS

In solid mechanics, particles of a body (a closed system of particles) are identified by coordinates of the initial geometry. Equilibrium equations etc. can be written for any selection of the material coordinates, but a clever selection may simplify the setting.





A Cartesian (x, y, z) coordinate system with known derivatives of the basis vector, gradient operator etc. is always needed as a reference system.

BASIS VECTORS

In a Cartesian coordinate system, the position vector of a particle (x, y, z) is given by $\vec{r}(x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$, basis vectors by $\vec{e}_x = \partial\vec{r} / \partial x$, $\vec{e}_y = \partial\vec{r} / \partial x$, and $\vec{e}_z = \partial\vec{r} / \partial z$. In addition, the derivatives of the basis vectors vanish. In a curvilinear system, when a particle is identified by (α, β, γ)

Basis vectors: $\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix} = \begin{Bmatrix} (\partial\vec{r} / \partial\alpha) / |\partial\vec{r} / \partial\alpha| \\ (\partial\vec{r} / \partial\beta) / |\partial\vec{r} / \partial\beta| \\ (\partial\vec{r} / \partial\gamma) / |\partial\vec{r} / \partial\gamma| \end{Bmatrix} = [F] \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = [F]^{-1} \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}.$

Basis vector derivatives: $\dfrac{\partial}{\partial\eta} \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix} = (\dfrac{\partial}{\partial\eta}[F])[F]^{-1} \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix} \quad \eta \in \{\alpha, \beta, \gamma\}$

- The starting point is the position vector of a material point in the reference system $\vec{r}(\alpha,\beta,\gamma) = x(\alpha,\beta,\gamma)\vec{i} + y(\alpha,\beta,\gamma)\vec{j} + z(\alpha,\beta,\gamma)\vec{k}$ expressed in terms of (α,β,γ) identifying the particles. Basis vectors of the curvilinear $(\alpha,\beta,\gamma)-$ system

$$\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix} = \begin{Bmatrix} (\partial\vec{r} / \partial\alpha)/ |\partial\vec{r} / \partial\alpha| \\ (\partial\vec{r} / \partial\beta)/ |\partial\vec{r} / \partial\beta| \\ (\partial\vec{r} / \partial\gamma)/ |\partial\vec{r} / \partial\gamma| \end{Bmatrix} = [F]\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \quad \Leftrightarrow \quad \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = [F]^{-1}\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}.$$

- Calculating the derivative on both sides with respect to $\eta \in \{\alpha,\beta,\gamma\}$ and retaining the basis of the curvilinear system gives

$$\frac{\partial}{\partial\eta}\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix} = \frac{\partial}{\partial\eta}[F]\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = (\frac{\partial}{\partial\eta}[F])[F]^{-1}\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}. \quad \Leftarrow$$

GRADIENT OPERATOR

As a vector gradient in invariant with respect to the coordinate system. Change of the basis and the quantities used for particle identification affects, however, the representation

$$(x,y,z)\text{:}\quad \nabla = \begin{Bmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \partial / \partial x \\ \partial / \partial y \\ \partial / \partial z \end{Bmatrix} = \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z} \;,$$

$$(\alpha,\beta,\gamma)\text{:}\quad \nabla = \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}^{\mathrm{T}} [F]^{-\mathrm{T}} [H]^{-1} \begin{Bmatrix} \partial / \partial \alpha \\ \partial / \partial \beta \\ \partial / \partial \gamma \end{Bmatrix} \text{ where } [H] = \begin{bmatrix} \partial x / \partial \alpha & \partial y / \partial \alpha & \partial z / \partial \alpha \\ \partial x / \partial \beta & \partial y / \partial \beta & \partial z / \partial \beta \\ \partial x / \partial \gamma & \partial y / \partial \gamma & \partial z / \partial \gamma \end{bmatrix}.$$

Notice that $[F]$ and $[H]$ differ only in the scaling of the rows i.e. $[F] = [R]^{-1}[H]$!

- Using the chain rule, the relationships between coordinates and basis vectors and the (coordinate system) invariance of the gradient operator (it is a vector)

$$\begin{Bmatrix} \partial/\partial\alpha \\ \partial/\partial\beta \\ \partial/\partial\gamma \end{Bmatrix} = \begin{bmatrix} \partial x/\partial\alpha & \partial y/\partial\alpha & \partial z/\partial\alpha \\ \partial x/\partial\beta & \partial y/\partial\beta & \partial z/\partial\beta \\ \partial x/\partial\gamma & \partial y/\partial\gamma & \partial z/\partial\gamma \end{bmatrix} \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{Bmatrix} = [H] \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{Bmatrix} \Rightarrow$$

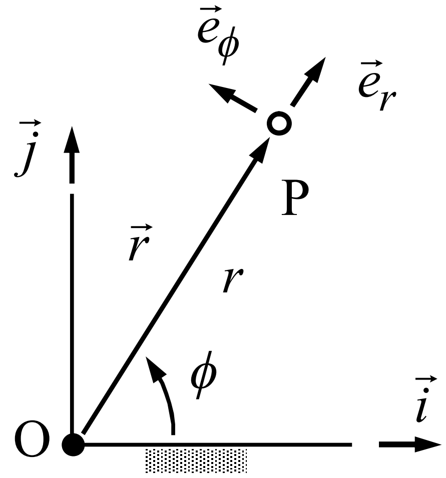
$$\nabla = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{Bmatrix} = \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}^{\mathrm{T}} [F]^{-\mathrm{T}} [H]^{-1} \begin{Bmatrix} \partial/\partial\alpha \\ \partial/\partial\beta \\ \partial/\partial\gamma \end{Bmatrix} \Leftrightarrow$$

$$\nabla = \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}^{\mathrm{T}} [F]^{-\mathrm{T}} [H]^{-1} \begin{Bmatrix} \partial/\partial\alpha \\ \partial/\partial\beta \\ \partial/\partial\gamma \end{Bmatrix} = \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}^{\mathrm{T}} ([H][F]^{\mathrm{T}})^{-1} \begin{Bmatrix} \partial/\partial\alpha \\ \partial/\partial\beta \\ \partial/\partial\gamma \end{Bmatrix}. \Leftarrow$$

POLAR COORDINATES (r,ϕ)

In a curvilinear rectangular Polar coordinate system, a particle is identified by its distance r from the origin and angle ϕ from a chosen line. Basis vectors, their derivatives, and the gradient operator are given by mapping $\vec{r}(r,\phi) = r\cos\phi\vec{i} + r\sin\phi\vec{j}$:

$$\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} = [F] \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix},$$



$$\frac{\partial}{\partial\phi} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \begin{Bmatrix} \vec{e}_\phi \\ -\vec{e}_r \end{Bmatrix} \quad \text{(otherwise zeros),}$$

$$\nabla = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \left([H][F]^{\mathrm{T}}\right)^{-1} \begin{Bmatrix} \partial/\partial r \\ \partial/\partial\phi \end{Bmatrix} = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial\phi}.$$

- The derivatives follow from the generic expression or in a more clear manner from steps (just to emphasize the idea)

$$\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} = [F] \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = [F]^{-1} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} \Rightarrow$$

$$\frac{\partial}{\partial\phi} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \left(\frac{\partial}{\partial\phi} \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}\right) \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} = \begin{bmatrix} -\sin\phi & \cos\phi \\ -\cos\phi & -\sin\phi \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} \quad \Rightarrow$$

$$\frac{\partial}{\partial\phi} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \begin{bmatrix} -\sin\phi & \cos\phi \\ -\cos\phi & -\sin\phi \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \begin{Bmatrix} \vec{e}_\phi \\ -\vec{e}_r \end{Bmatrix}. \quad \Leftarrow$$

- In writing the gradient expression, one needs the relationships between basis and partial derivatives in a Cartesian and polar coordinate systems:

$$\begin{Bmatrix} \partial/\partial r \\ \partial/\partial\phi \end{Bmatrix} = \begin{bmatrix} \partial x/\partial r & \partial y/\partial r \\ \partial x/\partial\phi & \partial y/\partial\phi \end{bmatrix} \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -r\sin\phi & r\cos\phi \end{bmatrix} \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \end{Bmatrix} = [H] \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \end{Bmatrix}.$$

Using the vector (operator) invariance with respect to the coordinate system used

$$\nabla = \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \end{Bmatrix} = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} ([H][F]^{\mathrm{T}})^{-1} \begin{Bmatrix} \partial/\partial r \\ \partial/\partial\phi \end{Bmatrix} \Rightarrow$$

$$\nabla = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} \cos\phi & \sin\phi \\ -r\sin\phi & r\cos\phi \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \right)^{-1} \begin{Bmatrix} \partial/\partial r \\ \partial/\partial\phi \end{Bmatrix} \Leftrightarrow$$

$$\nabla = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 0 \\ 0 & 1/r \end{bmatrix} \begin{Bmatrix} \partial/\partial r \\ \partial/\partial\phi \end{Bmatrix} = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r} \frac{\partial}{\partial\phi}. \quad \leftarrow$$

EXAMPLE. Derive the component forms of the balance law $\nabla\cdot\vec{\sigma}+\vec{f}=0$ in the polar coordinate system when stress and distributed force

$$\vec{\sigma}=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\sigma_{rr} & \sigma_{r\phi}\\ \sigma_{\phi r} & \sigma_{\phi\phi}\end{bmatrix}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix} \text{ and } \vec{f}=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}f_r\\ f_\phi\end{Bmatrix},$$

respectively. Derivatives of the basis vectors and the gradient operator of the polar coordinate system are

$$\frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}=\begin{Bmatrix}\vec{e}_\phi\\ -\vec{e}_r\end{Bmatrix} \text{ and } \nabla=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\partial/\partial r\\ \partial/(r\partial\phi)\end{Bmatrix}=\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{1}{r}\frac{\partial}{\partial\phi}.$$

Answer $\frac{1}{r}[\frac{\partial(r\sigma_{rr})}{\partial r}+\frac{\partial\sigma_{\phi r}}{\partial\phi}-\sigma_{\phi\phi}]+f_r=0$ and $\frac{1}{r}[\frac{\partial(r\sigma_{r\phi})}{\partial r}+\frac{\partial\sigma_{\phi\phi}}{\partial\phi}+\sigma_{\phi r}]+f_\phi=0$

- In polar coordinate system basis vectors depend on the angular coordinate. First, let us expand the stress divergence and consider the terms one-by-one by keeping the order of the basis vectors and position of the inner product:

$$\nabla\cdot\vec{\sigma} = (\vec{e}_r\frac{\partial}{\partial r} + \vec{e}_\phi\frac{1}{r}\frac{\partial}{\partial\phi})\cdot(\sigma_{rr}\vec{e}_r\vec{e}_r + \sigma_{r\phi}\vec{e}_r\vec{e}_\phi + \sigma_{\phi r}\vec{e}_\phi\vec{e}_r + \sigma_{\phi\phi}\vec{e}_\phi\vec{e}_\phi) \iff$$

$$\nabla\cdot\vec{\sigma} = \vec{e}_r\cdot\frac{\partial}{\partial r}\sigma_{rr}\vec{e}_r\vec{e}_r + \vec{e}_r\cdot\frac{\partial}{\partial r}\sigma_{r\phi}\vec{e}_r\vec{e}_\phi + \vec{e}_r\cdot\frac{\partial}{\partial r}\sigma_{\phi r}\vec{e}_\phi\vec{e}_r + \vec{e}_r\cdot\frac{\partial}{\partial r}\sigma_{\phi\phi}\vec{e}_\phi\vec{e}_\phi +$$

$$\vec{e}_\phi\cdot\frac{1}{r}\frac{\partial}{\partial\phi}\sigma_{rr}\vec{e}_r\vec{e}_r + \vec{e}_\phi\cdot\frac{1}{r}\frac{\partial}{\partial\phi}\sigma_{r\phi}\vec{e}_r\vec{e}_\phi + \vec{e}_\phi\cdot\frac{1}{r}\frac{\partial}{\partial\phi}\sigma_{\phi r}\vec{e}_\phi\vec{e}_r + \vec{e}_\phi\cdot\frac{1}{r}\frac{\partial}{\partial\phi}\sigma_{\phi\phi}\vec{e}_\phi\vec{e}_\phi \,.$$

- Let us consider the terms one-by-one by keeping the order of the basis vectors, position of the inner product, and taking into account the non-zero derivatives $\partial\vec{e}_r/\partial\phi = \vec{e}_\phi$ and $\partial\vec{e}_\phi/\partial\phi = -\vec{e}_r$,

$$\vec{e}_r \cdot \frac{\partial}{\partial r}\sigma_{rr}\vec{e}_r\vec{e}_r = \frac{\partial \sigma_{rr}}{\partial r}(\vec{e}_r \cdot \vec{e}_r)\vec{e}_r = \frac{\partial \sigma_{rr}}{\partial r}\vec{e}_r \ , \qquad \text{(basis vectors are orthonormal)}$$

$$\vec{e}_r \cdot \frac{\partial}{\partial r}\sigma_{r\phi}\vec{e}_r\vec{e}_\phi = \frac{\partial \sigma_{r\phi}}{\partial r}(\vec{e}_r \cdot \vec{e}_r)\vec{e}_\phi = \frac{\partial \sigma_{r\phi}}{\partial r}\vec{e}_\phi,$$

$$\vec{e}_r \cdot \frac{\partial}{\partial r}\sigma_{\phi r}\vec{e}_\phi\vec{e}_r = \frac{\partial \sigma_{\phi r}}{\partial r}(\vec{e}_r \cdot \vec{e}_\phi)\vec{e}_r = 0,$$

$$\vec{e}_r \cdot \frac{\partial}{\partial r}\sigma_{\phi\phi}\vec{e}_\phi\vec{e}_\phi = \frac{\partial \sigma_{\phi\phi}}{\partial r}(\vec{e}_r \cdot \vec{e}_\phi)\vec{e}_\phi = 0,$$

$$\vec{e}_\phi \cdot \frac{1}{r}\frac{\partial}{\partial \phi}\sigma_{rr}\vec{e}_r\vec{e}_r = \frac{1}{r}\frac{\partial \sigma_{rr}}{\partial \phi}(\vec{e}_\phi \cdot \vec{e}_r)\vec{e}_r + \frac{1}{r}\sigma_{rr}(\vec{e}_\phi \cdot \frac{\partial \vec{e}_r}{\partial \phi})\vec{e}_r + \frac{1}{r}\sigma_{rr}(\vec{e}_\phi \cdot \vec{e}_r)\frac{\partial \vec{e}_r}{\partial \phi} = \frac{1}{r}\sigma_{rr}\vec{e}_r,$$

$$\vec{e}_\phi \cdot \frac{1}{r}\frac{\partial}{\partial \phi}\sigma_{r\phi}\vec{e}_r\vec{e}_\phi = \frac{1}{r}\frac{\partial \sigma_{r\phi}}{\partial \phi}(\vec{e}_\phi \cdot \vec{e}_r)\vec{e}_\phi + \frac{1}{r}\sigma_{r\phi}(\vec{e}_\phi \cdot \frac{\partial \vec{e}_r}{\partial \phi})\vec{e}_\phi + \frac{1}{r}\sigma_{r\phi}(\vec{e}_\phi \cdot \vec{e}_r)\frac{\partial \vec{e}_\phi}{\partial \phi} = \frac{1}{r}\sigma_{r\phi}\vec{e}_\phi,$$

$$\vec{e}_\phi \cdot \frac{1}{r}\frac{\partial}{\partial \phi}\sigma_{\phi r}\vec{e}_\phi \vec{e}_r = \frac{1}{r}\frac{\partial \sigma_{\phi r}}{\partial \phi}(\vec{e}_\phi \cdot \vec{e}_\phi)\vec{e}_r + \frac{1}{r}\sigma_{\phi r}(\vec{e}_\phi \cdot \frac{\partial \vec{e}_\phi}{\partial \phi})\vec{e}_r + \frac{1}{r}\sigma_{\phi r}(\vec{e}_\phi \cdot \vec{e}_\phi)\frac{\partial \vec{e}_r}{\partial \phi}$$

$$= \frac{1}{r}\frac{\partial \sigma_{\phi r}}{\partial \phi}\vec{e}_r + \frac{1}{r}\sigma_{\phi r}\vec{e}_\phi,$$

$$\vec{e}_\phi \cdot \frac{\partial}{r\partial \phi}\sigma_{\phi\phi}\vec{e}_\phi \vec{e}_\phi = \frac{1}{r}\frac{\partial \sigma_{\phi\phi}}{\partial \phi}(\vec{e}_\phi \cdot \vec{e}_\phi)\vec{e}_\phi + \frac{1}{r}\sigma_{\phi\phi}(\vec{e}_\phi \cdot \frac{\partial \vec{e}_\phi}{\partial \phi})\vec{e}_\phi + \frac{1}{r}\sigma_{\phi\phi}(\vec{e}_\phi \cdot \vec{e}_\phi)\frac{\partial \vec{e}_\phi}{\partial \phi}$$

$$= \frac{1}{r}\frac{\partial \sigma_{\phi\phi}}{\partial \phi}\vec{e}_\phi - \frac{1}{r}\sigma_{\phi\phi}\vec{e}_r,$$

- Finally, by combining the terms

$$\nabla \cdot \vec{\sigma} = \frac{\partial \sigma_{rr}}{\partial r}\vec{e}_r + \frac{\partial \sigma_{r\phi}}{\partial r}\vec{e}_\phi + \frac{1}{r}\sigma_{rr}\vec{e}_r + \frac{1}{r}\sigma_{r\phi}\vec{e}_\phi + \frac{1}{r}\frac{\partial \sigma_{\phi r}}{\partial \phi}\vec{e}_r + \frac{1}{r}\sigma_{\phi r}\vec{e}_\phi + \frac{1}{r}\frac{\partial \sigma_{\phi\phi}}{\partial \phi}\vec{e}_\phi - \frac{1}{r}\sigma_{\phi\phi}\vec{e}_r$$

$$\nabla\cdot\vec{\sigma}=\frac{1}{r}\Big(r\frac{\partial\sigma_{rr}}{\partial r}+\sigma_{rr}+\frac{\partial\sigma_{\phi r}}{\partial\phi}-\sigma_{\phi\phi}\Big)\vec{e}_r+\frac{1}{r}\Big(r\frac{\partial\sigma_{r\phi}}{\partial r}+\sigma_{r\phi}+\sigma_{\phi r}+\frac{\partial\sigma_{\phi\phi}}{\partial\phi}\Big)\vec{e}_\phi \quad\Leftrightarrow$$

$$\nabla\cdot\vec{\sigma}=\frac{1}{r}\Big[\frac{\partial(r\sigma_{rr})}{\partial r}+\frac{\partial\sigma_{\phi r}}{\partial\phi}-\sigma_{\phi\phi}\Big]\vec{e}_r+\frac{1}{r}\Big[\frac{\partial(r\sigma_{r\phi})}{\partial r}+\frac{\partial\sigma_{\phi\phi}}{\partial\phi}+\sigma_{\phi r}\Big]\vec{e}_\phi .$$

- With the distributed force $\vec{f}=f_r\vec{e}_r+f_\phi\vec{e}_\phi$, the local form of the momentum balance $\nabla\cdot\vec{\sigma}+\vec{f}=0$ in the polar coordinate system

$$\frac{1}{r}\Big[\frac{\partial(r\sigma_{rr})}{\partial r}+\frac{\partial\sigma_{\phi r}}{\partial\phi}-\sigma_{\phi\phi}+rf_r\Big]\vec{e}_r+\frac{1}{r}\Big[\frac{\partial(r\sigma_{r\phi})}{\partial r}+\frac{\partial\sigma_{\phi\phi}}{\partial\phi}+\sigma_{\phi r}+rf_\phi\Big]\vec{e}_\phi=0. \quad \leftarrow$$

EXAMPLE. The small strain measure is obtained as the symmetric part of displacement gradient (a tensor then). Use the definition to find the components of the strain tensor (a) in Cartesian coordinate system and (b) in the polar system.

Answer (a) $\varepsilon_{xx} = \dfrac{\partial u_x}{\partial x}$, $\varepsilon_{yy} = \dfrac{\partial u_y}{\partial y}$, and $\varepsilon_{xy} = \varepsilon_{yx} = \dfrac{1}{2}(\dfrac{\partial u_y}{\partial x} + \dfrac{\partial u_x}{\partial y})$

(b) $\varepsilon_{rr} = \dfrac{\partial u_r}{\partial r}$, $\varepsilon_{\phi\phi} = \dfrac{1}{r}(\dfrac{\partial u_\phi}{\partial \phi} + u_r)$, and $\varepsilon_{r\phi} = \varepsilon_{\phi r} = \dfrac{1}{2}(\dfrac{1}{r}\dfrac{\partial u_r}{\partial \phi} - \dfrac{u_\phi}{r} + \dfrac{\partial u_\phi}{\partial r})$

- In Cartesian system, $\nabla = \vec{i}\partial/\partial x + \vec{j}\partial/\partial y$ and $\vec{u} = u_x\vec{i} + u_y\vec{j}$, therefore

$$\nabla\vec{u} = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y})(u_x\vec{i} + u_y\vec{j}) = \vec{i}\vec{i}\frac{\partial u_x}{\partial x} + \vec{i}\vec{j}\frac{\partial u_y}{\partial x} + \vec{j}\vec{i}\frac{\partial u_x}{\partial y} + \vec{j}\vec{j}\frac{\partial u_y}{\partial y}) \Rightarrow$$

$$(\nabla\vec{u})_{\mathrm{c}} = \vec{i}\vec{i}\frac{\partial u_x}{\partial x} + \vec{j}\vec{i}\frac{\partial u_y}{\partial x} + \vec{i}\vec{j}\frac{\partial u_x}{\partial y} + \vec{j}\vec{j}\frac{\partial u_y}{\partial y}$$

giving

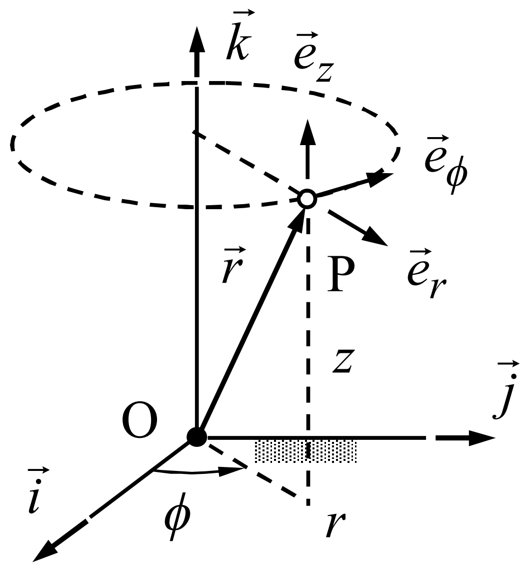
$$\vec{\varepsilon} = \frac{1}{2}[\nabla\vec{u} + (\nabla\vec{u})_{\mathrm{c}}] = \vec{i}\vec{i}\frac{\partial u_x}{\partial x} + \vec{j}\vec{j}\frac{\partial u_y}{\partial y} + \vec{i}\vec{j}\frac{1}{2}(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}) + \vec{j}\vec{i}\frac{1}{2}(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}).$$ ←

- In polar coordinates $\vec{u} = u_r\vec{e}_r + u_\phi\vec{e}_\phi$, and $\nabla = \vec{e}_r\partial/\partial r + \vec{e}_\phi\partial/(r\partial\phi)$, $\partial\vec{e}_r/\partial\phi = \vec{e}_\phi$ and $\partial\vec{e}_\phi/\partial\phi = -\vec{e}_r$. Otherwise calculation follows the steps used with the Cartesian coordinate system (one of the exercise problems).

CYLINDRICAL COORDINATES (r, ϕ, z)

A particle is identified by its distance r from the z-axis origin, angle ϕ from the x-axis and distance z from the xy-plane. Mapping $\vec{r} = r\cos\phi\vec{i} + r\sin\phi\vec{j} + z\vec{k}$ gives

$$\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix},$$



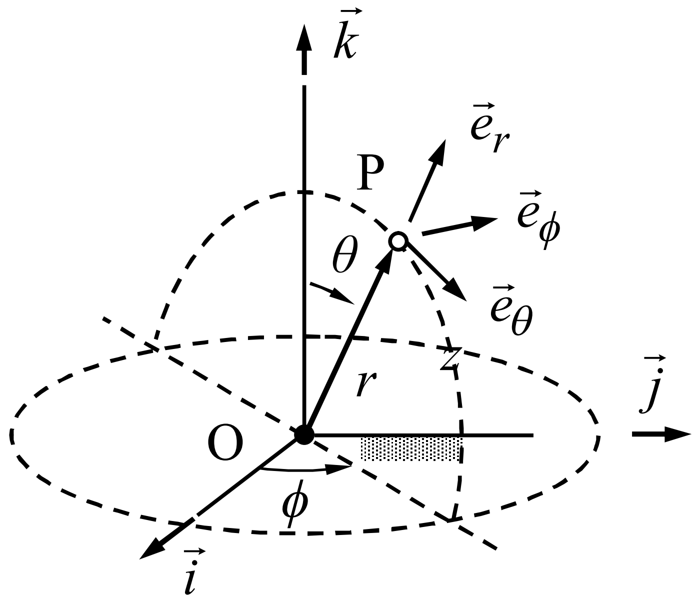
$$\frac{\partial}{\partial\phi}\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix} \text{ otherwise zeros,}$$

$$\nabla = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial\phi} + \vec{e}_z \frac{\partial}{\partial z}.$$

SPHERICAL COORDINATES (θ,ϕ,r)

A particle is identified by its distance r, angle ϕ from the x-axis, and angle θ from the z-axis. Mapping $\vec{r}(\theta,\phi,r) = r(\sin\theta\cos\phi\vec{i} + \sin\theta\sin\phi\vec{j} + \cos\theta\vec{k})$, gives

$$\begin{Bmatrix} \vec{e}_\theta \\ \vec{e}_\phi \\ \vec{e}_r \end{Bmatrix} = \begin{bmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix},$$



$$\frac{\partial}{\partial\phi}\begin{Bmatrix} \vec{e}_\theta \\ \vec{e}_\phi \\ \vec{e}_r \end{Bmatrix} = \begin{Bmatrix} \cos\theta\,\vec{e}_\phi \\ -\sin\theta\vec{e}_r - \cos\theta\vec{e}_\theta \\ \sin\theta\vec{e}_\phi \end{Bmatrix}, \quad \frac{\partial}{\partial\theta}\begin{Bmatrix} \vec{e}_\theta \\ \vec{e}_\phi \\ \vec{e}_r \end{Bmatrix} = \begin{Bmatrix} -\vec{e}_r \\ 0 \\ \vec{e}_\theta \end{Bmatrix},$$

$$\nabla = \vec{e}_\theta \frac{1}{r}\frac{\partial}{\partial\theta} + \vec{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} + \vec{e}_r \frac{\partial}{\partial r}.$$

- According to the generic recipe (here c ~ cos and s ~ sin)

$$\frac{\partial}{\partial r}\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix} = (\frac{\partial}{\partial r}[F])[F]^{-1}\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix} = \begin{Bmatrix}0\\ 0\\ 0\end{Bmatrix},$$

$$\frac{\partial}{\partial \phi}\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix} = (\frac{\partial}{\partial \phi}[F])[F]^{-1}\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix} = \begin{bmatrix}0 & \mathrm{c}\theta & 0\\ -\mathrm{c}\theta & 0 & -\mathrm{s}\theta\\ 0 & \mathrm{s}\theta & 0\end{bmatrix}\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix} = \begin{Bmatrix}\mathrm{c}\theta\,\vec{e}_\phi\\ -\mathrm{s}\theta\vec{e}_r - \mathrm{c}\theta\vec{e}_\theta\\ \mathrm{s}\theta\vec{e}_\phi\end{Bmatrix},$$

$$\frac{\partial}{\partial \theta}\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix} = (\frac{\partial}{\partial \theta}[F])[F]^{-1}\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix} = \begin{bmatrix}0 & 0 & -1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix} = \begin{Bmatrix}-\vec{e}_r\\ 0\\ \vec{e}_\theta\end{Bmatrix}.$$ ←

1.3 CURVES AND SURFACES

The domain Ω of an engineering (mathematical) model has usually lower dimension than the body $V \in \mathbb{R}^3$. The representation of the domain embedded in $\mathbb{R}^3$ may be curve (mid-curve of beam) or surface (mid-surface of shell).

Curve: $\vec{r}_0(\alpha) = x(\alpha)\vec{i} + y(\alpha)\vec{j} + z(\alpha)\vec{k} \qquad \alpha \in \Omega \subset \mathbb{R}$ 1 parameter

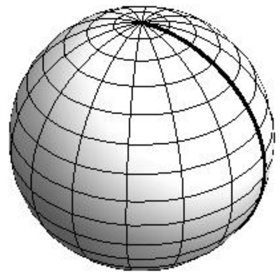
Surface: $\vec{r}_0(\alpha,\beta) = x(\alpha,\beta)\vec{i} + y(\alpha,\beta)\vec{j} + z(\alpha,\beta)\vec{k} \quad (\alpha,\beta) \in \Omega \subset \mathbb{R}^2$ 2 parameters

Shape of a mid-curve is defined by a one-parameter mapping and a mid-surface by a two-parameter mapping. In MEC-E8003, the coordinate curves of surfaces (defined by constant values of α or β) are assumed to be orthogonal (just to simplify the setting).

SOME MAPPINGS

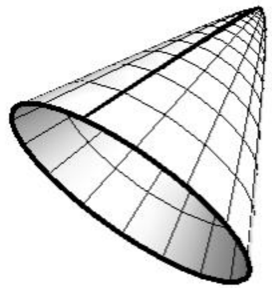
Coil $\vec{r}_0(\phi) = \vec{i}R\cos\phi + \vec{j}R\sin\phi + \vec{k}h\dfrac{\phi}{2\pi},$

Cylinder $\vec{r}_0(z,\phi) = R(\vec{i}\cos\phi + \vec{j}\sin\phi) + \vec{k}z$



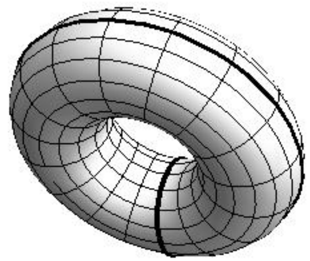
Cone $\vec{r}_0(z,s) = R(z)(\vec{i}\cos\phi + \vec{j}\sin\phi) + \vec{k}z$

Sphere $\vec{r}_0(\phi,\theta) = R(\vec{i}\sin\theta\cos\phi + \vec{j}\sin\theta\sin\phi + \vec{k}\cos\theta)$



Ellipsoid $\vec{r}_0(\phi,\theta) = R(\vec{i}\sin\theta\cos\phi + \vec{j}\sin\theta\sin\phi + \varepsilon\vec{k}\cos\theta)$

Hyperboloid $\vec{r}_0(\phi,\theta) = R(\vec{i}\sinh\theta\cos\phi + \vec{j}\sin h\theta\sin\phi - \varepsilon\vec{k}\cosh\theta)$



Torus $\vec{r}_0(\phi,\theta) = \vec{i}\cos\phi(R + r\cos\theta) + \vec{j}\sin\phi(R + r\cos\theta) + \vec{k}r\sin\theta$

CURVED BEAMS AND PLATES

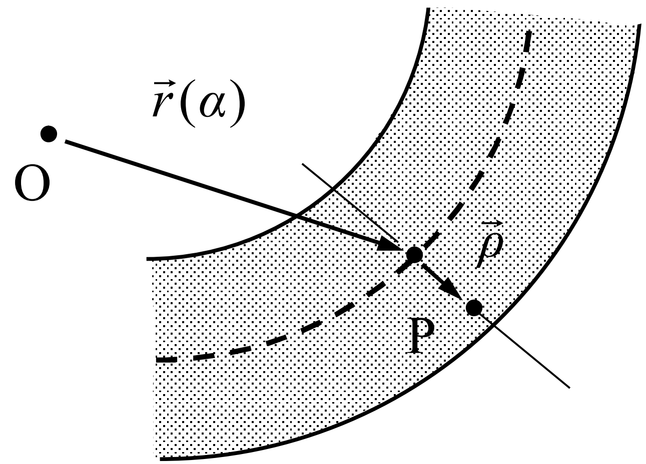
Curve or surface mapping identifies the particles on the mid-curve or mid-surface. Identification of all particles (P in the figure) of a thin body requires also the relative position vector $\vec{\rho}$:

relative

Beam mapping: $\vec{r}(\alpha,n,b)=\vec{r}_0(\alpha)+\underbrace{n\vec{e}_n(\alpha)+b\vec{e}_b(\alpha)}$

Shell mapping: $\vec{r}(\alpha,\beta,n)=\vec{r}_0(\alpha,\beta)+\underbrace{n\vec{e}_n(\alpha,\beta)}$

relative



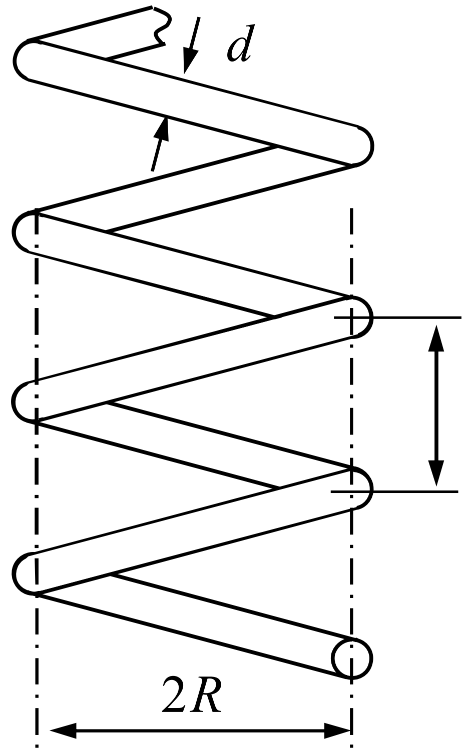
The mapping for the mid-curve or surface is used to define the basis vectors. In MEC-E8003 basis vectors are orthonormal to keep the setting as simple as possible (curved geometry induces some complications anyway)!

BEAM COORDINATES (s,n,b)

Particle is identified by distance s along the mid-curve and distances (n,b) from the curve.

Mapping $\vec{r}(s,n,b)=\vec{r}_0(s)+n\vec{e}_n(s)+b\vec{e}_b(s)$ gives

$$\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix}=\begin{Bmatrix}\partial\vec{r}_0/\partial s\\ (\partial\vec{e}_s/\partial s)/\left|\partial\vec{e}_s/\partial s\right|\\ \vec{e}_s\times\vec{e}_n\end{Bmatrix},$$



$$\frac{\partial}{\partial s}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix}=\begin{bmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix},$$

$$\nabla=\frac{\vec{e}_s}{1-n\kappa}[\frac{\partial}{\partial s}+\tau(b\frac{\partial}{\partial n}-n\frac{\partial}{\partial b})]+\vec{e}_n\frac{\partial}{\partial n}+\vec{e}_b\frac{\partial}{\partial b}.$$

- Beam (s,n,b) coordinate system is curvilinear and orthonormal. Therefore the matrix of the basis vector derivatives is anti-symmetric and expressible in the form

$$\frac{\partial}{\partial s}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix}=(\frac{\partial}{\partial s}[F])[F]^{-1}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix}=\begin{bmatrix}0 & \kappa_b & -\kappa_n\\ -\kappa_b & 0 & \kappa_s\\ \kappa_n & -\kappa_s & 0\end{bmatrix}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix}$$

containing geometrical quantities $\kappa_s=\tau$, $\kappa_n=0$, and $\kappa_b=\kappa=1/R$. Antisymmetry follows from $[F]^{\mathrm{T}}=[F]^{-1}$ and

$$\frac{\partial}{\partial s}([F][F]^{-1})=0 \qquad \Rightarrow\ (\frac{\partial}{\partial s}[F])[F]^{-1}=-[F]\frac{\partial}{\partial s}[F]^{-1}=-\left[(\frac{\partial}{\partial s}[F])[F]^{-1}\right]^{\mathrm{T}}.$$

- The generic expression of the gradient operator is given by (some manipulations are needed to end up with the expression)

$$\nabla = \begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix}^{\mathrm{T}} [F]^{-\mathrm{T}} [H]^{-1} \begin{Bmatrix} \partial / \partial s \\ \partial / \partial n \\ \partial / \partial b \end{Bmatrix} = \frac{\vec{e}_s}{1 - n\kappa} [\frac{\partial}{\partial s} + \tau (b \frac{\partial}{\partial n} - n \frac{\partial}{\partial b})] + \vec{e}_n \frac{\partial}{\partial n} + \vec{e}_b \frac{\partial}{\partial b} \ .$$

- The geometrical quantitities κ and τ follow from the expressions of the basis vector derivatives

$$\kappa = \vec{e}_n \cdot \frac{\partial}{\partial s} \vec{e}_s = -\vec{e}_s \cdot \frac{\partial}{\partial s} \vec{e}_n \ \text{ and } \ \tau = \vec{e}_b \cdot \frac{\partial}{\partial s} \vec{e}_n = -\vec{e}_n \cdot \frac{\partial}{\partial s} \vec{e}_b$$

or from the concept and expressions of curvature of curves and surfaces embedded in $\mathbb{R}^3$ discussed later.

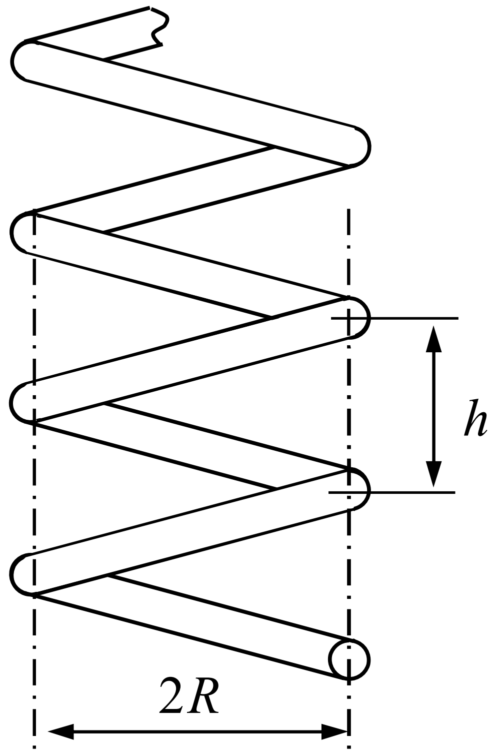
CURVATURE AND TORSION

Curvature κ is the amount by which a curve deviates from being *straight*. The radius of curvature $R = 1/\kappa$ at a point is given by the best fitting circle. Torsion τ describes the rate by which $\vec{e}_n$ and $\vec{e}_b$ rotate around the mid-curve.

Circle: $\kappa = \dfrac{1}{R}$ and $\tau = 0$

Twisted bar: $\kappa = 0$ and $\tau = \dfrac{2\pi}{h}$

Coil: $\kappa = \dfrac{R}{h^2 + R^2}$ and $\tau = \dfrac{h}{h^2 + R^2}$



- A circular bar of radius R has zero twist. The basis vectors of (x, y, z) and (s, n, b) coordinate systems differ by rotation with respect to the normal direction to the plane of circle (z here). With distance s along the mid-curve and $\phi = s / R$

$$\begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix} = \begin{bmatrix} -\sin\phi & \cos\phi & 0 \\ -\cos\phi & -\sin\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \quad \Rightarrow \quad \kappa = \vec{e}_n \cdot \frac{\partial}{\partial s} \vec{e}_s = \frac{1}{R}. \quad \Leftarrow$$

- Twisted bar has zero curvature. The basis vectors of (x, y, z) and (s, n, b) differ by rotation along the $x-$axis. With notation $\omega = 2\phi / h$

$$\begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega s & \sin\omega s \\ 0 & -\sin\omega s & \cos\omega s \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \quad \Rightarrow \quad \tau = \vec{e}_b \cdot \frac{\partial}{\partial s} \vec{e}_n = \frac{2\pi}{h}. \quad \Leftarrow$$

SHELL COORDINATES (α, β, n)

A particle is identified by mid-surface position (α, β) (generalized coordinates) and distance n in the normal direction. Mapping $\vec{r}(\alpha, \beta, n) = \vec{r}_0(\alpha, \beta) + n\vec{e}_n(\alpha, \beta)$ gives

$$\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} (\partial\vec{r}_0 / \partial\alpha) / \left|\partial\vec{r}_0 / \partial\alpha\right| \\ (\partial\vec{r}_0 / \partial\beta) / \left|\partial\vec{r}_0 / \partial\beta\right| \\ \vec{e}_\alpha \times \vec{e}_\beta \end{Bmatrix} = [F] \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix},$$

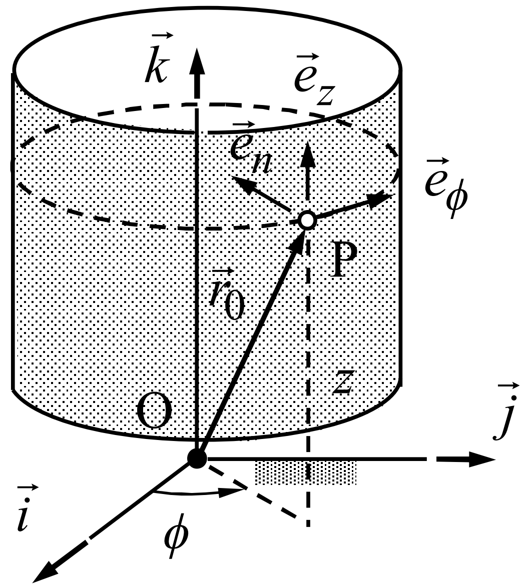
$$\frac{\partial}{\partial\eta} \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_n \end{Bmatrix} = (\frac{\partial}{\partial\eta}[F])[F]^{-1} \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_n \end{Bmatrix} \quad \eta \in \{\alpha, \beta, n\} \quad \text{and } \nabla = \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_n \end{Bmatrix}^{\mathrm{T}} [F]^{-\mathrm{T}} [H]^{-1} \begin{Bmatrix} \partial / \partial\alpha \\ \partial / \partial\beta \\ \partial / \partial n \end{Bmatrix}$$

In-surface basis vectors are assumed to be orthogonal i.e. $\vec{e}_\alpha \cdot \vec{e}_\beta = 0$.

CYLINDRICAL SHELL (z,ϕ,n)

A particle is identified by mid-surface position (z,ϕ) and distance n in the normal direction (inwards). Mid-surface mapping $\vec{r}_0(z,\phi)=\vec{i}R\cos\phi+\vec{j}R\sin\phi+\vec{k}z$ gives

$$\begin{Bmatrix}\vec{e}_z\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}=\begin{bmatrix}0 & 0 & 1\\ -\sin\phi & \cos\phi & 0\\ -\cos\phi & -\sin\phi & 0\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix},$$



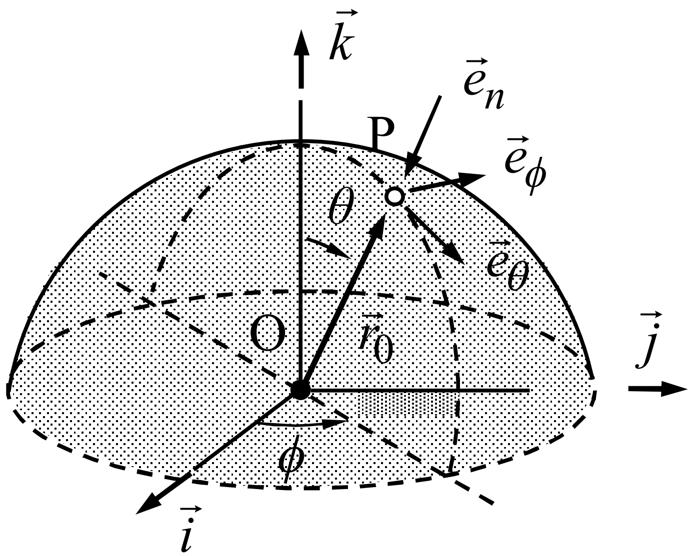
$$\frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_z\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}=\begin{Bmatrix}0\\ \vec{e}_n\\ -\vec{e}_\phi\end{Bmatrix}$$ zeros otherwise,

$$\nabla=\vec{e}_z\frac{\partial}{\partial z}+(\frac{R}{R-n})\frac{1}{R}\vec{e}_\phi\frac{\partial}{\partial\phi}+\vec{e}_n\frac{\partial}{\partial n}.$$

SPHERICAL SHELL (ϕ,θ,n)

A particle is identified by mid-surface position (ϕ,θ) and distance n in the normal direction (inwards). Mid-surface mapping $\vec{r}_0(\phi,\theta) = R(\vec{i}\sin\theta\cos\phi + \vec{j}\sin\theta\sin\phi + \vec{k}\cos\theta)$:

$$\begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} -\sin\phi & \cos\phi & 0 \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\cos\phi & -\sin\theta\sin\phi & -\cos\theta \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix},$$



$$\frac{\partial}{\partial\phi}\begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} \sin\theta\vec{e}_n - \cos\theta\vec{e}_\theta \\ \cos\theta\vec{e}_\phi \\ -\sin\theta\,\vec{e}_\phi \end{Bmatrix}, \quad \frac{\partial}{\partial\theta}\begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ \vec{e}_n \\ -\vec{e}_\theta \end{Bmatrix},$$

$$\nabla = \frac{R}{R-n}\frac{1}{R\sin\theta}\,\vec{e}_\phi\frac{\partial}{\partial\phi} + \frac{R}{R-n}\frac{1}{R}\vec{e}_\theta\frac{\partial}{\partial\theta} + \vec{e}_n\frac{\partial}{\partial n}.$$

1.4 CURVATURE

Curvature is the amount by which a geometric object deviates from being *flat,* or *straight* in the case of a curve. Curvature of a surface ($\kappa \sim 1/R$) at a point depends on the direction of a curve through the point.

Curvature: $\overleftrightarrow{\kappa}_{\mathrm{c}} = \nabla \vec{e}_n$ Definion!

Principal curvatures: $(\kappa_1, \vec{n}_1)$ and $(\kappa_2, \vec{n}_2)$ such that $\overleftrightarrow{\kappa} \cdot \vec{n} = \kappa \vec{n}$

Gaussian curvature: $K = \det[\kappa] = \kappa_1 \kappa_2$ Curvature measure!

Mean curvature: $H = \frac{1}{2} \nabla \cdot \vec{e}_n = \frac{1}{2} \overleftrightarrow{I} : \overleftrightarrow{\kappa} = \frac{1}{2}(\kappa_1 + \kappa_2)$ Another curvature measure!

Curvature concept has many somewhat different aspects and the related definitions!

EXAMPLE. A planar curve is defined by mapping

(a) $\vec{r}_0(\alpha) = x(\alpha)\vec{i} + y(\alpha)\vec{j}$ (generic parametric form of a planar curve)

(b) $\vec{r}_0(x) = x\vec{i} + y(x)\vec{j}$

Derive the expression of curvature starting with the definition $\overleftrightarrow{\kappa}_{\mathrm{c}} = \nabla\vec{e}_n$.

Answer: (a) $\overleftrightarrow{\kappa} = \vec{e}_\alpha\vec{e}_\alpha \dfrac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}$ (b) $\overleftrightarrow{\kappa} = \vec{e}_x\vec{e}_x \dfrac{y''}{(1 + y'^2)^{3/2}}$

- To use the definition, one needs the derivatives of the basis vectors and also the gradient operator of the curvilinear (α, n, b) system. With the Lagrange's notation of derivative with respect to α, $g = \sqrt{x'^2 + y'^2}$, $s_x = x'/g$, and $s_y = y'/g$

$$\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix} = \begin{Bmatrix} \vec{r}_0'/|\vec{r}_0'| \\ \vec{e}_\alpha'/|\vec{e}_\alpha'| \\ \vec{e}_\alpha \times \vec{e}_n \end{Bmatrix} = \begin{bmatrix} s_x & s_y & 0 \\ -s_y & s_x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = [F] \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \text{ and } [F]^{-1} = [F]^{\mathrm{T}} \Rightarrow$$

$$\begin{Bmatrix} \vec{e}_\alpha' \\ \vec{e}_n' \\ \vec{e}_b' \end{Bmatrix} = \begin{bmatrix} s_x' & s_y' & 0 \\ -s_y' & s_x' & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = [F]' \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \Rightarrow$$

$$\begin{Bmatrix} \vec{e}'_\alpha \\ \vec{e}'_n \\ \vec{e}'_b \end{Bmatrix} = [F]'[F]^{-1} \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix} = (s'_x s_y - s'_y s_x) \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix},$$

$$\begin{Bmatrix} \partial\vec{r}/\partial\alpha \\ \partial\vec{r}/\partial n \\ \partial\vec{r}/\partial b \end{Bmatrix} = \begin{Bmatrix} \partial(\vec{r}_0 + n\vec{e}_n + b\vec{e}_b)/\partial\alpha \\ \partial(\vec{r}_0 + n\vec{e}_n + b\vec{e}_b)/\partial n \\ \partial(\vec{r}_0 + n\vec{e}_n + b\vec{e}_b)/\partial b \end{Bmatrix} = \begin{bmatrix} x' - ns'_y & y' + ns'_x & 0 \\ -s_y & s_x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = [H] \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix},$$

giving the gradient operator expression

$$\nabla = \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix}^{\mathrm{T}} ([H][F]^{\mathrm{T}})^{-1} \begin{Bmatrix} \partial_\alpha \\ \partial_n \\ \partial_b \end{Bmatrix} = \vec{e}_\alpha \frac{1}{g + n(s_y s'_x - s_x s'_y)} \frac{\partial}{\partial\alpha} + \vec{e}_n \frac{\partial}{\partial n} + \vec{e}_b \frac{\partial}{\partial b}.$$

- The definition of curvature gives now (the mid-curve corresponds to $n=0$)

$$\vec{\vec{\kappa}}_{\rm c} = \nabla_0 \vec{e}_n = \vec{e}_\alpha \frac{1}{g}\frac{\partial \vec{e}_n}{\partial \alpha} + \vec{e}_n \frac{\partial \vec{e}_n}{\partial n} + \vec{e}_b \frac{\partial \vec{e}_n}{\partial b} = \vec{e}_\alpha \frac{1}{g}\vec{e}_n' = \vec{e}_\alpha \vec{e}_\alpha \frac{s_x' s_y - s_y' s_x}{g},$$

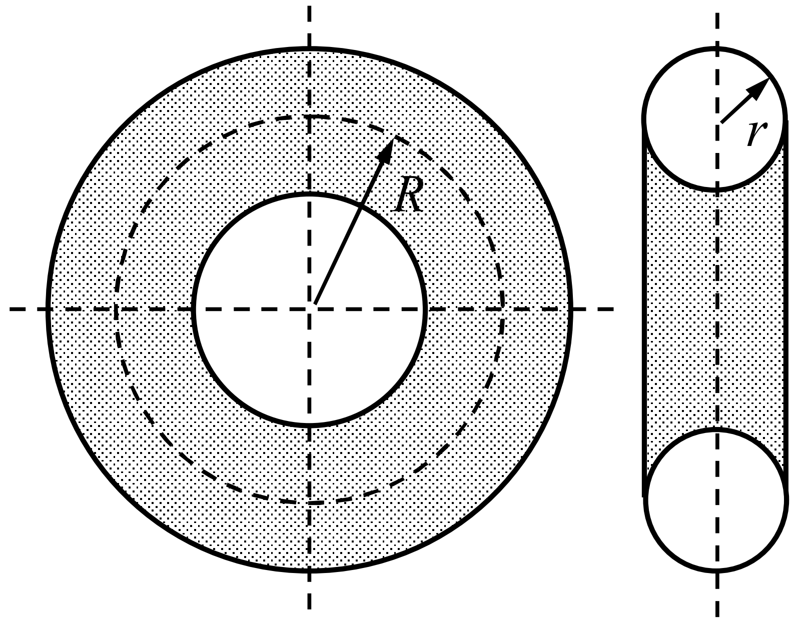
$$\vec{\vec{\kappa}} = \vec{e}_\alpha \vec{e}_\alpha \frac{s_x' s_y - s_y' s_x}{g} = \vec{e}_\alpha \vec{e}_\alpha \frac{x'' y' - y'' x'}{(x'^2 + y'^2)^{3/2}} \, .$$ ←

- For a circular curve defined by $\vec{r}_0(\phi) = R\cos\phi \vec{i} + R\sin\phi \vec{j}$ and $\alpha = \phi$, the outcome is $\vec{\vec{\kappa}} = \vec{e}_\phi \vec{e}_\phi / R$. Selection $\alpha = x$ gives $x' = 1$, $x'' = 0$ and

$$\vec{\vec{\kappa}} = \vec{e}_x \vec{e}_x \frac{y''}{(1 + y'^2)^{3/2}} .$$ ←

EXAMPLE. Consider torus surface (donut) having distance R from the center of the tube to the center of the torus and radius r of the tube. Derive the basis vectors, basis vector derivatives, gradient expression, and curvature in (ϕ,θ,n) coordinate system. The mapping defining the geometry, $\phi\in[0,2\pi]$ and $\theta\in[0,2\pi]$, is

$$\vec{r}_0(\phi,\theta)=(R+r\cos\theta)(\vec{i}\cos\phi+\vec{j}\sin\phi)+\vec{k}r\sin\theta .$$



Answer:

$$\nabla=\frac{1}{R+(n+r)\cos\theta}\vec{e}_\phi\frac{\partial}{\partial\phi}+\vec{e}_\theta\frac{1}{n+r}\frac{\partial}{\partial\theta}+\vec{e}_n\frac{\partial}{\partial n}\ ,\ \ \overleftrightarrow{\kappa}=\vec{e}_\phi\vec{e}_\phi\frac{\cos\theta}{R+(n+r)\cos\theta}+\vec{e}_\theta\vec{e}_\theta\frac{1}{n+r}$$

- Let us start with the relationship between the basis vectors. Definitions give

$$\begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} \equiv \begin{Bmatrix} (\partial\vec{r}_0/\partial\phi)/|\partial\vec{r}_0/\partial\phi| \\ (\partial\vec{r}_0/\partial\theta)/|\partial\vec{r}_0/\partial\theta| \\ \vec{e}_\phi \times \vec{e}_\theta \end{Bmatrix} = \begin{bmatrix} -\sin\phi & \cos\phi & 0 \\ -\cos\phi\sin\theta & -\sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & \sin\theta \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = [F] \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}$$

- Since the basis is orthonormal i.e. $[F]^{-1} = [F]^{\mathrm{T}}$, the partial derivatives of the basis vectors are given by

antisymmetric!

$$\frac{\partial}{\partial\phi} \begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} = (\frac{\partial}{\partial\phi}[F])[F]^{-1} \begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} 0 & \sin\theta & -\cos\theta \\ -\sin\theta & 0 & 0 \\ \cos\theta & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix}$$

$$\frac{\partial}{\partial\theta}\begin{Bmatrix}\vec{e}_\phi\\ \vec{e}_\theta\\ \vec{e}_n\end{Bmatrix}=(\frac{\partial}{\partial\theta}[F])[F]^{-1}\begin{Bmatrix}\vec{e}_\phi\\ \vec{e}_\theta\\ \vec{e}_n\end{Bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{bmatrix}\begin{Bmatrix}\vec{e}_\phi\\ \vec{e}_\theta\\ \vec{e}_n\end{Bmatrix}, \text{ and } \frac{\partial}{\partial n}\begin{Bmatrix}\vec{e}_\phi\\ \vec{e}_\theta\\ \vec{e}_n\end{Bmatrix}=0.$$

- The gradient expression in concerned with a generic material point so that the mapping between the curvilinear (ϕ,θ,n) coordinate system and the reference (x,y,z) coordinate system is written as $\vec{r}=\vec{r}_0+n\vec{e}_n$ (the mapping needs to define positions of all particles of the body not just those on the mid-surface). Relationship gives

$$[H]=\begin{bmatrix}-[R+(n+r)\cos\theta]\sin\phi & (R+(n+r)\cos\theta)\cos\phi & 0\\ -(n+r)\cos\phi\sin\theta & -(n+r)\sin\theta\sin\phi & (n+r)\cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin\phi & \sin\theta\end{bmatrix} \Leftrightarrow$$

$$[H]^{-1}=\begin{bmatrix} -\dfrac{\sin\phi}{R+(n+r)\cos\theta} & -\dfrac{\cos\phi\sin\theta}{n+r} & \cos\theta\cos\phi \\ \dfrac{\cos\phi}{R+(n+r)\cos\theta} & -\dfrac{\sin\theta\sin\phi}{n+r} & \cos\theta\sin\phi \\ 0 & \dfrac{\cos\theta}{n+r} & \sin\theta \end{bmatrix}.$$

- The generic formula for the gradient operator gives ($[F]=[F]^{-\mathrm{T}}$)

$$\nabla=\begin{Bmatrix}\vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n\end{Bmatrix}^{\mathrm{T}}[F]^{-\mathrm{T}}[H]^{-1}\begin{Bmatrix}\partial/\partial\phi \\ \partial/\partial\theta \\ \partial/\partial n\end{Bmatrix}=\frac{1}{R+(n+r)\cos\theta}\vec{e}_\phi\frac{\partial}{\partial\phi}+\vec{e}_\theta\frac{1}{n+r}\frac{\partial}{\partial\theta}+\vec{e}_n\frac{\partial}{\partial n}\,. \quad \Leftarrow$$

- Finally, curvature of the torus geometry becomes

$$\nabla \vec{e}_n = \frac{1}{R+(n+r)\cos\theta}\vec{e}_\phi \frac{\partial}{\partial \phi}\vec{e}_n + \vec{e}_\theta \frac{1}{n+r}\frac{\partial}{\partial \theta}\vec{e}_n + \vec{e}_n \frac{\partial}{\partial n}\vec{e}_n \quad \Rightarrow$$

$$\vec{\vec{\kappa}} = (\nabla \vec{e}_n)_\mathrm{c} = \vec{e}_\phi \vec{e}_\phi \frac{\cos\theta}{R+(n+r)\cos\theta} + \vec{e}_\theta \vec{e}_\theta \frac{1}{n+r}.$$ ←