6. Existence by the direct method in the calculus of variations

6.1. Newtonian spaces with zero boundary values

In order to be able to compare boundary values of Newtonian functions and to discuss the solution of the Dirichlet problem we shall need the Newtonian spaces with zero boundary values. Let $E \subset X$, then

$$N^{1,p}_0(E) = \{ u \in N^{1,p}(X) : u = 0 \text{ in } X \setminus E \}.$$

It can be shown that $N^{1,p}_0(E)$ is a closed subspace of $N^{1,p}(X)$ and thus a Banach space.

6.2. The Dirichlet problem.

Let $1 < p < \infty$, $\cancel{g}\, v \in N^{1,p}(X)$, $E \subset X$ a measurable set such that $\mu(X \setminus E) > 0$. A function $u \in N^{1,p}(X)$ is a minimizer of the p-Dirichlet integral in E with the boundary values v if

(i) $u - v \in N^{1,p}_0(E)$ and

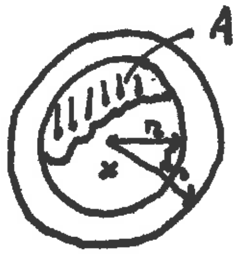
(ii) $\displaystyle\int_E g_u^p \, d\mu \le \int_E g_w^p \, d\mu$

for every function $w \in N^{1,p}(X)$ such that $w - v \in N^{1,p}_0(E)$. Here g_u and g_w are the minimal p-weak upper gradients of u and w respectively, see 2.24.

We shall show that the Dirichlet problem above has a unique solution if the space supports a p-Poincaré inequality. Next we prove a Sobolev inequality for functions that vanish on a large set.

6.3. Lemma. Assume that X is a complete metric measure space with a doubling measure and a weak p-Poincaré inequality. Assume that $u \in N^{1,p}(X)$ and let $A = \{y \in B(x,r) : |u(y)| > 0\}$. If $\mu(A) \leq \gamma\, \mu(B(x,r))$ for some γ with $0 < \gamma < 1$, there exists a constant C and an exponent $q > p$ as in Theorem 3.8 such that

$$\left(\fint_{B(x,r)} |u|^q \, d\mu \right)^{\frac{1}{q}} \leq C r \left(\fint_{B(x,\lambda r)} g_u^p \, d\mu \right)^{\frac{1}{p}}.$$



Proof: By Minkowski's inequality and Theorem 3.8 we have

$$\begin{aligned}
\left(\fint_{B(x,r)} |u|^q \, d\mu \right)^{\frac{1}{q}} &\leq \left(\fint_{B(x,r)} |u - u_{B(x,r)}|^q \, d\mu \right)^{\frac{1}{q}} + |u_{B(x,r)}| \\
&\leq C r \left(\fint_{B(x,\lambda r)} g_u^p \, d\mu \right)^{\frac{1}{p}} + |u_{B(x,r)}|.
\end{aligned}$$

Hölder's inequality implies

$$\begin{aligned}
|u_{B(x,r)}| &\leq \fint_{B(x,r)} |u| \, d\mu = \frac{1}{\mu(B(x,r))} \int_A |u| \, d\mu \\
&\leq \frac{1}{\mu(B(x,r))} \left(\int_A |u|^q \, d\mu \right)^{\frac{1}{q}} \mu(A)^{1-\frac{1}{q}} \\
&= \left(\frac{\mu(A)}{\mu(B(x,r))} \right)^{1-\frac{1}{q}} \left(\fint_{B(x,r)} |u|^q \, d\mu \right)^{\frac{1}{q}}
\end{aligned}$$

$|u| = 0$ in $B(x,r) \setminus A$

$$\le \gamma^{1-\frac{1}{q}}\Big(\fint_{B(x,r)}|u|^q\,d\mu\Big)^{\frac{1}{q}}.$$

Thus

$$\big(1-\gamma^{1-\frac{1}{q}}\big)\Big(\fint_{B(x,r)}|u|^q\,d\mu\Big)^{\frac{1}{q}} \le cr\Big(\fint_{B(x,\lambda r)} g_u^p\,d\mu\Big)^{\frac{1}{p}}$$

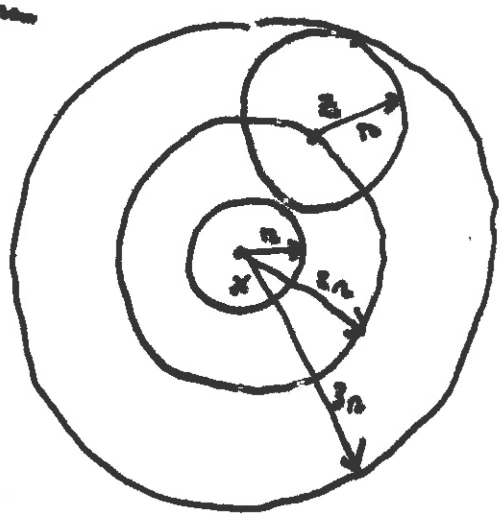
from which the claim follows since $0<\gamma<1$. $\square$

6.4. <u>Remark</u>. Lemma 6.3 gives a Sobolev inequality for functions in $N^{1,p}_0(B(x,r))$. To be more precise, there exists a constant c and an exponent $q>p$ such that

$$\Big(\fint_{B(x,r)}|u|^q\,d\mu\Big)^{\frac{1}{q}} \le cr\Big(\fint_{B(x,r)} g_u^p\,d\mu\Big)^{\frac{1}{p}}$$

for every $u\in N^{1,p}_0(B(x,r))$ whenever $0<r\le \frac{\operatorname{diam}(X)}{3}$.

To see this, we observe that there has to be a point $z\in\partial B(x,2r)$. Otherwise it is easy to construct a function that violates the Poincaré inequality, see Theorem 3.3. Then



$$B(z,r)\subset\{y\in B(x,3r): |u(y)|=0\}$$

and thus

$$A=\{y\in B(x,3r): |u(y)|>0\}$$
$$\subset B(x,3r)\setminus B(z,r).$$

This implies

$$\mu(A)\le\mu(B(x,3r)\setminus B(z,r))$$
$$=\mu(B(x,3r))-\mu(B(z,r)),$$

where by Lemma 1.11

$$\frac{\mu(B(z,r))}{\mu(B(x,3r))} \geq 4^{-\delta}\left(\frac{r}{3r}\right)^{\delta}, \quad \delta \geq \log_2 c_d .$$

Thus

$$\mu(A) \leq (1-12^{-\delta})\,\mu(B(x,3r))$$

and we may apply Lemma 6.3 to obtain

$$c\left(\fint_{B(x,3r)} |u|^q \, d\mu\right)^{\frac{1}{q}} \leq C r \left(\fint_{B(x,3\lambda r)} g_u^p \, d\mu\right)^{\frac{1}{p}}$$

$\qquad\qquad\qquad\qquad\qquad\qquad$ $\shortparallel \leftarrow g_u = 0$ in $X \setminus B(x,r)$

$$\vee\!\mid$$

$$\left(\fint_{B(x,r)} |u|^q \, d\mu\right)^{\frac{1}{q}} \qquad\qquad C r \left(\fint_{B(x,r)} g_u^p \, d\mu\right)^{\frac{1}{p}}$$

6.5. Theorem. Let $1<p<\infty$, $w \in N^{1,p}(X)$ and $E \subset X$ a bounded and measurable set such that $\mu(X \setminus E) > 0$. Then the Dirichlet problem in 6.2 has a unique solution, that is, there exists a function $u \in N^{1,p}(X)$ such that $u - w \in N^{1,p}_0(E)$ and

$$\int_E g_u^p \, d\mu \leq \int_E g_v^p \, d\mu$$

for every $v \in N^{1,p}(X)$ with $v - w \in N^{1,p}_0(E)$.

__Remark.__

$$\int_E g_u^p \, d\mu = \inf\left\{ \int_E g_v^p \, d\mu : v \in N^{1,p}(X),\ v - w \in N^{1,p}_0(E) \right\}$$

<u>Proof</u>: Let

$$I = \inf\left\{ \int_E g_w^p \, d\mu : w \in N^{1,p}(X),\ w - v \in N_0^{1,p}(E) \right\}.$$

We may choose $w = v$ to see that $0 \le I < \infty$. Let $(u_i)_{i\in\mathbb{N}}$ be a minimising sequence of $u_i \in N^{1,p}(X)$ with $u_i - v \in N_0^{1,p}(E)$ such that

$$\lim_{i\to\infty} \int_E g_{u_i}^p \, d\mu = I.$$

It follows that $(g_{u_i})_{i\in\mathbb{N}}$ is a bounded sequence in $L^p(E)$.

Since $E \subset X$ is bounded, there exists a ball $B(x,r) \supset E$ such that $\mu(B(x,r)\setminus E) > 0$. If X is bounded we may choose $B(x,r) = X$.

By Lemma 6.3 ~~with $q>p$~~ we have

$$\int_X |u_i - v|^p \, d\mu = \int_{B(x,2r)} |u_i - v|^p \, d\mu$$

$u_i - v = 0$ in $X \setminus B(x,r)$

$$\le \left(\int_{B(x,2r)} |u_i - v|^q \, d\mu \right)^{\frac{p}{q}} \mu(B(x,2r))^{1-\frac{p}{q}}$$

Hölder, $q > p$

$$\le C r^p \fint_{B(x,2\lambda r)} g_{u_i - v}^p \, d\mu \; \mu(B(x,2r))$$

Lemma 6.3

$$\mu(\{ y \in B(x,2r) : |u_i(y) - v(y)| > 0 \}) \le \mu(E) = \mu(B(x,2r)) - \mu(B(x,2r)\setminus E) \le \mu(B(x,2r)) - \underbrace{\mu(B(x,r)\setminus E)}_{>0} = \gamma \mu(B(x,$$

$$\le C r^p \int_{B(x,2\lambda r)} g_{u_i - v}^p \, d\mu$$

$$= C r^p \int_X g_{u_i - v}^p \, d\mu.$$

$g_{u_i - v} = 0$ in $X \setminus B(x,r)$

Thus

$$\left(\int_E |u_i|^p \, d\mu\right)^{\frac{1}{p}} \leq \left(\int_E |v|^p \, d\mu\right)^{\frac{1}{p}} + \left(\int_E |u_i - v|^p \, d\mu\right)^{\frac{1}{p}}$$

$u_i = v + (u_i - u)$

$$\leq \|v\|_{L^p(E)} + C r \left(\int_E g_{u_i - v}^p \, d\mu\right)^{\frac{1}{p}}$$

the Poincaré on the previous page

$$\leq \|v\|_{L^p(E)} + C r \left(\int_E (g_{u_i} + g_v)^p \, d\mu\right)^{\frac{1}{p}}$$

$$\leq \|v\|_{L^p(E)} + C r \left(\|g_{u_i}\|_{L^p(E)} + \|g_v\|_{L^p(E)}\right)$$

and consequently $(u_i)_{i\in\mathbb{N}}$ is a bounded sequence in $L^p(E)$. Since $L^p(E)$ is reflexive for $1<p<\infty$, there exists a subsequence denoted by $(u_i)_{i\in\mathbb{N}}$ such that $u_i \to u$ weakly in $L^p(E)$. Since $(g_{u_i})_{i\in\mathbb{N}}$ is a bounded sequence in $L^p(E)$ we may pass to a subsequence once more and conclude that

$$u_i \to u \quad \text{and} \quad g_{u_i} \to g \text{ weakly in } L^p(E).$$

We shall use the following Mazur's lemma: Let X be a normed space and assume that $x_i \to x$ weakly in X, then there exists a sequence of convex combinations of x_i

$$\tilde{x}_i = \sum_{j=i}^{n_i} \alpha_{i,j} x_j, \quad \alpha_{i,j} \geq 0, \quad \sum_{j=i}^{n_i} \alpha_{i,j} = 1$$

such that $\tilde{x}_i \to x$ in the norm of X, see [HKST 2.3].

We apply Mazur's lemma for the sequence (u_i, g_{u_i}) in $L^p(E)\times L^p(E)$. Since $(u_i, g_{u_i}) \to (u, g_{\#})$ weakly in $L^p(E)\times L^p(E)$, there exists a sequence of convex combinations

$$\tilde{u}_i = \sum_{j=i}^{m_i} \alpha_{i,j} u_j \quad \text{and} \quad \tilde{g}_i = \sum_{j=i}^{m_i} \alpha_{i,j} g_{u_j}$$

such that $\tilde{u}_i \to u$ in $L^p(E)$ and $\tilde{g}_i \to g_{\#}$ in $L^p(E)$. Here $\tilde{g}_i$ is a p-weak upper gradient of $\tilde{u}_i$, see 2.12. Note that it is essential that the coefficients in the convex combinations are the same for $\tilde{u}_i$ and $\tilde{g}_i$. Fuglede's lemma 2.8 implies that $g_{\#}$ is a p-weak upper gradient of u [Proposition 2.2 in Björns' book]. Since $u \in L^p(E)$ and $g_{\#} \in L^p(E)$, we have $u \in N^{1,p}(E)$. Since $u_i - v \in N^{1,p}_0(E)$, we have $\tilde{u}_i - v \in N^{1,p}_0(E)$. It follows that $u - v \in N^{1,p}_0(E)$. This implies

$$I \le \int_E g_u^p \, d\mu \underset{\substack{\uparrow \\ g_u \text{ is the minimal } p\text{-weak upper gradient}}}{\le} \int_E g^p \, d\mu$$

$$= \lim_{i\to\infty} \int_E \tilde{g}_i^p \, d\mu$$

$$\underset{\substack{\uparrow \\ t \mapsto t^p \text{ is convex}}}{\le} \lim_{i\to\infty} \sum_{j=i}^{m_i} \alpha_{i,j} \int_E g_{u_j}^p \, d\mu \underset{\substack{\uparrow \\ \sum_{j=i}^{m_i} \alpha_{i,j} = 1, \; i=1,2,\dots}}{=} I.$$

Thus

$$I = \int_E g_u^p \, d\mu.$$

This shows that u is a minimizer.

It remains to prove the uniqueness. Assume that u_1 and u_2 are minimizers. Let $A = \{x \in E : g_{u_1}(x) \neq g_{u_2}(x)\}$ and assume that $\mu(A) > 0$. Let

$$\tilde{u} = \frac{u_1 + u_2}{2}.$$

Then $\tilde{u} \in N^{1,p}(X)$ with $\tilde{u} - v \in N_0^{1,p}(E)$. Moreover

$$\tilde{g} = \frac{g_{u_1} + g_{u_2}}{2}$$

is a p-weak upper gradient of $\tilde{u}$. Thus

$$I \le \int_E \tilde{g}^p \, d\mu = \int_A \tilde{g}^p \, d\mu + \int_{E \setminus A} \tilde{g}^p \, d\mu$$

$$= \underbrace{\int_A \left(\tfrac{1}{2} g_{u_1} + \tfrac{1}{2} g_{u_2}\right)^p d\mu}_{< \frac{1}{2}\int_A g_{u_1}^p \, d\mu + \frac{1}{2}\int_A g_{u_2}^p \, d\mu} + \underbrace{\int_{E \setminus A} \left(\tfrac{1}{2} g_{u_1} + \tfrac{1}{2} g_{u_2}\right)^p d\mu}_{\le \frac{1}{2}\int_{E\setminus A} g_{u_1}^p \, d\mu + \frac{1}{2}\int_{E\setminus A} g_{u_2}^p \, d\mu}$$

$t \mapsto t^p$ strictly convex   convexity

$$< \frac{1}{2} \int_E g_{u_1}^p \, d\mu + \frac{1}{2} \int_E g_{u_2}^p \, d\mu$$

$$= \frac{1}{2}(I + I) = I \quad ↯$$

Thus $\mu(A) = 0$ which implies that $g_{u_1} = g_{u_2}$ almost everywhere in E.

We shall show that $g_{u_1 - u_2} = 0$ almost everywhere. By the Poincaré inequality as on page 6/5 we have

$$\int_X |u_1 - u_2|^p \, d\mu \le C r^p \int_X g_{u_1 - u_2}^p \, d\mu = 0.$$

To show that $g_{u_1-u_2}=0$ almost everywhere in E, we consider

$$u=\max\{u_1,u_2\}.$$

Then

$$u-v=\max\{u_1,u_2\}-v=\max\{u_1-v,u_2-v\}\in N_0^{1,p}(E).$$

By Lemma 2.21 it can be shown that

$$g_u\le g_{u_1}\chi_{\{u_1>u_2\}}+g_{u_2}\chi_{\{u_1\le u_2\}}$$

$$\underset{g_{u_1}=g_{u_2}\text{ a.e. in }E}{=}g_{u_1}\chi_{\{u_1>u_2\}}+g_{u_1}\chi_{\{u_1\le u_2\}}=g_{u_1}$$

since $g_{u_1}=g_{u_2}$ almost everywhere in E. Thus

$$I\underset{u\text{ is admissible}}{\le}\int_E g_u^p\,d\mu\le\int_E g_{u_1}^p\,d\mu=I$$

and

$$\int_E\underbrace{(g_{u_1}^p-g_u^p)}_{\ge 0}\,d\mu=I-I=0,$$

which implies $g_u=g_{u_1}=g_{u_2}$ almost everywhere in E.

Let

$$A=\{x\in E:\ u_1(x)<u(x),\ g_{u_1}(x)>0\}.$$

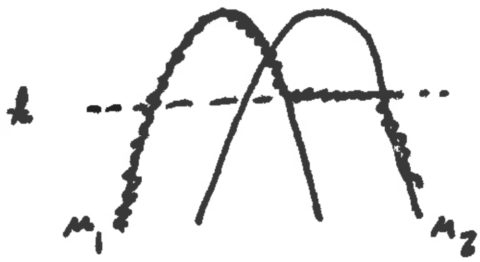
We claim that $\mu(A)=0$. If not, then the set

$$A_t=\{x\in A: u_1(x)<t<u(x)\}$$

has positive measure for some $t\in\mathbb{R}$, that is, $\mu(A_t)>0$.

Let

$$\tilde{u}(x) = \begin{cases} u(x), & u(x) \le t, \\ t, & u_1(x) < t < u(x), \\ u_1(x), & u_1(x) \ge t. \end{cases}$$



Then $\tilde{u}(x) = \max\{u_1(x), \min\{u_2(x), t\}\}$ with

$$\tilde{u} - v \le \max\{u_1, u_2\} - v = \max\{u_1 - v, u_2 - v\} \in N_0^{1,p}(E)$$

and

$$\tilde{u} - v \ge u_1 - v \in N_0^{1,p}(E).$$

This implies $\tilde{u} - u \in N_0^{1,p}(E)$, see [Lemma 2.37, Björn]. Thus

$$I \le \int_E g_{\tilde{u}}^p \, d\mu = \int_{E \setminus A_t} g_{\tilde{u}}^p \, d\mu + \underbrace{\int_{A_t} g_{\tilde{u}}^p \, d\mu}_{= 0, \text{ since } g_{\tilde{u}} = 0 \text{ a.e. in } A_t}$$

$$= \int_{E \setminus A_t} g_{u_1}^p \, d\mu$$

$\begin{cases} g_u(x) \text{ equals } g_{u_1}(x) \text{ or } g_{u}(x) \text{ in } E \setminus A_t \\ g_{u_1}(x) = g_{u_2}(x) \text{ a.e. in } E \end{cases}$

$$< \int_{E \setminus A_t} g_{u_1}^p \, d\mu + \underbrace{\int_{A_t} g_{u_1}^p \, d\mu}_{> 0, \text{ since } g_{u_1} > 0 \text{ in } A_t}$$

$$= \int_E g_{u_1}^p \, d\mu = I \quad \lightning$$

Thus $\mu(A)=0$. This implies that for almost every $x \in E$ either $u_1(x) \geq u(x)$ or $g_{u_1}(x)=0$. If

$$u_1(x) \geq u(x) = \max\{u_1(x), u_2(x)\}$$

then $u_1(x) \geq u_2(x)$. By switching the roles of u_1 and u_2 we obtain that for almost every $x \in E$ either $u_2(x) \geq u_1(x)$ or $g_{u_2}(x)=0$. Since $g_{u_1}(x) = g_{u_2}(x)$ for almost every $x \in E$, we conclude that for almost every $x \in E$ either $u_1(x) = u_2(x)$ or $g_{u_1}(x) = g_{u_2}(x) = 0$. Since $g_{u_1-u_2} = 0$ almost everywhere on the set where $u_1 = u_2$ and

$$g_{u_1-u_2} \leq g_{u_1} + g_{u_2} = 2 g_{u_1} = 0$$

on the set where $g_{u_1} = g_{u_2} = 0$, we conclude that $g_{u_1-u_2}=0$ almost everywhere in E. $\square$

6.6. <u>Remark</u>. It is a well known result in functional analysis that if B is a reflexive Banach space and $I: B \to \mathbb{R}$ is a convex, lower semicontinuous and coercive operator, then there exists an element u in B that minimizes I. Here I is said to be convex, if

$$I(tu + (1-t)v) \leq t I(u) + (1-t) I(v)$$

for every $t \in [0,1]$ and $u, v \in B$. The operator I is said to be lower semicontinuous if

$$I(u) \leq \liminf_{i \to \infty} I(u_i)$$

whenever $u_i \to u$ in B and coercive if $I(u_i) \to \infty$ whenever $\|u_i\|_B \to \infty$.

Moreover, if I is strictly convex, then the minimizer is unique. Lower semicontinuity can be replaced with lower semicontinuity with respect to sequential weak convergence. However, it is not clear how to apply this general result here as such. A result of Cheeger shows that $N^{1,p}(X)$ is reflexive if the measure is doubling and the space supports a p-Poincaré inequality. As a closed subspace of $N^{1,p}(X)$, the space $N^{1,p}_0(G)$ is reflexive as well.