Exercise 1 - Sample Solutions *6 March 2019*

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1. (a) (A repetition of basic electromagnetic field theory course. Remind yourself of Maxwell equations and vector calculus—for example, David K. Cheng's book *Field and Wave Electromagnetics*.)

A Hertzian dipole which is located in the origin of the coordinate system in free space (there are no sources except this dipole), and which has the z-directed dynamic dipole moment IL, radiates electromagnetic vectorial field which depends on the spherical coordinates (r,θ,φ) in the following way. The electric field $\mathbf{E}$ has two components:

$$\mathbf{E}(\mathbf{r}) = \mathbf{u}_r E_r + \mathbf{u}_\theta E_\theta,$$

which read

$$E_r(\mathbf{r}) = \mathrm{j}\omega\mu_0\, 2IL\,\cos\theta \left(\frac{1}{\mathrm{j}kr} + \frac{1}{(\mathrm{j}kr)^2}\right)\frac{e^{-\mathrm{j}kr}}{4\pi r} \quad (1)$$

$$E_\theta(\mathbf{r}) = \mathrm{j}\omega\mu_0\, IL\,\sin\theta \left(1 + \frac{1}{\mathrm{j}kr} + \frac{1}{(\mathrm{j}kr)^2}\right)\frac{e^{-\mathrm{j}kr}}{4\pi r} \quad (2)$$

with the angular frequency of radiation ω and the free-space wave number $k = \omega\sqrt{\mu_0\varepsilon_0}$.

An alternative representation of an electric multipole field is by utilizing the multipole decomposition that appears in David Jackson's book "Classical Electrodynamics". There the electric field of a z directed dipole distribution reads:

$$\mathbf{E} = \frac{-\mathrm{j}a_\mathrm{e}}{\omega\varepsilon_0}\nabla\times\left(h_1^{(2)}(kr)\mathbf{X}_{10}(\theta,\varphi)\right) \quad (3)$$

where a_e is the scattering amplitude and $\mathbf{X}_{10}(\theta,\varphi) = -\frac{\mathrm{j}}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\mathbf{u}_\varphi$. The function $h_1^{(2)}(kr)$ is the spherical Hankel function of the second kind and first order.

i. Evaluate the field components (r,θ) of expression (3). Compare them with Eqs. (1) and (2). What do you observe?

ii. The series expressions of the spherical Hankel function is

$$h_n^{(2)}(x) = (\mathrm{j})^{n+1}\frac{e^{-\mathrm{j}x}}{x}\sum_{m=0}^{n}\frac{(-\mathrm{j})^m}{m!(2x)^m}\frac{(n+m)!}{(n-m)!} \quad (4)$$

Substitute the Hankel function in the results of question i: what do you observe now?

iii. Finally, evaluate a_e as a function of IL. Which is the unit of a_e?

Write down each step in your calculation. Use the vector differential calculus formulas for curl in the spherical coordinate system:

$$\nabla\times\mathbf{f} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{u}_r & r\mathbf{u}_\theta & r\sin\theta\mathbf{u}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ f_r & rf_\theta & r\sin\theta f_\varphi \end{vmatrix}$$

(b) Given the following complex vector $\mathbf{E}$ (expanded with cartesian unit vectors $\mathbf{u}_{x,y,z}$, and where j is the imaginary unit):

$$\mathbf{E} = (2+\mathrm{j})\mathbf{u}_x + \mathbf{u}_y - 3\mathrm{j}\,\mathbf{u}_z$$

In time domain, the vector draws an ellipse. Find

i. the directions of the major and minor axes of the ellipse,
ii. the amplitudes of the major and minor axes, and
iii. the axis ratio.

(c) Study carefully sections 1.1–1.4 from the textbook. Remember that the "ordinary" algebraic rules for real vectors may not apply for complex vectors.)

Analyze the properties of an arbitrary complex vector **b**. This vector can be split into a linearly and a circularly polarized part with expressions

$$\mathbf{b}_{\mathrm{CP}} = \frac{\mathbf{b}\times(\mathbf{b}\times\mathbf{b}^*) + \mathrm{j}\sqrt{\mathbf{b}\cdot\mathbf{b}}\ \mathbf{b}\times\mathbf{b}^*}{|\mathbf{b}|^2 + |\mathbf{b}\cdot\mathbf{b}|} \tag{5}$$

and

$$\mathbf{b}_{\mathrm{LP}} = \frac{\sqrt{\mathbf{b}\cdot\mathbf{b}}}{|\mathbf{b}|^2 + |\mathbf{b}\cdot\mathbf{b}|}\left(\sqrt{\mathbf{b}^*\cdot\mathbf{b}^*}\ \mathbf{b} + \sqrt{\mathbf{b}\cdot\mathbf{b}}\ \mathbf{b}^* - \mathrm{j}\mathbf{b}\times\mathbf{b}^*\right) \tag{6}$$

Check that the above vectors satisfy the requirements

i. $\mathbf{b} = \mathbf{b}_{\mathrm{LP}} + \mathbf{b}_{\mathrm{CP}}$
ii. $\mathbf{b}_{\mathrm{LP}} \times \mathbf{b}_{\mathrm{LP}}^* = 0$ (property of linear polarization)
iii. $\mathbf{b}_{\mathrm{CP}} \cdot \mathbf{b}_{\mathrm{CP}} = 0$ (property of circular polarization)
iv. $\mathbf{b}_{\mathrm{LP}} \cdot \mathbf{b}_{\mathrm{CP}} = 0$ (decoupling of components)
v. $|\mathbf{b}|^2 = |\mathbf{b}_{\mathrm{CP}}|^2 + |\mathbf{b}_{\mathrm{LP}}|^2$ (power orthogonality)

As a numerical example, verify the above requirements using $\mathbf{b} = \mathbf{E} = (2+\mathrm{j})\mathbf{u}_x + \mathbf{u}_y - 3\mathrm{j}\,\mathbf{u}_z$ from the previous problem.

Sample Solutions

1. (a) i. In a straightforward manner and after some careful calculations we obtain

$$\mathbf{E} = -\frac{a_\mathrm{e}}{2\omega\varepsilon_0}\sqrt{\frac{3}{2\pi}}\nabla\times\left(h_1^{(2)}(kr)\sin\theta\mathbf{u}_\varphi\right) \tag{7}$$

We evaluate the curl in spherical coordinate system, i.e.,

$$\begin{aligned}
\nabla\times\left(h_1^{(2)}(kr)\sin\theta\mathbf{u}_\varphi\right) &= \frac{2r\sin\theta}{r^2\sin\theta}h_1^{(2)}(kr)\cos\theta\mathbf{u}_r - \frac{r\sin\theta}{r^2\sin\theta}\frac{\partial}{\partial r}\left(rh_1^{(2)}(kr)\right)\sin\theta\mathbf{u}_\theta \\
&= 2\frac{h_1^{(2)}(kr)}{r}\cos\theta\mathbf{u}_r - \left(\frac{h_1^{(2)}(kr)}{r} + \frac{d}{dr}h_1^{(2)}(kr)\right)\sin\theta\mathbf{u}_\theta \\
&= 2\frac{h_1^{(2)}(kr)}{r}\cos\theta\mathbf{u}_r - \left(\frac{h_1^{(2)}(kr)}{r} + kh_0^{(2)}(kr) - 2\frac{h_1^{(2)}(kr)}{r}\right)\mathbf{u}_\theta \\
&= 2\frac{h_1^{(2)}(kr)}{r}\cos\theta\mathbf{u}_r - \frac{1}{r}\left(krh_0^{(2)}(kr) - h_1^{(2)}(kr)\right)\mathbf{u}_\theta
\end{aligned}$$

Combining this to the final results we obtain:

$$\mathbf{E} = -\sqrt{\frac{3}{2\pi}}\frac{a_\mathrm{e}}{2r\omega\varepsilon_0}\left(2h_1^{(2)}(kr)\cos\theta\mathbf{u}_r - \left(krh_0^{(2)}(kr) - h_1^{(2)}(kr)\right)\mathbf{u}_\theta\right) \tag{8}$$

Comparison between the components of Eq. (1), (2), and Eq. (8) reveals that the new representation is more compact and easy to memorize since it requires only the use of the Hankel functions!

ii. Employing the given expression for Hankel function we have:

$$h_1^{(2)}(kr) = -\mathrm{j}\left(\frac{1}{\mathrm{j}kr} + \frac{1}{(\mathrm{j}kr)^2}\right)e^{-\mathrm{j}kr} \tag{9}$$

and

$$h_1^{(2)}(kr) - krh_0^{(2)}(kr) = -\mathrm{j}\left(1 + \frac{1}{\mathrm{j}kr} + \frac{1}{(\mathrm{j}kr)^2}\right)e^{-\mathrm{j}kr} \tag{10}$$

By combining the above to Eq.(8) we obtain

$$\mathbf{E} = \mathrm{j}\sqrt{\frac{3}{2\pi}}\frac{a_\mathrm{e}}{2r\omega\varepsilon_0}\left(2\cos\theta\left(\frac{1}{\mathrm{j}kr} + \frac{1}{(\mathrm{j}kr)^2}\right)e^{-\mathrm{j}kr}\mathbf{u}_r - \left(1 + \frac{1}{\mathrm{j}kr} + \frac{1}{(\mathrm{j}kr)^2}\right)e^{-\mathrm{j}kr}\mathbf{u}_\theta\right) \tag{11}$$

We notice that this form is quite close to our initial dipole field form. Hence know we are more confident that indeed the Hankel version of a dipole field describes the same quantities.

iii. Funally, by a simple comparisson between the component of Eq. (1), (2), and Eq. (8) we obtain

$$a_\mathrm{e} = \sqrt{\frac{\pi}{6}}k^2 Il \tag{12}$$

with units A/m since the dipole moment Il has A·m.

(b) The product $\mathbf{E}\cdot\mathbf{E}$ is not real, which means that the real and imaginary parts of $\mathbf{E}$ do not correspond to the axes of the ellipse. Following section 1.4 in Methods, a new vector can be defined, i.e.,

$$\mathbf{b} = \left|\sqrt{\mathbf{E}\cdot\mathbf{E}}\right| \frac{\mathbf{E}}{\sqrt{\mathbf{E}\cdot\mathbf{E}}} \tag{13}$$

describing the axial representation of the given $\mathbf{E}$ vector.

i. Indeed the real part of the above expression gives the major axis direction vector

$$\mathbf{b}_{\text{major}} = \left|\sqrt{\mathbf{E}\cdot\mathbf{E}}\right| \Re\left\{\frac{\mathbf{E}}{\sqrt{\mathbf{E}\cdot\mathbf{E}}}\right\} = \{1.605, 0.331, -2.83\} \tag{14}$$

with direction angles $\vartheta_{\text{major}} = 149.9°$ and $\varphi_{\text{major}} = 11.648°$, while the imaginary part the minor axis

$$\mathbf{b}_{\text{minor}} = \left|\sqrt{\mathbf{E}\cdot\mathbf{E}}\right| \Im\left\{\frac{\mathbf{E}}{\sqrt{\mathbf{E}\cdot\mathbf{E}}}\right\} = \{-1.556, -0.943, -0.993\} \tag{15}$$

with direction angles $\vartheta_{\text{minor}} = 61.382°$ and $\varphi_{\text{minor}} = 31.23°$, respectively.

ii. The amplitude of the above major and minor vectors are

$$\left|\mathbf{b}_{\text{major}}\right| = 3.271$$

and

$$|\mathbf{b}_{\text{minor}}| = 2.073$$

respectively

iii. Finally the axial ratio is

$$\frac{\left|\mathbf{b}_{\text{major}}\right|}{|\mathbf{b}_{\text{minor}}|} = 1.577$$

(c) For simplicity we assume that $|\mathbf{b}|^2+|\mathbf{b}\cdot\mathbf{b}|=1$. Also it is useful to invoke the following vector properties: Given a complex vector $\mathbf{a}$ we have

$$\mathbf{a}\times\mathbf{a}=0 \tag{16}$$

$$\mathbf{a}\times(\mathbf{a}\times\mathbf{a}^*)=\mathbf{a}|\mathbf{a}|^2-\mathbf{a}^*(\mathbf{a}\cdot\mathbf{a}) \tag{17}$$

$$\mathbf{a}\cdot(\mathbf{a}\times\mathbf{a}^*)=0 \tag{18}$$

and any other combinations that after certain permutations exhibit the product $\mathbf{a}\times\mathbf{a}$

i.

$$\begin{aligned}\mathbf{b}_{\mathrm{CP}}+\mathbf{b}_{\mathrm{LP}} &= \mathbf{b}\times\left(\mathbf{b}\times\mathbf{b}^*\right)+\mathrm{j}\sqrt{\mathbf{b}\cdot\mathbf{b}}\,\mathbf{b}\times\mathbf{b}^*+\sqrt{\mathbf{b}\cdot\mathbf{b}}\sqrt{\mathbf{b}^*\cdot\mathbf{b}^*}\,\mathbf{b}+\sqrt{\mathbf{b}\cdot\mathbf{b}}\sqrt{\mathbf{b}\cdot\mathbf{b}}\,\mathbf{b}^*-\mathrm{j}\sqrt{\mathbf{b}\cdot\mathbf{b}}\,\mathbf{b}\times\mathbf{b}^*\\ &= \mathbf{b}|\mathbf{b}|^2-\mathbf{b}^*(\mathbf{b}\cdot\mathbf{b})+\sqrt{\mathbf{b}\cdot\mathbf{b}}\sqrt{\mathbf{b}^*\cdot\mathbf{b}^*}\,\mathbf{b}+(\mathbf{b}\cdot\mathbf{b})\mathbf{b}^*\\ &= \mathbf{b}\left(|\mathbf{b}|^2+|\mathbf{b}\cdot\mathbf{b}|\right)=\mathbf{b}\end{aligned}$$

ii.

$$\begin{aligned}\mathbf{b}_{\mathrm{LP}}\times\mathbf{b}_{\mathrm{LP}}^* = \sqrt{\mathbf{b}\cdot\mathbf{b}}\sqrt{\mathbf{b}^*\cdot\mathbf{b}^*}\Bigg(&\underbrace{\sqrt{\mathbf{b}^*\cdot\mathbf{b}^*}\mathbf{b}\times\mathbf{b}^*\sqrt{\mathbf{b}\cdot\mathbf{b}}}_{=A}+\sqrt{\mathbf{b}^*\cdot\mathbf{b}^*}\,\underbrace{\mathbf{b}\times\mathbf{b}}_{=0}\sqrt{\mathbf{b}^*\cdot\mathbf{b}^*}\\ &-\underbrace{\mathrm{j}\sqrt{\mathbf{b}^*\cdot\mathbf{b}^*}\,\mathbf{b}\times(\mathbf{b}^*\times\mathbf{b})}_{=B}+\sqrt{\mathbf{b}\cdot\mathbf{b}}\sqrt{\mathbf{b}\cdot\mathbf{b}}\underbrace{\mathbf{b}^*\times\mathbf{b}^*}_{=0}+\underbrace{\sqrt{\mathbf{b}\cdot\mathbf{b}}\sqrt{\mathbf{b}^*\cdot\mathbf{b}^*}\mathbf{b}^*\times\mathbf{b}}_{=-A}\\ &-\underbrace{\mathrm{j}\sqrt{\mathbf{b}\cdot\mathbf{b}}\,\mathbf{b}^*\times(\mathbf{b}^*\times\mathbf{b})}_{=C}+\underbrace{\mathrm{j}\sqrt{\mathbf{b}\cdot\mathbf{b}}\,(\mathbf{b}\times\mathbf{b}^*)\times\mathbf{b}^*)}_{=C}\\ &+\underbrace{\mathrm{j}\sqrt{\mathbf{b}^*\cdot\mathbf{b}^*}\,(\mathbf{b}\times\mathbf{b}^*)\times\mathbf{b})}_{=B}+\underbrace{(\mathbf{b}\times\mathbf{b}^*)\times(\mathbf{b}^*\times\mathbf{b})}_{-\mathbf{a}\times\mathbf{a}=0}\Bigg)=0\end{aligned}$$

iii.

$$\mathbf{b}_{\mathrm{CP}}\cdot\mathbf{b}_{\mathrm{CP}}=\left(\mathbf{b}|\mathbf{b}|^2-\mathbf{b}^*(\mathbf{b}\cdot\mathbf{b})+\mathrm{j}\sqrt{\mathbf{b}\cdot\mathbf{b}}\,\mathbf{b}\times\mathbf{b}^*\right)\cdot\left(\mathbf{b}|\mathbf{b}|^2-\mathbf{b}^*(\mathbf{b}\cdot\mathbf{b})+\mathrm{j}\sqrt{\mathbf{b}\cdot\mathbf{b}}\,\mathbf{b}\times\mathbf{b}^*\right)= \tag{19}$$

$$\mathbf{b}\cdot\mathbf{b}|\mathbf{b}|^4-\mathbf{b}\cdot\mathbf{b}|\mathbf{b}|^4+\underbrace{\mathrm{j}\sqrt{\mathbf{b}\cdot\mathbf{b}}|\mathbf{b}|^2\mathbf{b}\cdot(\mathbf{b}\times\mathbf{b}^*)}_{=0}-\mathbf{b}^*\cdot\mathbf{b}(\mathbf{b}\cdot\mathbf{b})|\mathbf{b}|^2 \tag{20}$$

$$+\mathbf{b}^*\cdot\mathbf{b}^*(\mathbf{b}\cdot\mathbf{b})^2-\underbrace{\mathrm{j}\sqrt{\mathbf{b}\cdot\mathbf{b}}\,\mathbf{b}\cdot\mathbf{b}\,\mathbf{b}^*\cdot(\mathbf{b}\times\mathbf{b}^*)}_{=0}+\underbrace{\mathrm{j}\sqrt{\mathbf{b}\cdot\mathbf{b}}\,(\mathbf{b}\times\mathbf{b}^*)\cdot\mathbf{b}|\mathbf{b}|^2}_{=0} \tag{21}$$

$$+\underbrace{\mathrm{j}\sqrt{\mathbf{b}\cdot\mathbf{b}}\,\mathbf{b}\cdot\mathbf{b}\,(\mathbf{b}\times\mathbf{b}^*)\cdot\mathbf{b}}_{=0}-\sqrt{\mathbf{b}\cdot\mathbf{b}}\sqrt{\mathbf{b}\cdot\mathbf{b}}(\mathbf{b}\times\mathbf{b}^*)\cdot(\mathbf{b}\times\mathbf{b}^*) \tag{22}$$

$$=-(\mathbf{b}\cdot\mathbf{b})|\mathbf{b}|^4+\mathbf{b}^*\cdot\mathbf{b}^*(\mathbf{b}\cdot\mathbf{b})^2-\mathbf{b}\cdot\mathbf{b}(\mathbf{b}^*\cdot\underbrace{(\mathbf{b}\times\mathbf{b}^*\times\mathbf{b})}_{=\mathbf{b}^*(\mathbf{b}\cdot\mathbf{b})-\mathbf{b}|\mathbf{b}|^2})=0 \tag{23}$$

iv.

$$\begin{aligned}\mathbf{b}_{\mathrm{LP}}\cdot\mathbf{b}_{\mathrm{CP}} &= \sqrt{\mathbf{b}\cdot\mathbf{b}}\left(|\mathbf{b}\cdot\mathbf{b}|\mathbf{b}+(\mathbf{b}\cdot\mathbf{b}^*-\mathrm{j}\sqrt{\mathbf{b}\cdot\mathbf{b}}\,\mathbf{b}\times\mathbf{b}^*\right)\cdot\left(\mathbf{b}|\mathbf{b}|^2-\mathbf{b}^*(\mathbf{b}\cdot\mathbf{b})+\mathrm{j}\sqrt{\mathbf{b}\cdot\mathbf{b}}\,\mathbf{b}\times\mathbf{b}^*\right)\\ &= \sqrt{\mathbf{b}\cdot\mathbf{b}}\Bigg(|\mathbf{b}\cdot\mathbf{b}||\mathbf{b}|^2\mathbf{b}\cdot\mathbf{b}-|\mathbf{b}\cdot\mathbf{b}||\mathbf{b}|^2\mathbf{b}\cdot\mathbf{b}+\mathrm{j}|\mathbf{b}\cdot\mathbf{b}|\sqrt{\mathbf{b}\cdot\mathbf{b}}\,\underbrace{\mathbf{b}\cdot\mathbf{b}\times\mathbf{b}^*}_{=0}+(\mathbf{b}\cdot\mathbf{b})|\mathbf{b}|^4\\ &-(\mathbf{b}\cdot\mathbf{b})\mathbf{b}^*\cdot(\mathbf{b}\cdot\mathbf{b})\mathbf{b}^*+\mathrm{j}(\mathbf{b}\cdot\mathbf{b})\sqrt{\mathbf{b}\cdot\mathbf{b}}\mathbf{b}^*\cdot\mathbf{b}\times\mathbf{b}^*-\mathrm{j}\sqrt{\mathbf{b}\cdot\mathbf{b}}|\mathbf{b}|^2\mathbf{b}\times\mathbf{b}^*\cdot\mathbf{b}\\ &+\mathrm{j}\sqrt{\mathbf{b}\cdot\mathbf{b}}(\mathbf{b}\cdot\mathbf{b})\mathbf{b}\times\mathbf{b}^*\cdot\mathbf{b}^*+\mathbf{b}\cdot\mathbf{b}\,\mathbf{b}\times\mathbf{b}^*\cdot\mathbf{b}\times\mathbf{b}^*\\ &= \mathbf{b}\cdot\mathbf{b}|\mathbf{b}|^4-(\mathbf{b}\cdot\mathbf{b})^2\mathbf{b}^*\cdot\mathbf{b}^*+\underbrace{\mathbf{b}\cdot\mathbf{b}\,\mathbf{b}\times\mathbf{b}^*\cdot\mathbf{b}\times\mathbf{b}^*}_{(\mathbf{b}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{b})(\mathbf{b}^*\cdot\mathbf{b}^*)-(\mathbf{b}\cdot\mathbf{b})|\mathbf{b}|^4}=0\end{aligned}$$

v. Finally

$$|\mathbf{b}| = \mathbf{b}\cdot\mathbf{b}^* = (\mathbf{b}_{\mathrm{CP}} + \mathbf{b}_{\mathrm{LP}})\cdot\left(\mathbf{b}^*_{\mathrm{CP}} + \mathbf{b}^*_{\mathrm{LP}}\right) = |\mathbf{b}_{\mathrm{CP}}| + |\mathbf{b}_{\mathrm{LP}}| + 2\Re\{\mathbf{b}_{\mathrm{LP}}\cdot\mathbf{b}^*_{\mathrm{CP}}\}$$

we need to prove that

$$\mathbf{b}_{\mathrm{LP}}\cdot\mathbf{b}^*_{\mathrm{CP}} = 0$$

Similarly as before we expand. The important step here is that

$$\sqrt{\mathbf{b}\cdot\mathbf{b}}\sqrt{\mathbf{b}^*\cdot\mathbf{b}^*}\;\mathbf{b}\cdot\left(\mathbf{b}^*\times(\mathbf{b}^*\times\mathbf{b})\right) = |\mathbf{b}\cdot\mathbf{b}||\mathbf{b}\times\mathbf{b}^*|^2$$

and

$$\mathrm{j}\sqrt{\mathbf{b}\cdot\mathbf{b}}\;(\mathbf{b}\times\mathbf{b}^*)\cdot\left(\mathrm{j}\sqrt{\mathbf{b}^*\cdot\mathbf{b}^*}\;(\mathbf{b}^*\times\mathbf{b})\right) = -|\mathbf{b}\cdot\mathbf{b}||\mathbf{b}\times\mathbf{b}^*|^2$$

hence these terms cancels and $\mathbf{b}_{\mathrm{LP}}\cdot\mathbf{b}^*_{\mathrm{CP}} = 0$. Therefore

$$|\mathbf{b}|^2 = |\mathbf{b}_{\mathrm{CP}}|^2 + |\mathbf{b}_{\mathrm{LP}}|^2$$

The numerical example can be straightforwardly utilized from the above.