Agenda today

- Introduction to prescriptive modeling

- Linear optimization models through three examples:
  1. Production and inventory optimization
  2. Distribution system design
  3. Stochastic optimization

- Beyond linear optimization

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Part 1: Introduction to prescriptive modeling

Data Science for Business II

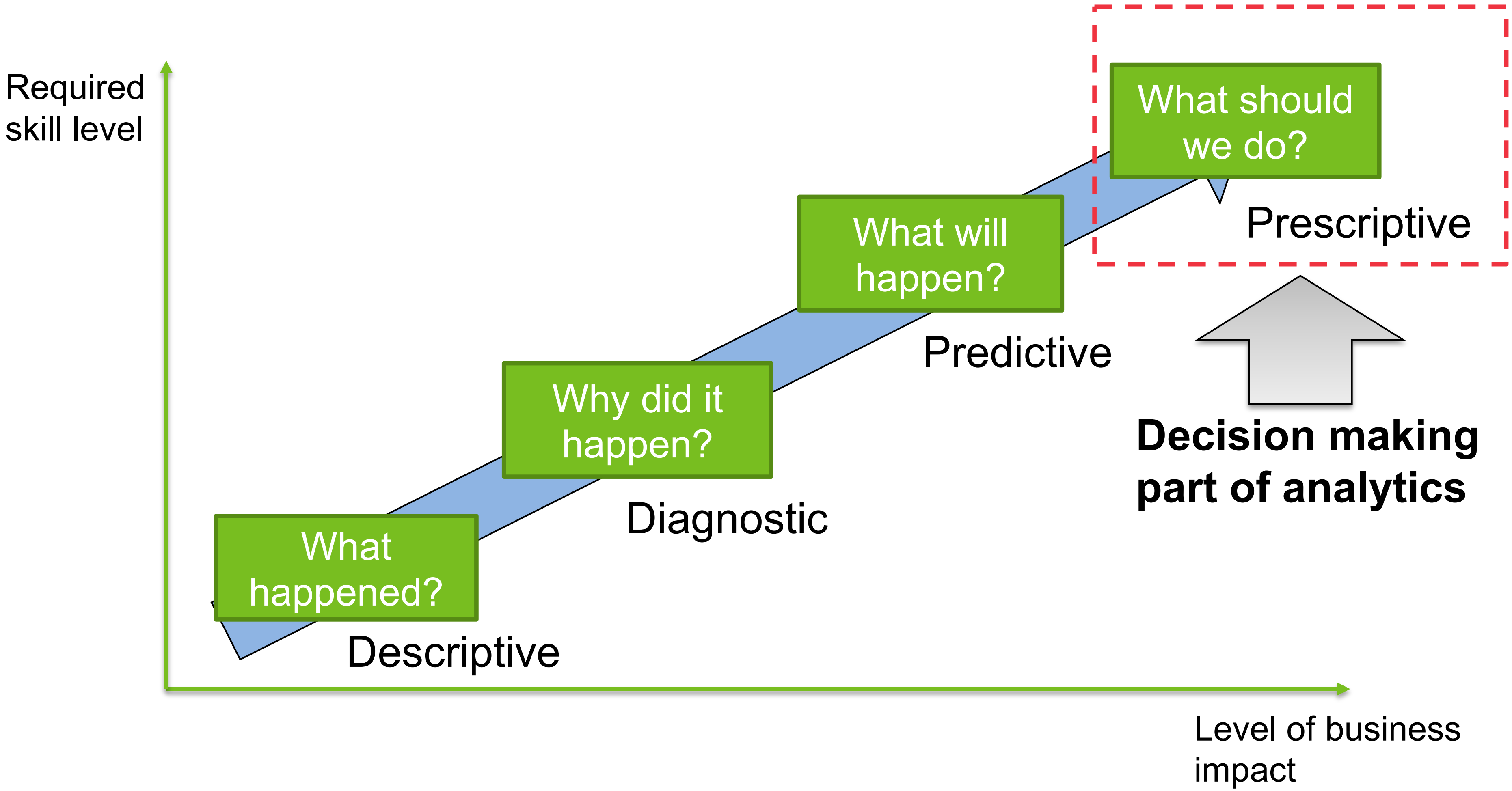
Pekka Malo & Eeva Vilkkumaa

Business Analytics

Business Analytics is the scientific process of transforming **data** into insight for making **better decisions**

Definition by the Institute for Operations Research and the Management Sciences (INFORMS)

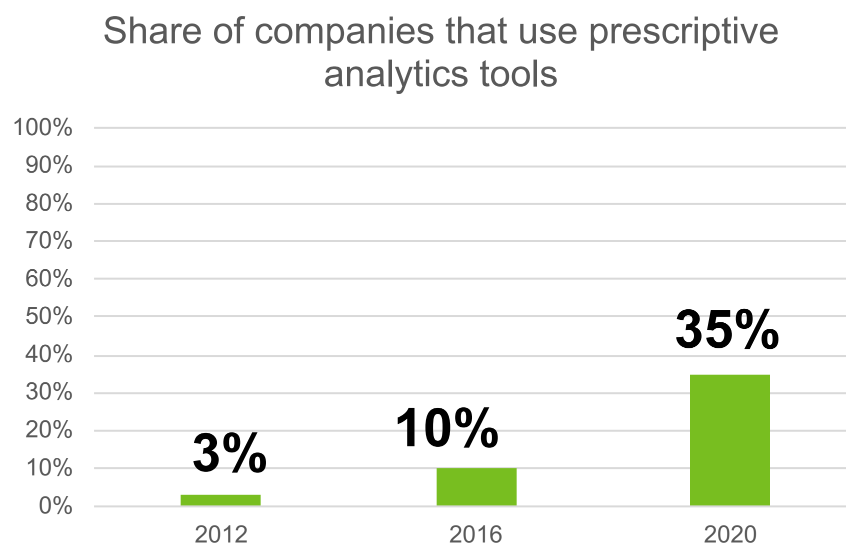
Four types of analytics



Source: Kart, L. "Advancing analytics", Gartner Inc. (2012)

Prescriptive analytics

- Prescriptive analytics tools help to quickly evaluate trillions of possible combinations of choices, and select the combination that makes the best use of scarce resources
- Hence, prescriptive analytics provides the largest business value
- Yet, it is only now gaining widespread adoption



Source: ”Forecast snapshot: Prescriptive analytics, worldwide”, Gartner Inc. (2016)

Optimization for prescriptive analytics

Objective: What do I want to achieve?

Mathematical representation:
Objective function $f(\boldsymbol{x})$ to be maximized or minimized

Decisions: What do I need to decide on? (And what can I decide on?)

Mathematical representation:
Vector $\boldsymbol{x}$ of decision variables

Restrictions: What do I have to adhere to?

Mathematical representation:
Constraints on the feasible set of decision variables $\boldsymbol{x}$

Inputs: What do I know? (And what would I need to know?) How certain is the information?

Mathematical representation:
Set $\boldsymbol{\theta}$ of model parameters

Mathematical optimization models

- Example: A company manufactures two products consisting entirely of three raw materials A, B and C. The shares of the raw materials in both products as well as their availabilities are shown in the table below. What are the optimal production quantities for the two products, when the profit from product 1 is 2 €/kg and that from product 2 is 3 €/kg?

- This can be formulated as an optimization problem:

$$\max 2x_1 + 3x_2$$

$$\text{s.t.} \quad 0.1x_1 + 0.55x_2 \leq 2500$$
$$0.7x_1 + 0.4x_2 \leq 3000$$
$$0.2x_1 + 0.05x_2 \leq 800$$
$$x_1, x_2 \geq 0$$

Objective function

Constraints

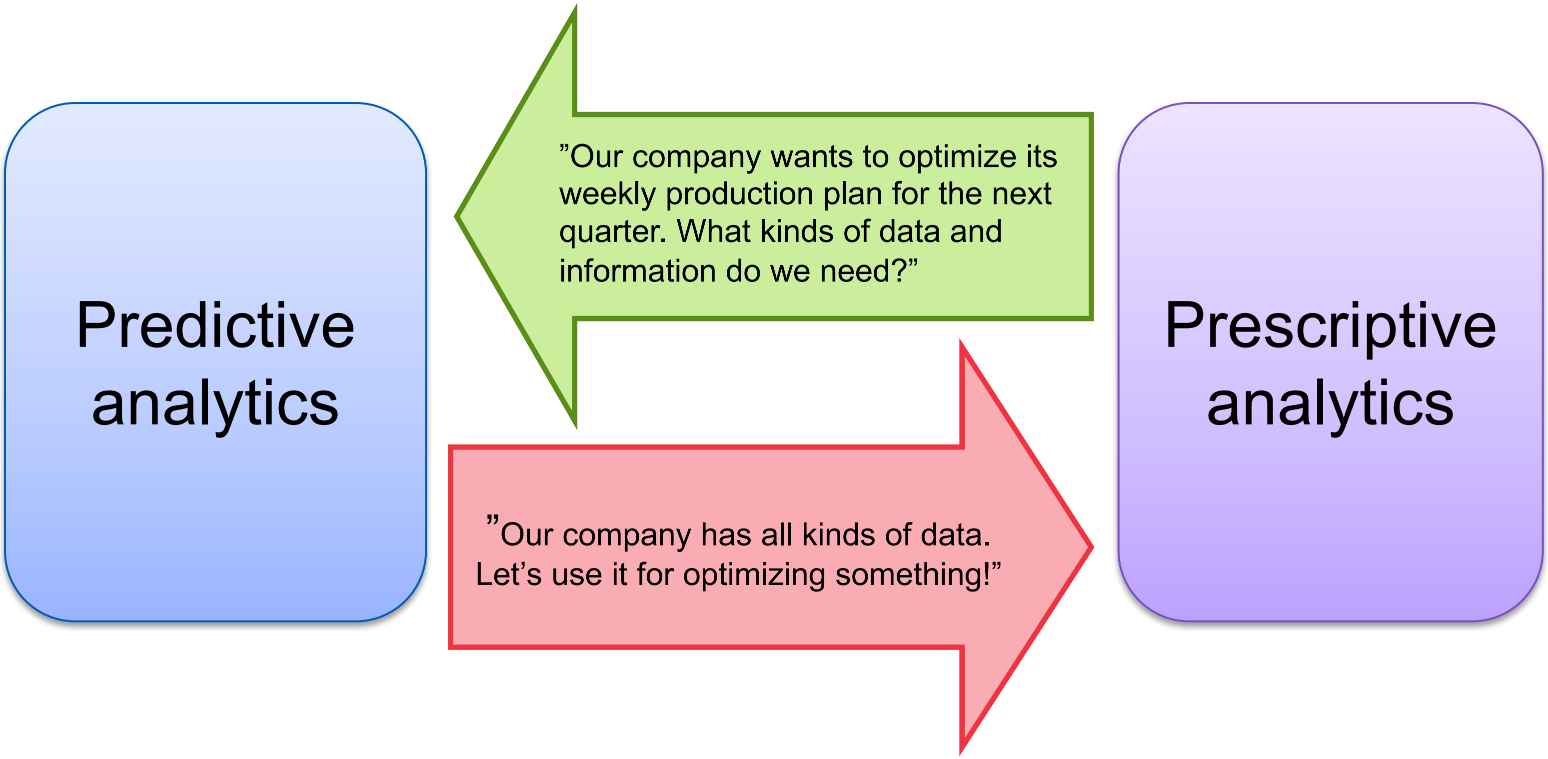
Decision variables x_1, x_2

	A	B	C	Profit
Product 1	10%	70%	20%	2
Product 2	55%	40%	5%	3
Availability	2500 kg	3000 kg	800 kg	

Parameters

Predictive vs. prescriptive analytics

- The parameters $\boldsymbol{\theta}$ of the prescriptive optimization model are often obtained through predictive analytics
- Yet, predictive analytics efforts should be planned to serve the objectives of the prescriptive model – not the other way around!



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Part 2: Linear optimization models

Production and inventory optimization
Distribution system design
Stochastic optimization

Linear optimization problems

- On this course, we will focus on **linear programming (LP) problems** in which both the objective function and constraints are linear in the decision variables

$$
\begin{aligned}
\max\ & 2x_1 + 3x_2 \\
\text{s.t.}\quad & 0.1x_1 + 0.55x_2 \leq 2500 \\
& 0.7x_1 + 0.4x_2 \leq 3000 \\
& 0.2x_1 + 0.05x_2 \leq 800 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

	A	B	C	Profit
Product 1	10%	70%	20%	2
Product 2	55%	40%	5%	3
Availability	2500 kg	3000 kg	800 kg	

- In this part we will demonstrate the flexibility of linear models to tackle business problems

Example 1: Production and inventory optimization

Contois Carpets is a small manufacturer of carpeting for home and office installations. Production capacity, estimated demand, production cost and inventory holding cost are shown in the below table. Contois wants to determine how many square meters of carpeting to produce each quarter to minimize the total production and inventory cost for the four-quarter period.

Quarter	Production capacity (m^2)	Estimated demand (m^2)	Production cost (€/m^2)	Inventory cost (€/m^2)
1	600	400	2	0.25
2	300	500	5	0.25
3	500	400	3	0.25
4	400	400	3	0.25

Source: Anderson et al. 2000, An Introduction to Management Science – Quantitative Approaches to Decision Making, South-Western College Publishing.

Example 1: Production and inventory optimization

Decision variables:

- x_t: Amount of carpet produced in quarter $t = 1, \dots, 4$
- s_t: Amount of carpet in inventory in quarter $t = 1, \dots, 4$

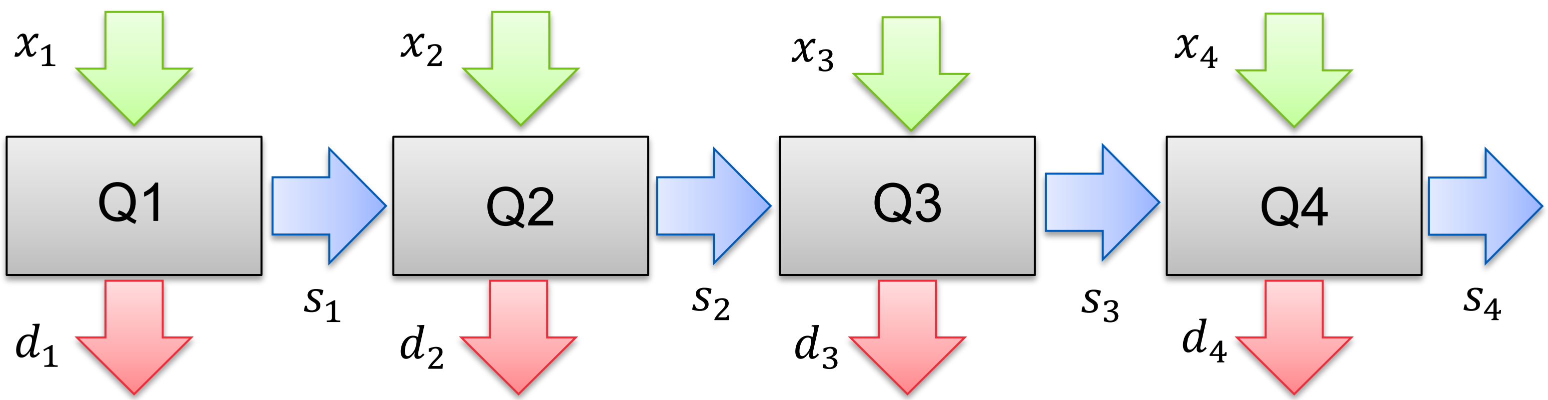
Objective function (total production & inventory cost):

$$2x_1 + 5x_2 + 3x_3 + 3x_4 + 0.25(s_1 + s_2 + s_3 + s_4)$$

Example 1: Production and inventory optimization

Constraints:

- Production amounts x_t are bounded by production capacities
- Production and inventory amounts are linked by the demand d_t in each quarter:



$d_1 = x_1 - s_1,\quad d_2 = s_1 + x_2 - s_2,\quad d_3 = s_2 + x_3 - s_3,\quad d_4 = s_3 + x_4 - s_4$

Example 1: Production and inventory optimization

LP formulation:

$$
\begin{aligned}
\min_{x_t, s_t} \quad & 2x_1 + 5x_2 + 3x_3 + 3x_4 + 0.25(s_1 + s_2 + s_3 + s_4) \\
\text{s.t.} \quad & x_1 \leq 600 \\
& x_2 \leq 300 \\
& x_3 \leq 500 \\
& x_4 \leq 400 \\
& x_1 - s_1 = 400 \\
& x_2 + s_1 - s_2 = 500 \\
& x_3 + s_2 - s_3 = 400 \\
& x_4 + s_3 - s_4 = 400 \\
& x_t, s_t \geq 0, \qquad t = 1, \ldots, 4
\end{aligned}
$$

Minimize total production & inventory cost

Production capacity constraints

Inventory / demand constraints

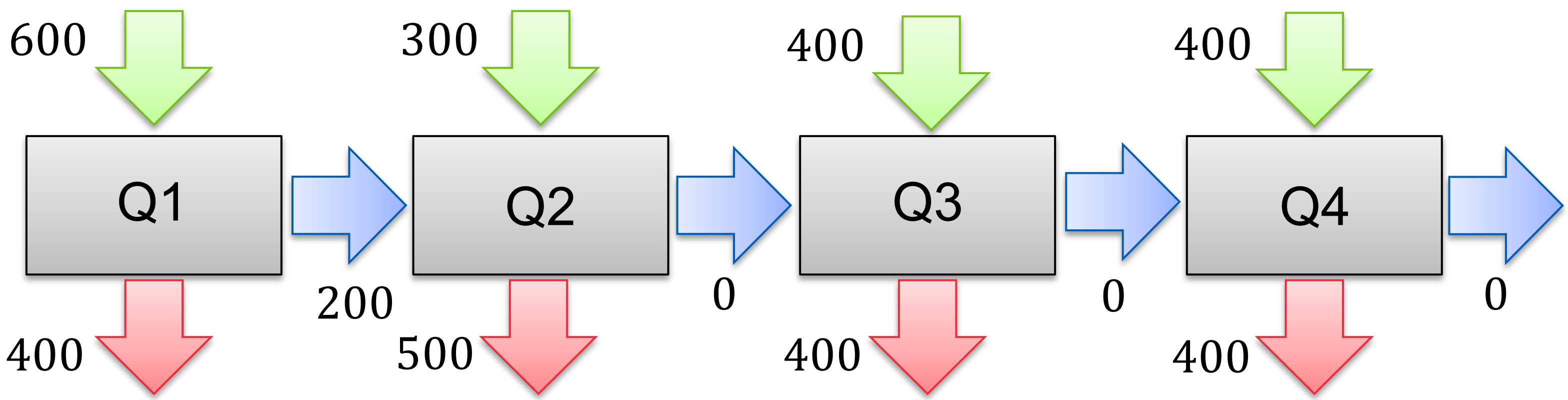
Non-negativity constraints

Example 1: Production and inventory optimization

The problem was solved with gurobi in Python (to be discussed in tutorial)

Solution:

- Optimal production amounts: $(x_1, x_2, x_3, x_4) = (600, 300, 400, 400)$
- Optimal inventory amounts: $(s_1, s_2, s_3, s_4) = (200, 0, 0, 0)$
- Optimal cost: 5,150 €



Example 1: Lessons learned

- The use of so-called *auxiliary decision variables* can be helpful in formulating the optimization problem

- E.g., in the production and inventory problem, the inventory variables could be eliminated by writing

$$s_1 = x_1 - 400 \qquad s_2 = x_2 + s_1 - 500 = x_1 + x_2 - 900$$

$$s_3 = x_1 + x_2 + x_3 - 1300 \qquad s_4 = x_1 + x_2 + x_3 + x_4 - 1700$$

- There is a trade-off between ease of formulation and the number of decision variables + constraints

- Yet, in continuous LP problems the number of decision variables + constraints is rarely a problem from the computational point of view

Example 2: Distribution system design

The Martin-Beck Company operates a plant in St. Louis with an annual capacity of 30,000 units. Product is shipped from the plant to regional distribution centers located in Boston, Atlanta, and Houston. Because of an anticipated increase in demand, Martin-Beck plans to increase capacity by constructing a new plant in one or more of the following cities: Dallas, Fort Worth, Denver, or Kansas City. Dallas and Fort Worth are very close to one another, whereby the company does not want to have a plant in both cities. The estimated annual fixed costs and capacities for the four proposed plants are as follows:

Proposed plant	Annual Fixed Cost	Annual Capacity
Dallas	$175,000	10,000
Fort Worth	$300,000	20,000
Denver	$375,000	30,000
Kansas City	$500,000	40,000

Source: Anderson et al. 2000, An Introduction to Management Science – Quantitative Approaches to Decision Making, South-Western College Publishing.

Example 2: Distribution system design

The company's long-range planning group has developed forecasts of the anticipated annual demand at the distribution centers as follows:

Distribution center	Annual demand
Boston	30,000
Atlanta	20,000
Houston	20,000

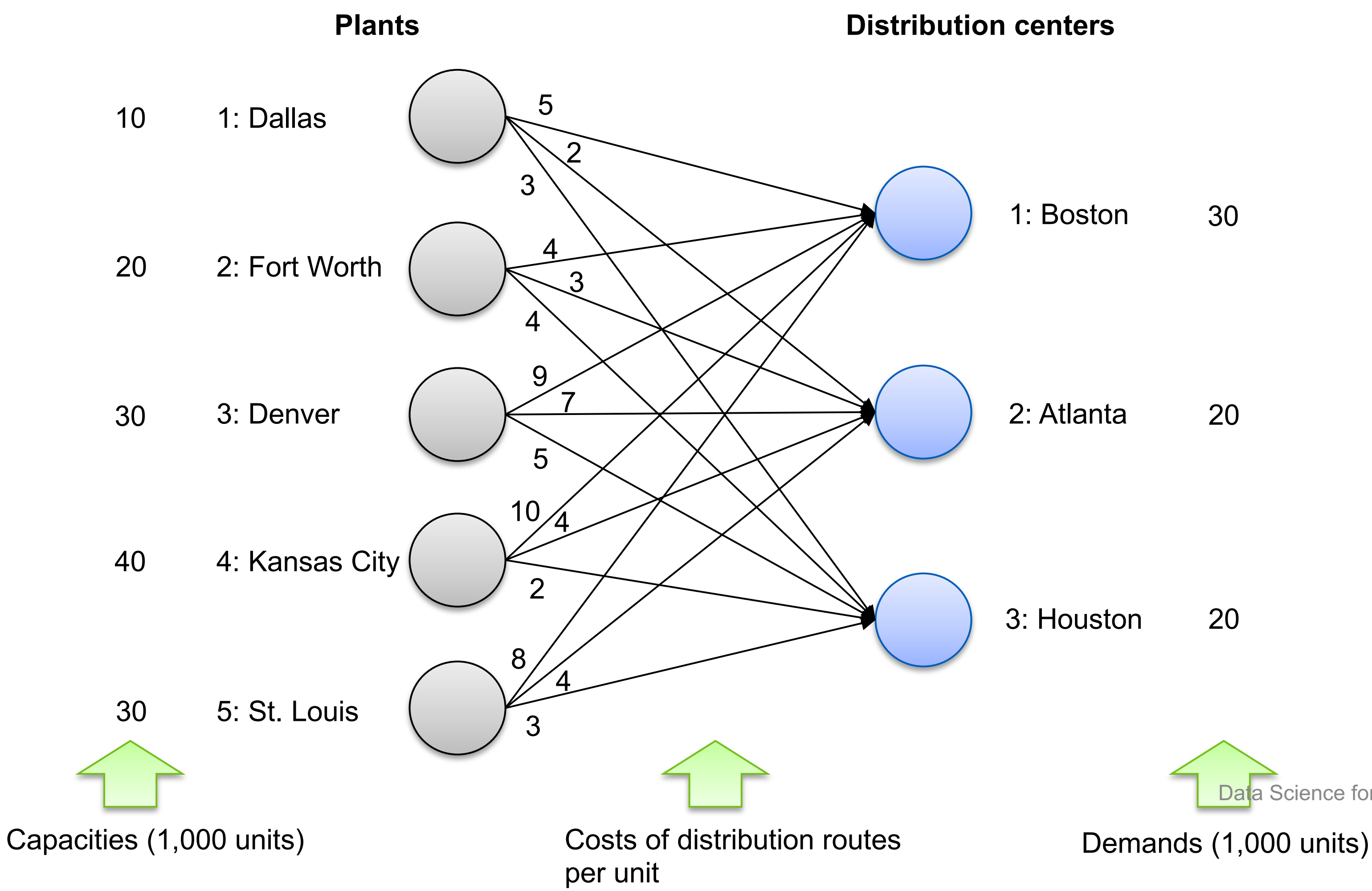
The shipping cost per unit from each plant to each distribution center are as follows:

	Distribution centers		
Plant site	Boston	Atlanta	Houston
Dallas	5	2	3
Fort Worth	4	3	4
Denver	9	7	5
Kansas City	10	4	2
St. Louis	8	4	3

In which city/cities should the Martin-Beck Company construct its new plant/plants?

Example 2: Distribution system design

Network representation of the problem:



Example 2: Distribution system design

Decision variables:

- $y_i \in \{0,1\}$: A binary variable indicating whether a plant is built in city i ($y_i = 1$) or not ($y_i = 0$)
- $x_{ij} \in \mathbb{R}^+$: A continuous variable indicating the amount shipped from plant in city i to distribution center in city j
- A linear optimization problem with both discrete and continuous decision variables is referred to as a **Mixed-Integer Linear Programming (MILP)** problem

Objective function:

- Annual shipping cost: $5x_{11} + 2x_{12} + 3x_{13} + 4x_{21} + 3x_{22} + 4x_{23} + 9x_{31} + 7x_{32} + 5x_{33} + 10x_{41} + 4x_{42} + 2x_{43} + 8x_{51} + 4x_{52} + 3x_{53}$

- Annual fixed costs of operating the new plant / plants:

$$175000y_1 + 300000y_2 + 375000y_3 + 500000y_4$$

Example 2: Distribution system design

Constraints:

- Shipping amounts are bounded by production capacities
- Shipping from a given plant i can only happen if the plant has been built
- E.g., capacity constraint for a plant in Dallas:

Sum of shipments from Dallas ($i = 1$)

$$x_{11} + x_{12} + x_{13} \leq 10000y_1$$

Sum of shipments is (smaller than or) equal to 0, if there is no plant in Dallas ($y_1 = 0$)

Sum of shipments is smaller than or equal to 10,000 units, if there is a plant in Dallas ($y_1 = 1$)

- Shipments to distribution center j from different plants i must equal the demand at distribution center j
- E.g., demand constraint for a distribution center in Boston:

Sum of shipments to Boston ($j = 1$)

$$x_{11} + x_{21} + x_{31} + x_{41} + x_{51} = 30000$$

Demand in Boston ($j = 1$) is 30,000 units

- There cannot be a plant in both Dallas ($i = 1$) & Fort Worth ($i = 2$):

$$y_1 + y_2 \leq 1$$

Example 2: Distribution system design

MILP formulation:

Minimize total transportation & operational costs

$$\min_{x_t, i_t} \quad 5x_{11} + 2x_{12} + 3x_{13} + 4x_{21} + 3x_{22} + 4x_{23} + 9x_{31} + 7x_{32} + 5x_{33} + 10x_{41} + 4x_{42} + 2x_{43} + 8x_{51} + 4x_{52} + 3x_{53} + 175000y_1 + 300000y_2 + 375000y_3 + 500000y_4$$

s.t.

$$
\begin{aligned}
& x_{11} + x_{12} + x_{13} - 10000y_1 \le 0 \\
& x_{21} + x_{22} + x_{23} - 20000y_2 \le 0 \\
& x_{31} + x_{32} + x_{33} - 30000y_3 \le 0 \\
& x_{41} + x_{42} + x_{43} - 40000y_4 \le 0 \\
& x_{51} + x_{52} + x_{53} \le 30000
\end{aligned}
$$

Plant capacity constraints

$$
\begin{aligned}
& x_{11} + x_{21} + x_{31} + x_{41} + x_{51} = 30000 \\
& x_{12} + x_{22} + x_{32} + x_{42} + x_{52} = 20000 \\
& x_{13} + x_{23} + x_{33} + x_{43} + x_{53} = 20000
\end{aligned}
$$

Demand constraints

$$y_1 + y_2 \le 1$$

No plant in both Da & FW

$$
\begin{aligned}
& x_{ij} \ge 0, \quad i = 1, \dots, 5, \quad j = 1, \dots, 3 \\
& y_i \in \{0,1\}, \quad i = 1, \dots, 5
\end{aligned}
$$

Non-negativity & binary constraints

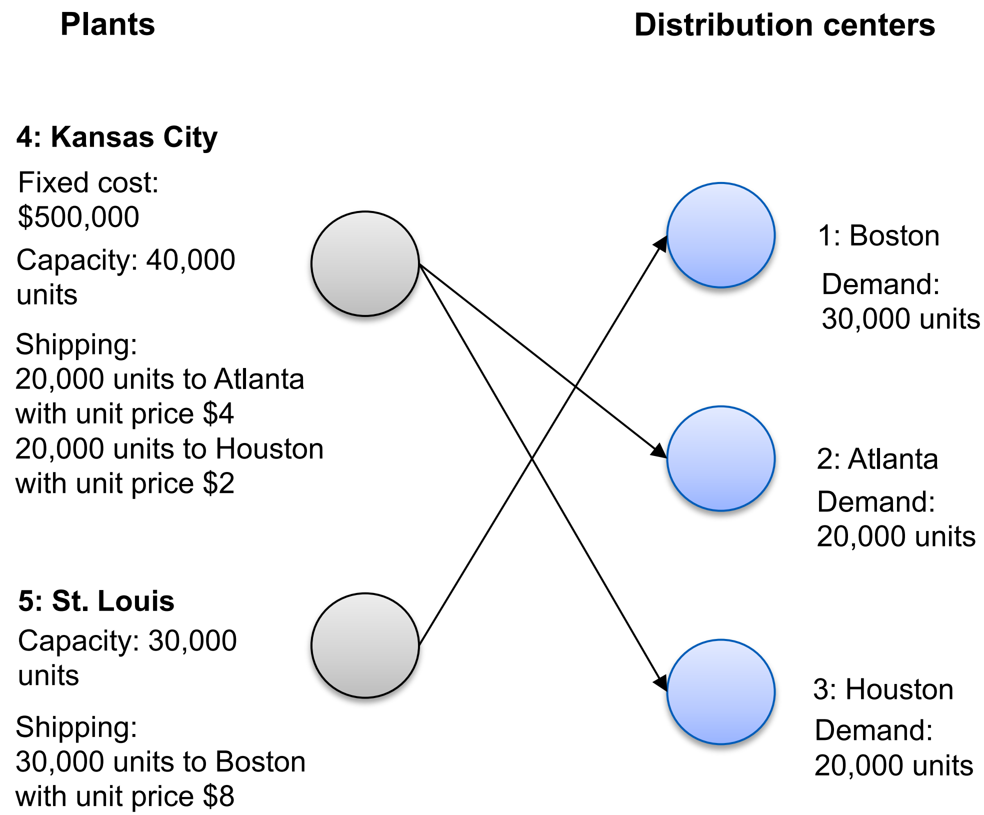
Example 2: Distribution system design

Solution:

- Optimal plant locations: $(y_1, y_2, y_3, y_4) = (0,0,0,1) \rightarrow$ $x_{ij} = 0$ for all i=1,2,3
- Optimal shipping amounts (in 1000 units)

$$(x_{41}, x_{42}, x_{43}) = (0, 20, 20)$$
$$(x_{51}, x_{52}, x_{53}) = (30, 0, 0)$$

- Optimal cost: \$860,000



Example 2: Lessons learned

- The selection of decision variables is not always obvious: To optimize plant locations (y_i), we also need to optimize shipping (x_{ij})

- Binary decision variables provide flexibility for linear optimization models
  - Conditional constraints: ”Production is at most 10,000 units if a plant is built, and otherwise zero” $\rightarrow x_{11} + x_{12} + x_{13} \leq 10000y_1$
  - Multiple choice constraints: ”Choose exactly one city in which to build a plant” $\rightarrow y_1 + y_2 + y_3 + y_4 = 1$
  - Mutually exclusive constraints: ”There should be no plant in both Dallas and Fort Worth” $\rightarrow y_1 + y_2 \leq 1$
  - k out of n constraints: ”At most two plants should be built” $\rightarrow y_1 + y_2 + y_3 + y_4 \leq 2$

- MILP problems are considerably more difficult to solve than LPs

- Yet, faster and more efficient algorithms for solving MILPs are constantly being developed

Example 3: Stochastic optimization

- In the previous two examples, point estimates for uncertain demands, costs etc. were taken at face value
- Often, such estimates produced by predictive analytics are uncertain
- One way to model parameter uncertainty is through a **scenario tree**
- Scenario trees are also useful, when decisions can be made **sequentially** after having observed how the uncertainties unfold
- Optimal decision sequences or strategies can be solved by **stochastic optimization** methods

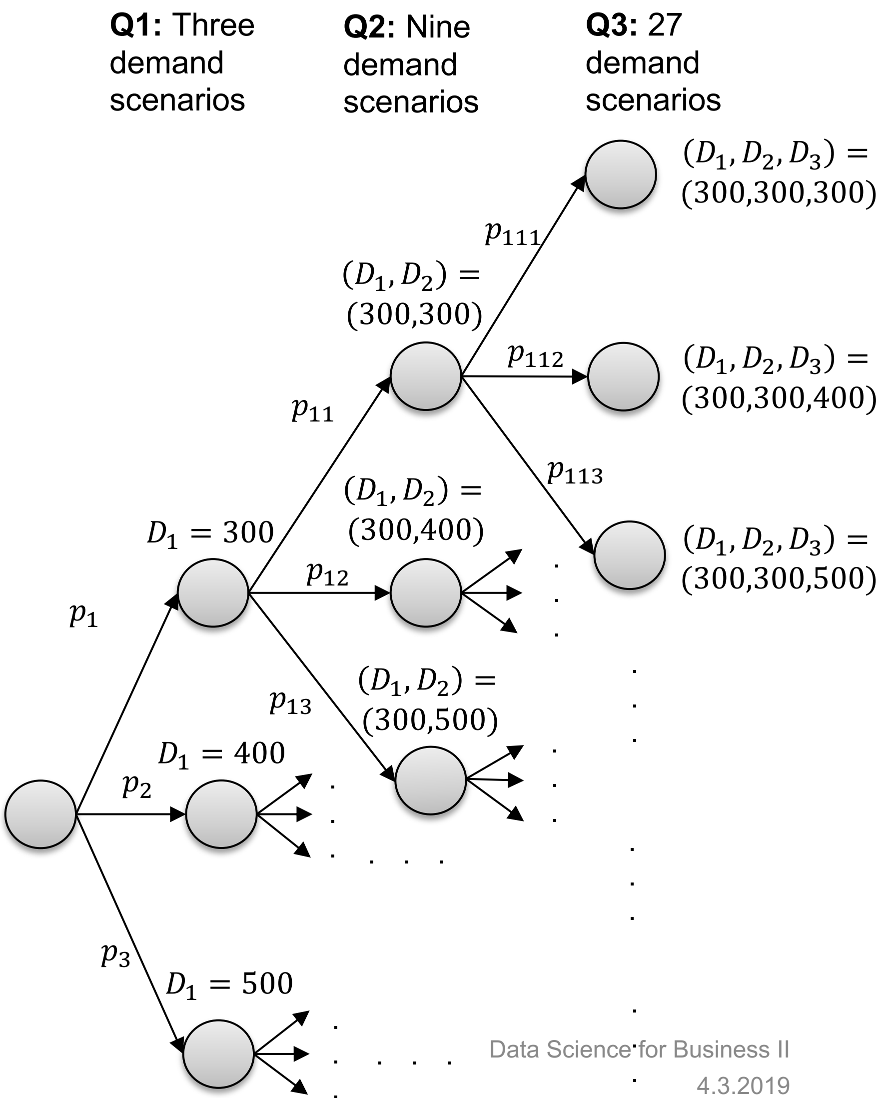
Example 3: Stochastic optimization

Let us revisit Example 1 (production and inventory optimization at Contois Carpets). For illustrative purposes, we will only focus on the first three periods. Assume that the production capacity, production cost and inventory cost are as follows:

Quarter	Production capacity (m^2)	Production cost (€/m^2)	Inventory cost (€/m^2)
1	700	2	0.25
2	400	5	0.25
3	500	3	0.25

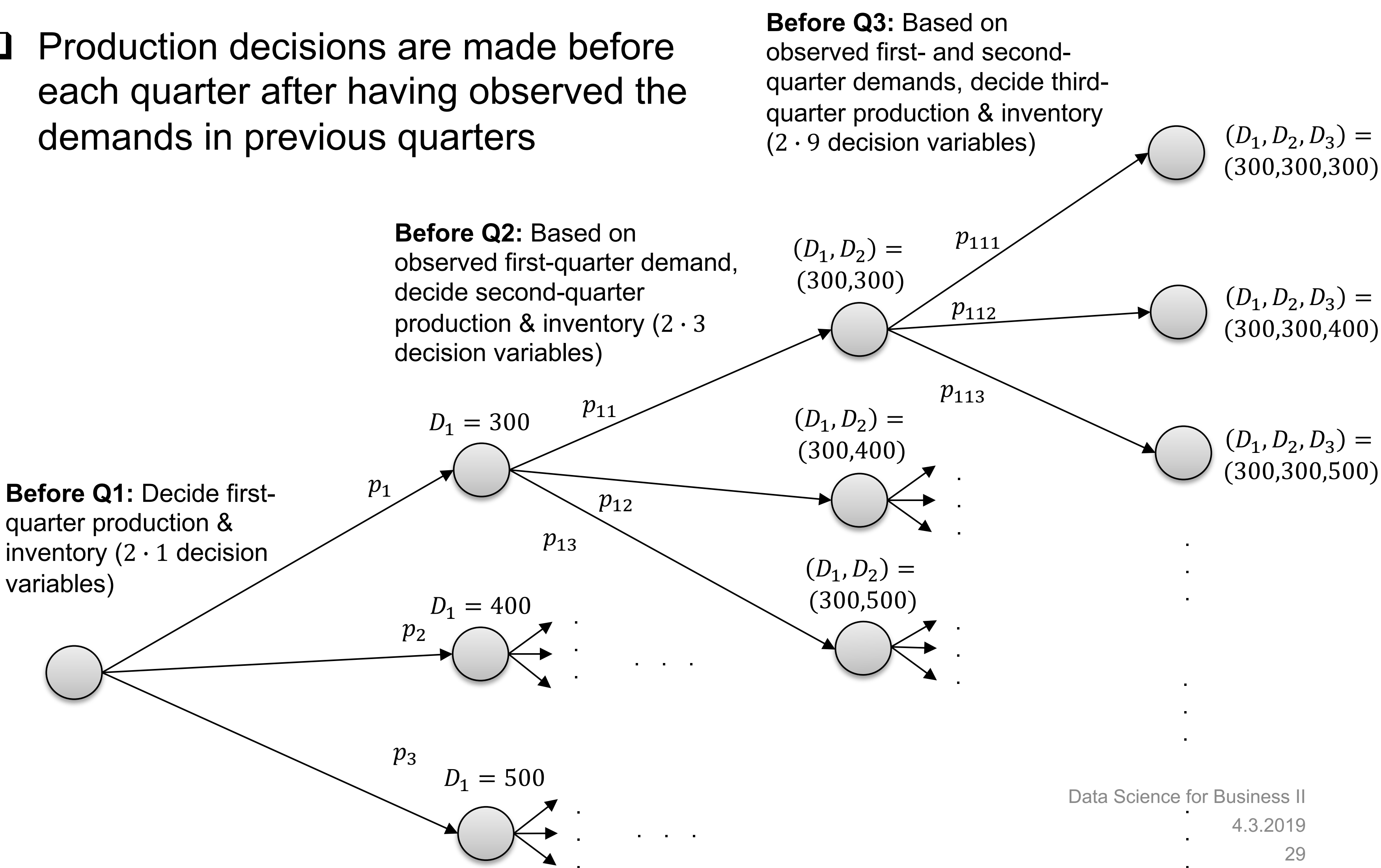
Example 3: Stochastic optimization

- Demand in each quarter is a **discrete random variable** $D_t \in \{d_1, d_2, d_3\} = \{300,400,500\}$.
- The probabilities of different demand values in each quarter depend on the observed demands in previous quarters.
- The evolution of the demand can be represented by a **scenario tree**, where
  - p_i = probability that the demand in quarter 1 is d_i
  - p_{ij} = probability that the demand in quarter 2 is d_j on the condition that the demand in quarter 1 was d_i
  - p_{ijk} = probability that the demand in quarter 3 is d_k on the condition that the demands in quarters 1 and 2 were d_i and d_j, respectively



Example 3: Stochastic optimization

- Production decisions are made before each quarter after having observed the demands in previous quarters



Example 3: Stochastic optimization

- Scenario probabilities are obtained by simulating trajectories using a time-series model for the uncertain demand
  - Demand = 300 m^2
  - Demand = 400 m^2
  - Demand = 500 m^2

p_i	p_{ij}	p_{ijk}
20 %	40 %	60 %
		30 %
		10 %
	35 %	40 %
		40 %
		20 %
	25 %	10 %
		30 %
		60 %
60 %	20 %	35 %
		45 %
		20 %
	60 %	20 %
		60 %
		20 %
	20 %	20 %
		45 %
		35 %
20 %	15 %	30 %
		45 %
		25 %
	30 %	35 %
		35 %
		30 %
	55 %	15 %
		20 %
		65 %

Example 3: Stochastic optimization

Decision variables:

- x: Amount of carpet produced in quarter 1 (one variable)
- x_i: Amount of carpet produced in quarter 2, given that the demand in quarter 1 was d_i (three variables)
- x_{ij}: Amount of carpet produced in quarter 3, given that the demands in quarters 1 and 2 were d_i and d_j, respectively (nine variables)

- s: Expected amount of carpet in inventory in quarter 1 (one variable)
- s_i: Expected amount of carpet in inventory in quarter 2, given that the demand in quarter 1 was d_i (three variables)
- s_{ij}: Expected amount of carpet in inventory in quarter 3, given that the demands in quarters 1 and 2 were d_i and d_j, respectively (nine variables)

→ **26 decision variables**

Example 3: Stochastic optimization

Objective function (total expected production & inventory cost):

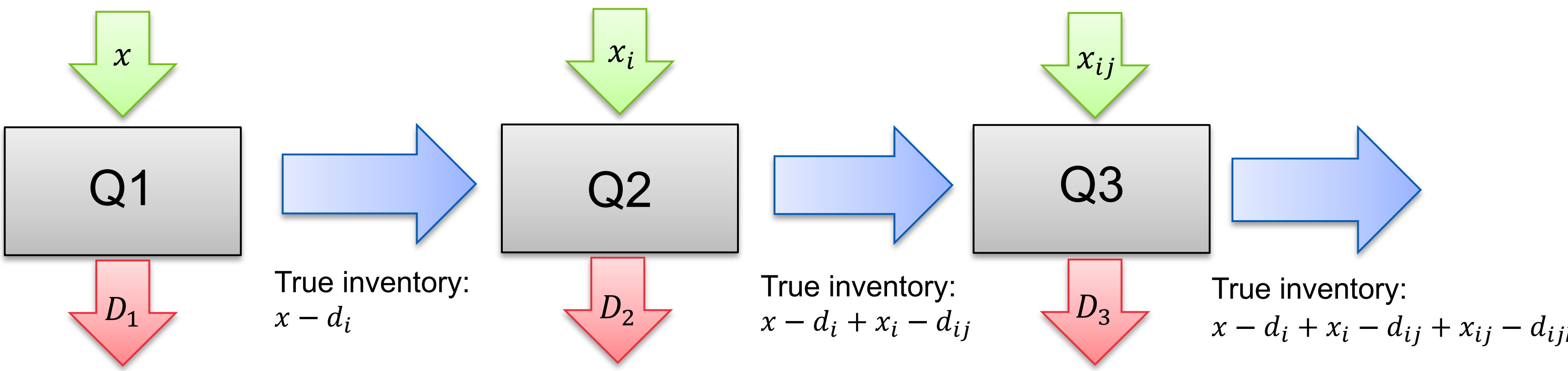
$$2x + 5\sum_{i=1}^{3} p_i x_i + 3\sum_{i=1}^{3} p_i \sum_{j=1}^{3} p_{ij} x_{ij} + 0.25\left(s + \sum_{i=1}^{3} p_i s_i + \sum_{i=1}^{3} p_i \sum_{j=1}^{3} p_{ij} s_{ij}\right)$$

- Expected second-quarter production
- Expected third-quarter production
- Expected second-quarter inventory
- Expected third-quarter inventory

Example 3: Stochastic optimization

Constraints:

- Production amounts x, x_i, x_{ij} are bounded by production capacities
- Production and inventory amounts are linked by the observed demands in previous quarters, and expected demand in the present quarter:



$E[D_1] = x - s,$

Match production amount x and expected inventory amount s with expected demand $E[D_1] = \sum_{i=1}^{3} p_i d_i$

$E[D_2|D_1 = d_i] = x - d_i + x_i - s_i,$

Match production amount x_i, expected inventory amount s_i, and observed inventory amount $x - d_i$ with expected demand $E[D_2|D_1 = d_i] = \sum_{j=1}^{3} p_{ij} d_{ij}$

$$E[D_3|D_1 = d_i, D_2 = d_j] = x - d_i + x_i - d_{ij} + x_{ij} - s_{ij},$$

Match production amount x_{ij}, expected inventory amount s_{ij}, and observed inventory amount $x - d_i + x_i - d_{ij}$ with expected demand $E[D_3|D_1 = d_i, D_2 = d_j] = \sum_{k=1}^{3} p_{ijk} d_{ijk}$

Example 3: Stochastic optimization

Minimize total expected production & inventory cost

LP formulation:

$$\min_{x_t, s_t} \; 2x + 5\sum_{i=1}^{3} p_i x_i + 3\sum_{i=1}^{3} p_i \sum_{j=1}^{3} p_{ij} x_{ij} + 0.25\left(s + \sum_{i=1}^{3} p_i s_i + \sum_{i=1}^{3} p_i \sum_{j=1}^{3} p_{ij} s_{ij}\right)$$

$$\begin{aligned}
\text{s.t.} \quad & x && \leq 700 \\
& x_i && \leq 400 \\
& x_{ij} && \leq 500 \\
& x - s && = \sum_{i=1}^{3} p_i d_i \\
& x + x_i - s_i && = d_i + \sum_{j=1}^{3} p_{ij} d_{ij} \\
& x + x_i + x_{ij} - s_{ij} && = d_i + d_{ij} + \sum_{k=1}^{3} p_{ijk} d_{ijk} \\
& x, x_i, x_{ij}, s, s_i, s_{ij} \geq 0
\end{aligned}$$

Production capacity constraints (13 pc.)

Inventory constraints (13 pc.)

Non-negativity constraints (26 pc.)

Example 3: Stochastic optimization

Results:

- Demand = 300 m^2 (lightest green)
- Demand = 400 m^2 (medium green)
- Demand = 500 m^2 (dark green)

❑ Expected cost of the optimal production strategy: 3260.45 €

Quarter	Production capacity (m^2)	Production cost (€/m^2)	Inventory cost (€/m^2)
1	700	2	0.25
2	400	5	0.25
3	500	3	0.25

Q1	Q2	Q3
$x = 700$, $s = 300$	$x_1 = 50$, $s_1 = 65$	$x_{11} = 200$, $s_{11} = 0$
		$x_{12} = 330$, $s_{12} = 0$
		$x_{13} = 500$, $s_{13} = 0$
	$x_2 = 115$, $s_2 = 15$	$x_{21} = 270$, $s_{21} = 0$
		$x_{22} = 385$, $s_{22} = 0$
		$x_{23} = 500$, $s_{23} = 0$
	$x_3 = 250$, $s_3 = 10$	$x_{31} = 245$, $s_{31} = 0$
		$x_{32} = 345$, $s_{32} = 0$
		$x_{33} = 500$, $s_{33} = 0$

Agenda for tutorial

- ❑ We will familiarize ourselves with gurobi optimization package for Python through the examples presented today
- ❑ We will take a look at Assignment 2
- ❑ Questions or comments?

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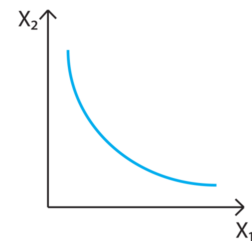
Part 3: Beyond linear optimization

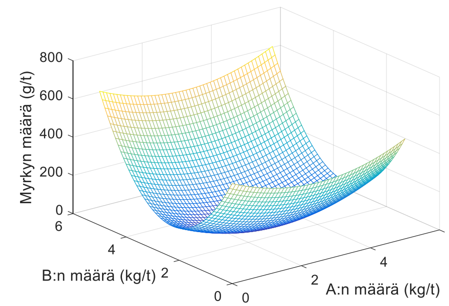
To be taken on a nice-to-know basis

Convex optimization problems

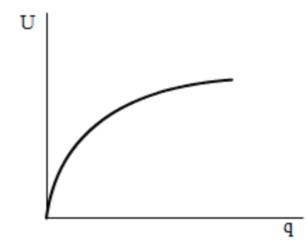
- A more general class of problems consists of *convex optimization problems,* in which
  - The objective function f is convex if minimized, or concave if maximized
  - Inequality constraints $g_i(\boldsymbol{x}) \leq 0$ are convex
  - Equality constraints $h_i(\boldsymbol{x}) = 0$ are linear
- Like linear problems, these problems are fairly easy to solve because a local optimum is also a global optimum

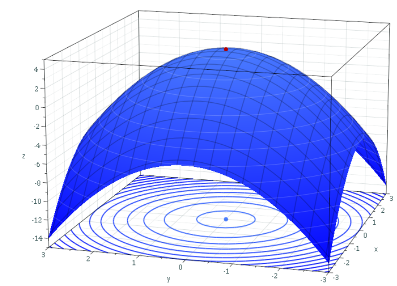
Convex functions



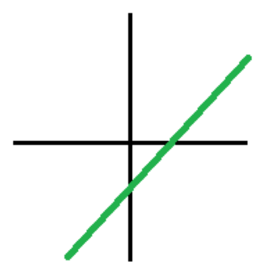


Concave functions

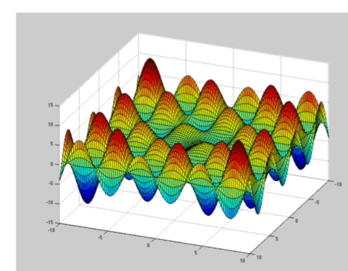




Convex & concave = linear function



Neither convex nor concave



Convex optimization problems

- Convex models can be used to accommodate risk considerations, either through minimizing risk or imposing constraints on risk
- E.g., Markowitz portfolio model:
  - Consider n risky assets $i = 1, \ldots, n$ with expected returns r_i and covariances σ_{ij}
  - Find the portfolio of assests (represented by shares w_i of funds allocated to each asset) that minimizes portfolio risk subject to a constraint on the expected portfolio return

$$\begin{aligned} \min_{w_i} \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} \\ \text{s.t.} \quad & \sum_{i=1}^{n} w_i r_i = R \\ & \sum_{i=1}^{n} w_i = 1 \\ & w_i \geq 0 \end{aligned}$$

Non-convex problems

- Many algorithms have been developed to find the global optimum for non-convex problems

- The performance of these algorithms depends on the problem – no 'one size fits all' algorithm exists
  – See http://www.denizyuret.com/2015/03/alec-radfords-animations-for.html