Nonlinear dynamics & chaos

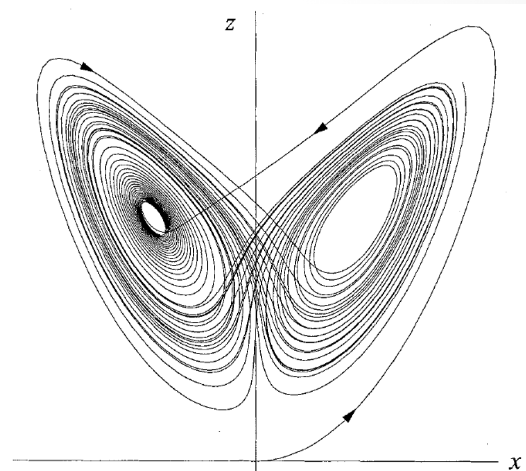
One-Dimensional Maps

Lecture IX

Recap

Last time: Lorenz' equations and the butterfly – a strange attractor.

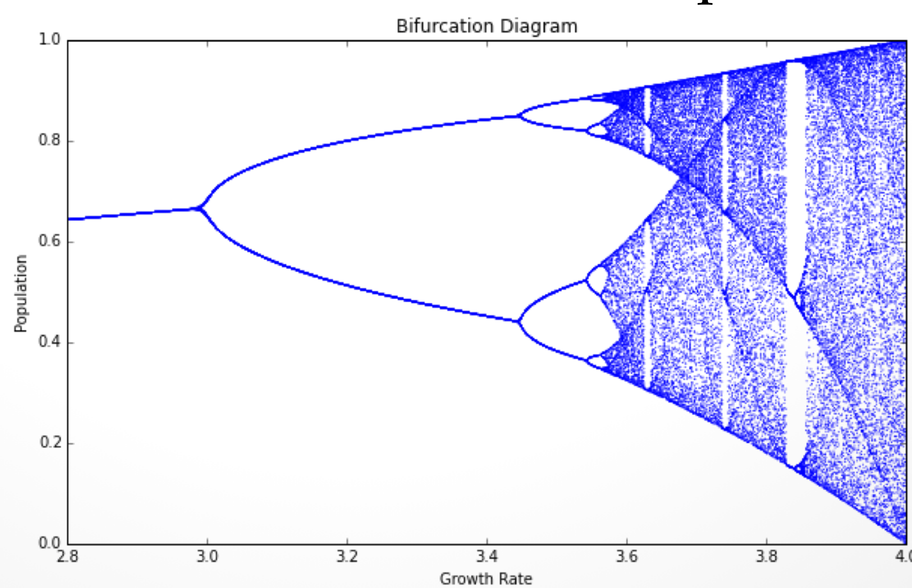
$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= rx - y - xz\\ \dot{z} &= xy - bz\end{aligned}$$



Continuous time.

This time: Discrete time and another iconic plot.

(Note! It says 'bifurcation diagram' in the header. Strictly, this is an orbit diagram, since unstable branches are not plotted.)



One-Dimensional Maps

New class of dynamical systems: time is discrete, not continuous.

Example of one-dimensional map:

$$x_{n+1} = \cos x_n$$

Sequence x_0, x_1, x_2, … is the orbit starting from x_0.

Maps arise:

1) *As tools for analyzing differential equations*. Ex. Lorenz map shows that the Lorenz attractor is not just a long-period limit cycle.
2) *As models of natural phenomena*: digital electronics, economics and finance theory, impulsively driven mechanics, …
3) *As simple examples of chaos*: points hop along their orbits → non-smooth behavior, wilder dynamics.

Successful predictions of routes to chaos by using maps.

Fixed Points and Cobwebs

$$x_{n+1} = f(x_n)$$

f is a smooth function from the real line to itself.

Fixed point x^*

$$f(x^*) = x^*$$

$$x_n = x^* \quad \rightarrow \quad x_{n+1} = f(x_n) = f(x^*) = x^*$$

Stability of x^* (consider an orbit sweeping past x^*: $x_n = x^* + \eta_n$)

$$x^* + \eta_{n+1} = x_{n+1} = f(x^* + \eta_n) = f(x^*) + f'(x^*)\eta_n + O(\eta_n^2)$$

$$\eta_{n+1} = f'(x^*)\eta_n + O(\eta_n^2)$$

$$\eta_{n+1} \sim f'(x^*)\eta_n$$

Fixed Points and Cobwebs

Linearized system

$$\eta_{n+1} \sim f'(x^*)\eta_n$$

$\lambda = f'(x^*)$ is the eigenvalue or multiplier.

$$\eta_1 \sim \lambda\eta_0,\ \eta_2 \sim \lambda\eta_1 \sim \lambda^2\eta_0, \ldots \rightarrow \quad \eta_n \sim \lambda^n\eta_0$$

$$\text{If } |\lambda| = |f'(x^*)| < 1 \quad \rightarrow \quad \eta_n \rightarrow 0,\ \text{as } n \rightarrow \infty$$

x^* is linearly stable.

If $|\lambda| = |f'(x^*)| > 1 \rightarrow x^*$ is linearly unstable.

Conclusions from linearization also hold for the nonlinear map, except in the marginal case $|\lambda| = |f'(x^*)| = 1$; here the neglected higher-order terms $O(\eta_n^2)$ determine the local stability.

Example

$$x_{n+1} = x_n^2$$

Fixed points

$$x^* = (x^*)^2 \quad \rightarrow \quad x^* = 0,\ x^* = 1$$

$$\lambda = f'(x^*) = 2x^*$$

$x^* = 0$ is stable ($|\lambda| = 0 < 1$), $x^* = 1$ is unstable ($|\lambda| = 2 > 1$).

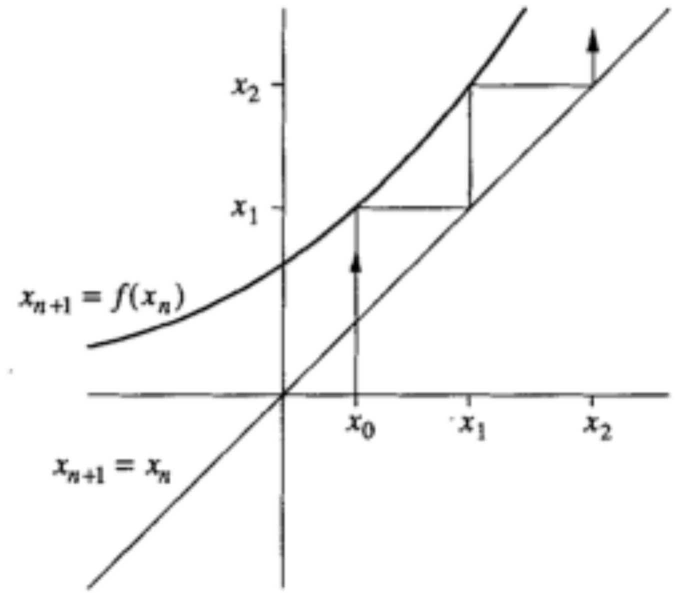
Fixed points with $\lambda = 0$ are superstable: convergence is extremely fast. $\eta_{n+1} = \lambda\eta_n + O(\eta_n^2)$ $\rightarrow$ perturbations decay like

$$\eta_1 \sim \eta_0^2,\ \eta_2 \sim \eta_1^2 \sim \eta_0^4, \ldots \rightarrow \quad \eta_n \sim \eta_0^{2^n}$$

Cobwebs

$x_{n+1} = f(x_n)$, initial condition x_0.

1) Draw a vertical line from x_0 until it intersects the graph of f: the height of the intersection point is x_1.
2) Draw horizontal line from current point to the diagonal.
3) Move vertically to the curve again
4) Etc.



The process is repeated n times to generate the first n points in the orbit.

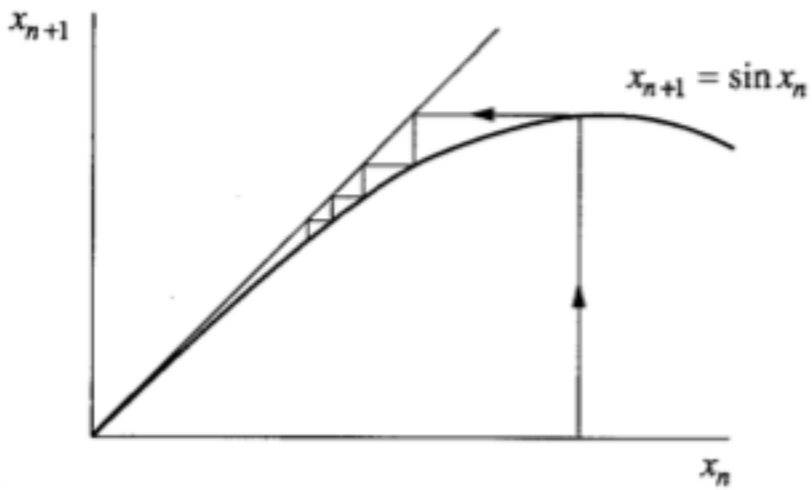
Cobwebs are particularly useful when linear analysis fails.

Example I

$$x_{n+1} = \sin x_n$$

Fixed points

$$x^* = 0 \quad \rightarrow \quad f'(x^*) = \cos x^* = 1$$



⬉ Marginal case: Linear analysis is inconclusive.

The cobweb shows that $x^* = 0$ is locally stable.

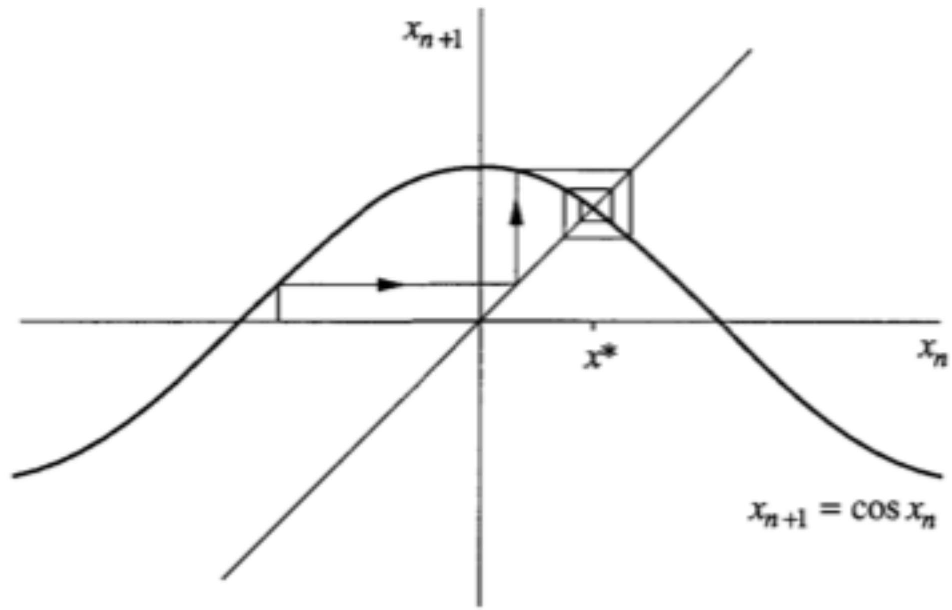
$x^* = 0$ is in fact globally stable, since any x is sent to the interval $-1 \leq x \leq 1$, from which it converges to the fixed point.

Example II

$$x_{n+1} = \cos x_n$$

Fixed points

$$x^* = \cos x^* \quad \rightarrow \quad x^* = 0.739...$$



$x^* = 0.739$ is globally stable. Since $\lambda = -\sin x^* = -0.6735.. < 0$, convergence occurs via damped oscillations, as opposed to the monotonic behavior observed when $\lambda > 0$.

Logistic Map: Numerics

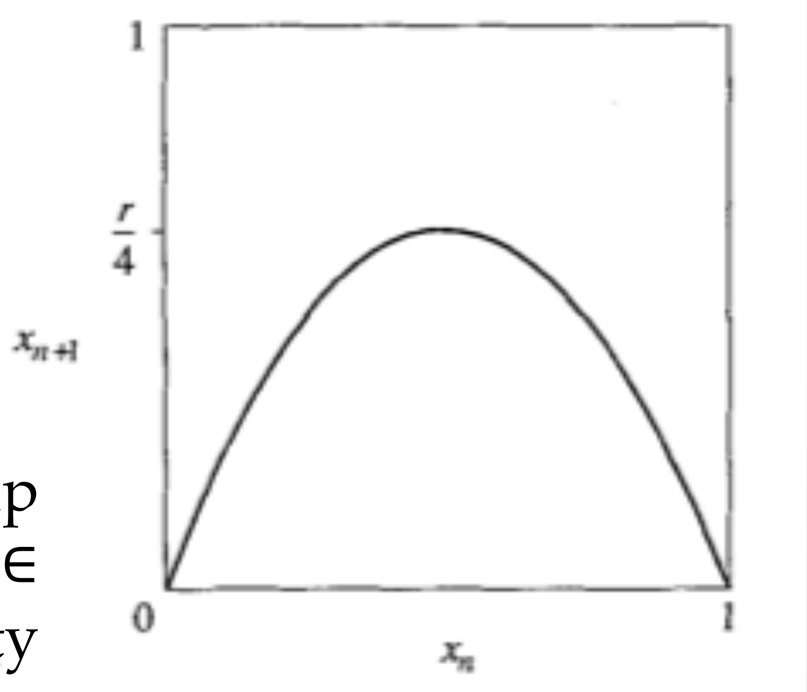
Logistic map (Robert May (1976))

$$x_{n+1} = r x_n (1 - x_n)$$

$x_n \geq 0$, $r \geq 0$.

Discrete-time analog of the logistic equation for population growth, $\dot{N} = rN\left(1 - \frac{N}{K}\right)$.

Focus: Choose $0 \leq r \leq 4 \rightarrow$ the map sends points $x \in [0,1]$ to points $x \in [0,1]$ (map of population density into population density).



Period-Doubling

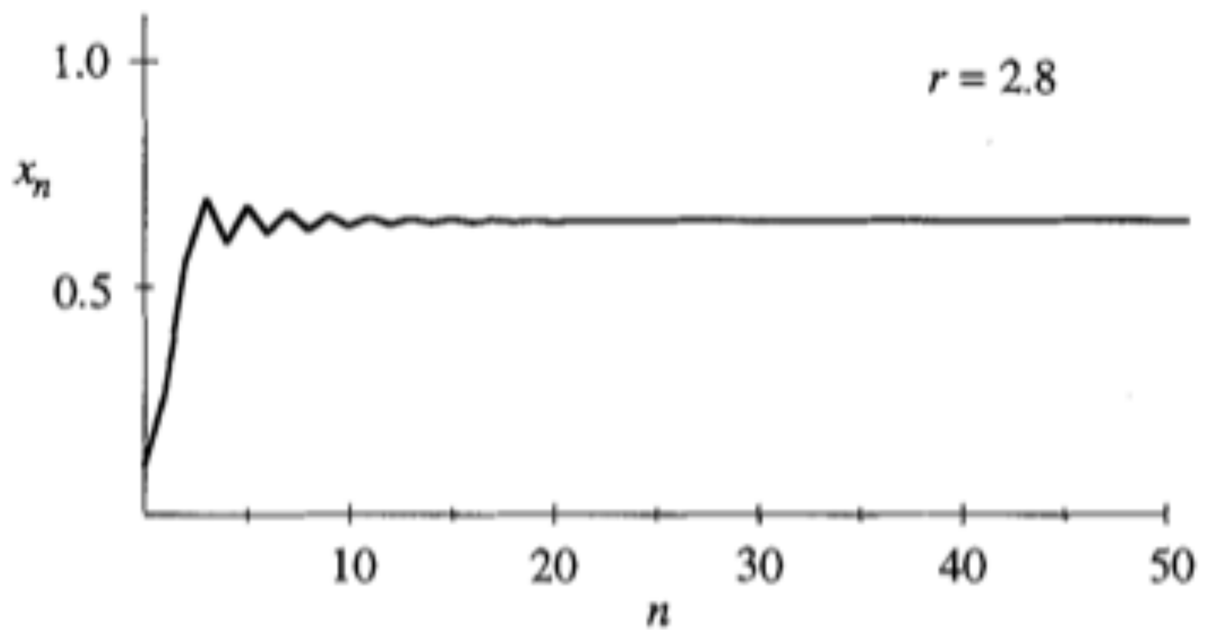
$$x_{n+1} = rx_n(1 - x_n)$$

Fix r and choose some initial population x_0. What happens?

For $r \leq 1$, $x_n \to 0$ as $n \to \infty$. (Extinction.)

For $1 < r \leq 3$, the population grows and reaches a non-zero steady state.

Time series. (Discrete points (n, x_n) connected by segments for clarity.)

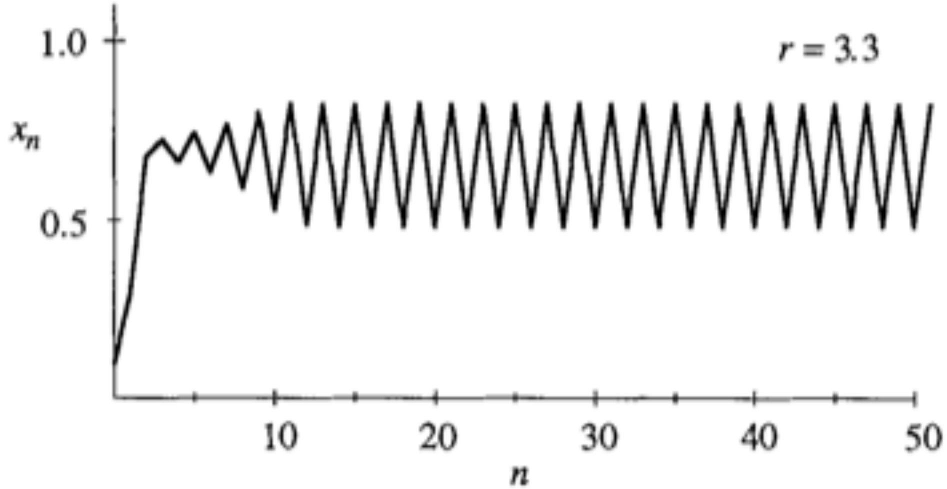


Period-Doubling

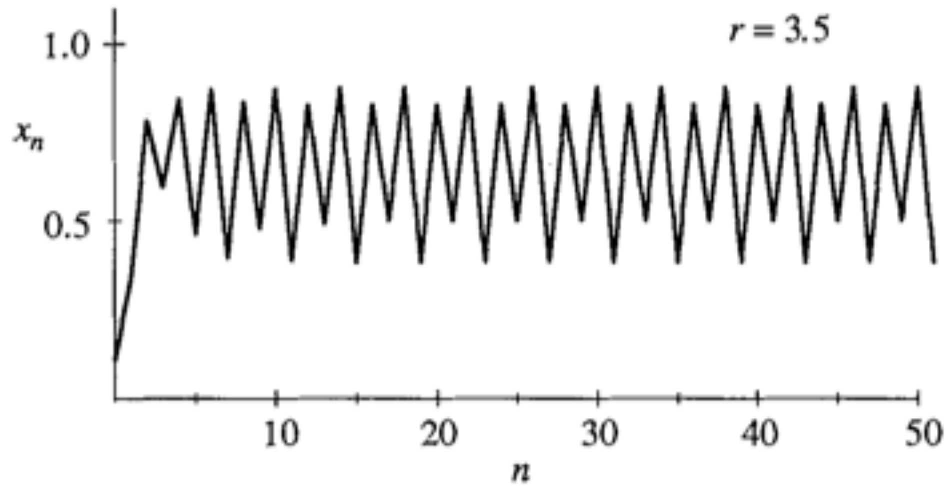
$$x_{n+1} = rx_n(1 - x_n)$$

$r > 3$?

For $r = 3.3$, the population oscillates between two values and x_n repeats every two iterations (period-2 cycle).



For $r = 3.5$, the population oscillates between four values (period-4 cycle)→ the period has doubled!



Period-Doubling

Further period-doublings occur for larger r (computer experiments).

r_n denotes the r-value where a 2^n-cycle first appears

$r_1 = 3$	(period 2 is born)
$r_2 = 3.449...$	4
$r_3 = 3.54409...$	8
$r_4 = 3.5644...$	16
$r_5 = 3.568759...$	32
$\vdots$	$\vdots$
$r_\infty = 3.569946...$	∞

Successive bifurcations occur after shorter and shorter intervals in r.

The sequence $\{r_n\}$ converges to a limiting value $r_\infty = 3.569946$

Convergence is essentially geometric:

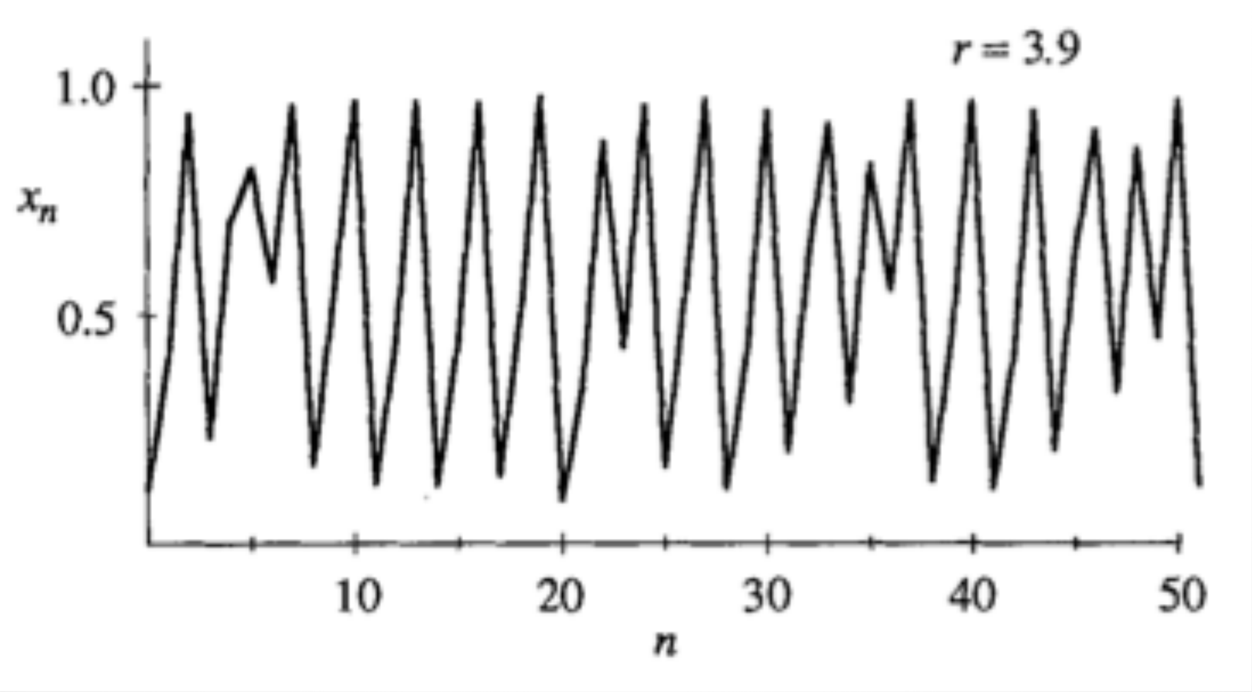
$$\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669...$$

Chaos and Periodic Windows

What happens for $r > r_\infty$?

For many values of r the sequence $\{x_n\}$ never settles down to a fixed point or a periodic orbit: a discrete-time version of chaos.

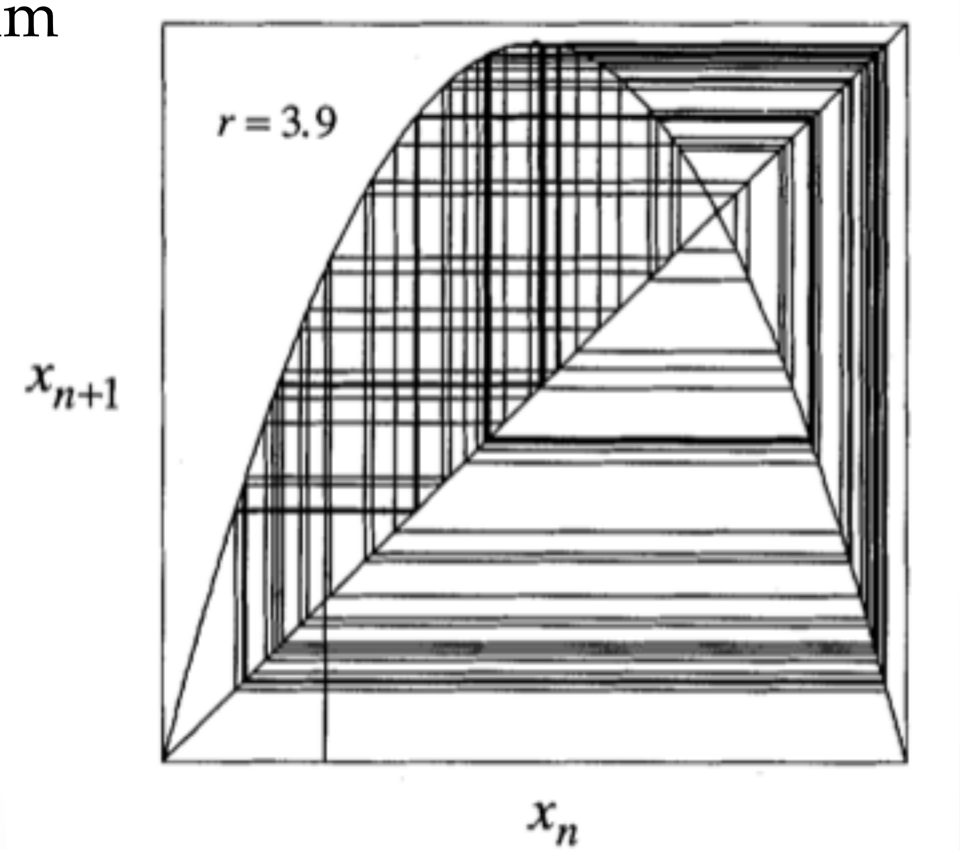
$$r = 3.9$$



Chaos and Periodic Windows

$r = 3.9$

The cobweb diagram



To see the the long-term behaviour for all r we plot the **orbit diagram ➔**

Chaos and Periodic Windows

Orbit diagram

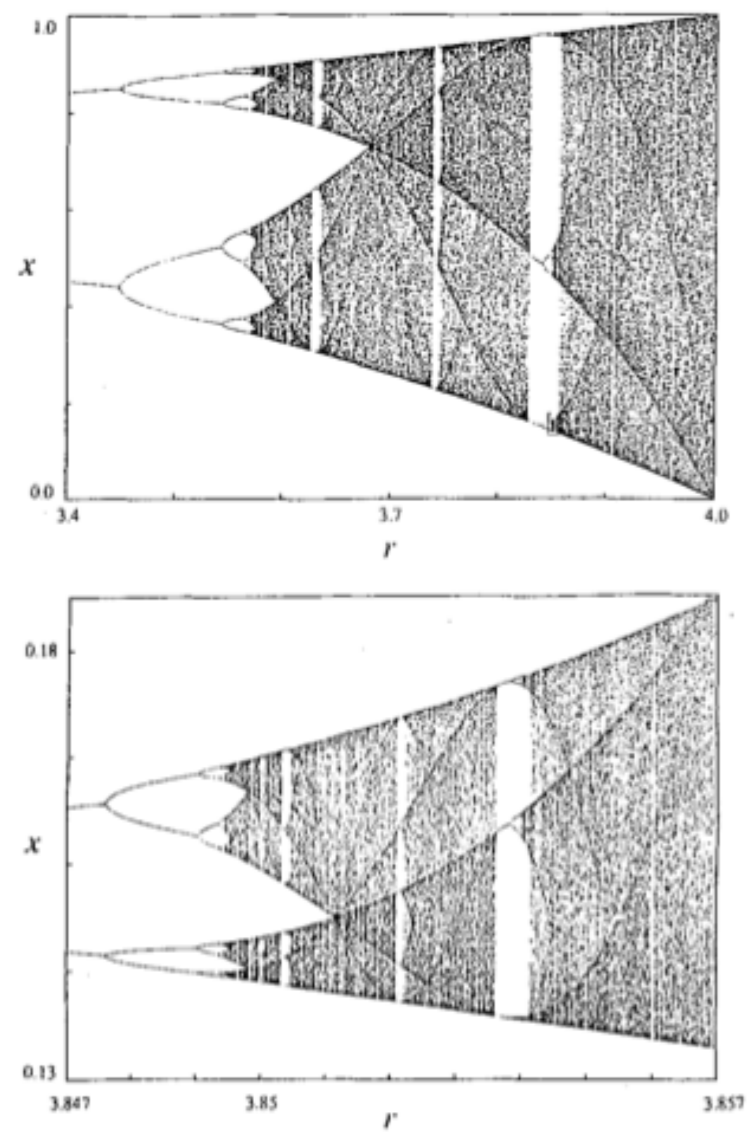
Branches indicate periodic solutions. Periodic doublings for $r < r_\infty \approx 3.57$.

When $r > r_\infty$ the map becomes chaotic and the attractor changes from a finite to an infinite set of points.

Mixture of order and chaos: for $r > r_\infty$ one also finds periodic windows!

Large window starting at about $r \approx 3.83$ contains a stable period-3 cycle.

Zooming into the window one finds a miniature copy of the full orbit diagram!



See https://www.youtube.com/watch?v=PtfPDfoF-iY

Example

$$x_{n+1} = rx_n(1 - x_n)$$

$0 \le r \le 4,\ 0 \le x_n \le 1$

Fixed points

$$x^* = f(x^*) = rx^*(1 - x^*) \quad \rightarrow \quad x^* = 0,\ 1 - 1/r$$

$$x^* = 0, \quad \forall r\,;\ x^* = 1 - 1/r, \quad r \ge 1$$

$$f'(x^*) = r - 2rx^* \quad \rightarrow \quad f'(0) = r\,; f'(1 - 1/r) = 2 - r$$

The origin is stable for $r < 1$ and unstable for $r > 1$.

$x^* = 1 - 1/r$ is stable for $1 < r < 3$ ($-1 < 2 - r < 1$) and unstable for $r > 3$.

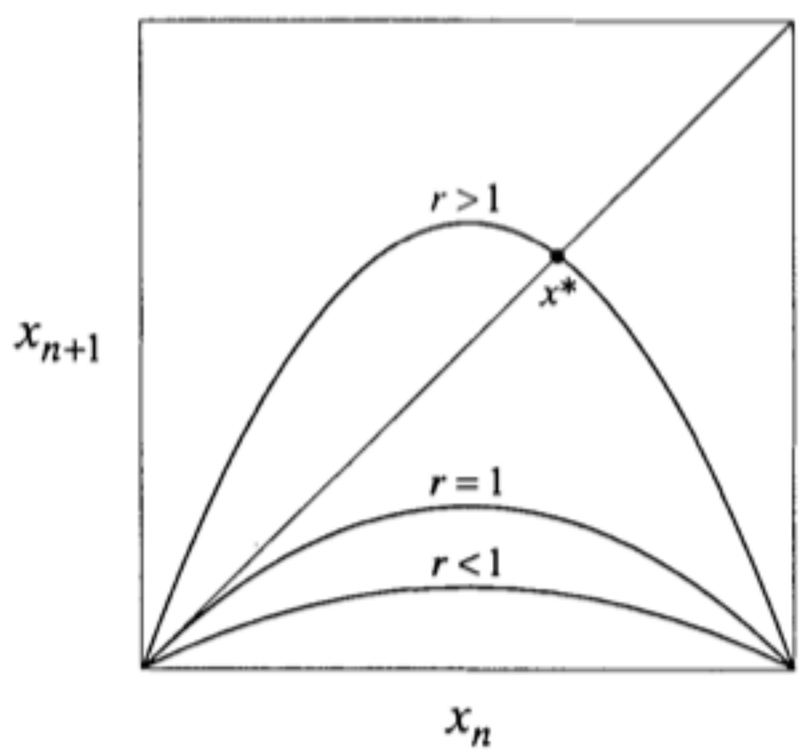
Example

$$x_{n+1} = rx_n(1 - x_n)$$

$0 \leq r \leq 4, \ 0 \leq x_n \leq 1$

Graphical analysis

1) For $r < 1$ the parabola lies below the diagonal → the origin is the only fixed point.
2) For $r > 1$ the parabola crosses the diagonal at another point $x^* = 1 - 1/r$, while the origin loses stability (transcritical bifurcation).
3) When $r > 3$ $x^* = 1 - 1/r$ loses stability (flip bifurcation).



Example

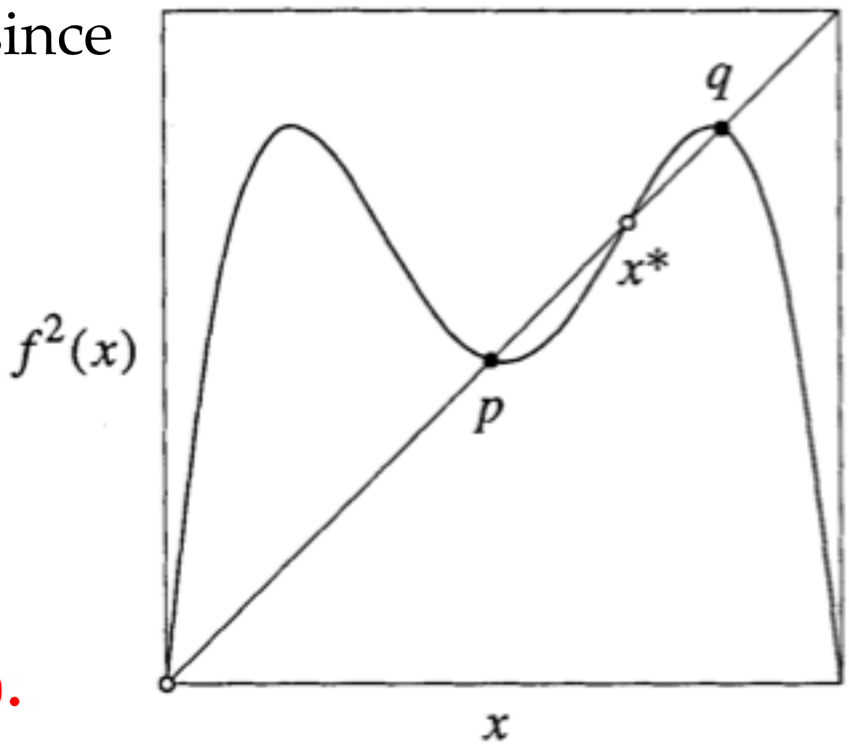
The logistic map has a 2-cycle for all $r > 3$.

2-cycle: there exist two points p and q such that $f(p) = q$ and $f(q) = p$. Equivalently,
$f\,[f(p)] = f(q) = p \rightarrow f^{\,2}(p) = p \rightarrow p$ is a fixed point of the second-iterated map $f^{\,2}(x) = f\,[f(x)]$.

$f^{\,2}(x) = x$ is a quartic polynomial (since $f(x)$ is quadratic).
$x^* = 0$ and $x^* = 1 - 1/r$ are trivial solutions, as they solve
$f(x) = x \rightarrow f^{\,2}(x) = x$.
Other two solutions:

$$p, q = \frac{r + 1 \pm \sqrt{(r-3)(r+1)}}{2r}$$

2-cycle exists for all $r > 3$ ($p, q \in \mathbb{R}$).



Example

Details just to make sure…

$$f^2(x) - x = f(f(x)) - x = f(rx(1-x)) - x$$
$$= r[rx(1-x)][1 - rx(1-x)] - x$$
$$= r^2x(1-x)[1 - rx(1-x)] - x$$

We know that $f^2(x) - x = h(x)x[x - \left(1 - \frac{1}{r}\right)]$, because $f^2(x^*) - x^* = 0$ and $x^* = 0$, and $x^* = 1 - \frac{1}{r}$. Do the long division of $f^2(x) - x$ by $x[x - \left(1 - \frac{1}{r}\right)] \rightarrow$ quadratic $h(x)$.

Solve $h(x^* = p, q) = 0. \rightarrow$

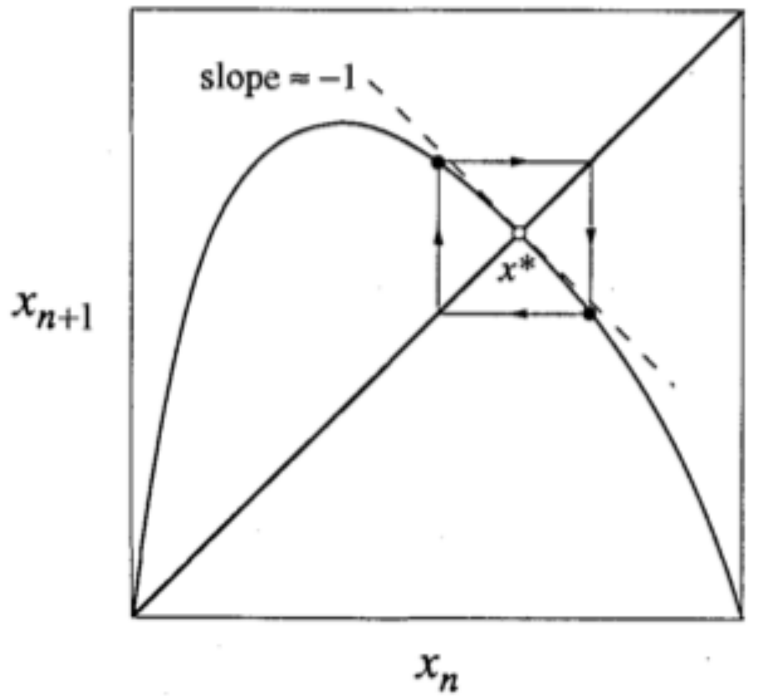
$$p, q = \frac{r + 1 \pm \sqrt{(r-3)(r+1)}}{2r}$$

Example

At $r = 3$ $x^* = p = q = 1 - \frac{1}{r} = 2/3$. The 2-cycle bifurcates continuously from x^*(flip bifurcation).

Near the fixed point $f'(x^*) \sim -1$.

If $f(x)$ is concave down at the fixed point, the cobweb tends to produce a small stable 2-cycle close to x^*.



Example

Show that the 2-cycle is stable for $3 < r < 1+\sqrt{6} \approx 3.449$ (= the numerically obtained value for r_2 at the birth of a 4-cycle).

To analyze the stability of a cycle, reduce the problem to a question about the stability of the relevant fixed point:
Both p and q are solutions of $f^2(x) = x$, that is, fixed points for f^2.
Accordingly, we compute the multiplier

$$\lambda = \frac{d}{dx}[f(f(x))]_{x=p} = f'[f(p)]f'(p) = f'(q)f'(p)$$

By symmetry of the final term the same λ is obtained at $x = q$.
Makes sense: The p and q branches must bifurcate simultaneously.

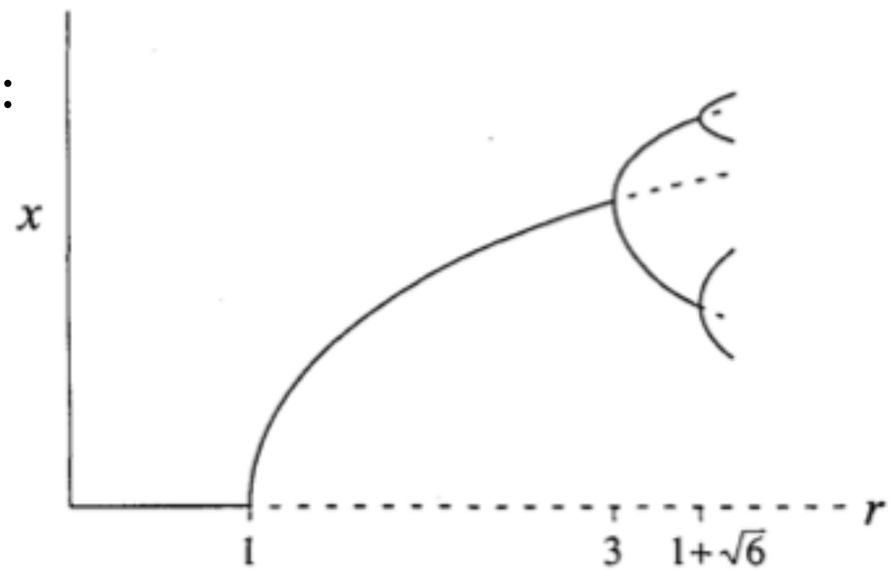
Example

$$\Rightarrow \qquad \begin{aligned} \lambda = &\quad r(1-2q)r(1-2p) \\ = &\quad r^2[1-2(p+q)+4pq] \\ = &\quad r^2[1-2(r+1)/r+4(r+1)/r^2] \\ = &\quad 4+2r-r^2 \end{aligned}$$

The 2-cycle is linearly stable for

$$|4+2r-r^2|<1 \quad \to \quad 3<r<1+\sqrt{6}$$

Partial bifurcation diagram:

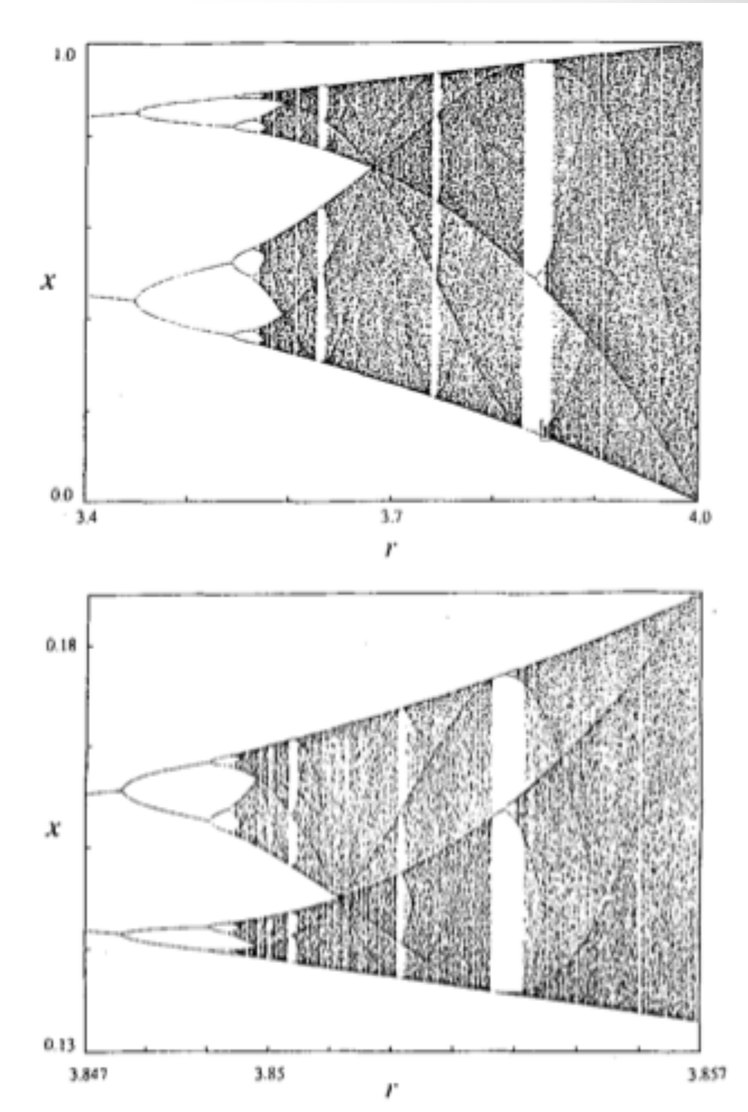


Periodic Windows

There are periodic windows after the onset of chaos, $r > r_\infty \approx 3.5699$.

For many values of r the sequence $\{x_n\}$ never settles down to a fixed or a periodic orbit. Instead the long-term behaviour is aperiodic: chaos. However, there are intervals in r where periodic motion prevails.

These **periodic windows** are interspersed between chaotic clouds of dots. For example: the large window starting near $r \approx 3.83$ (upper diagram) contains a stable 3-cycle. The blow-up (lower diagram) reveals self-similarity.



Periodic Windows

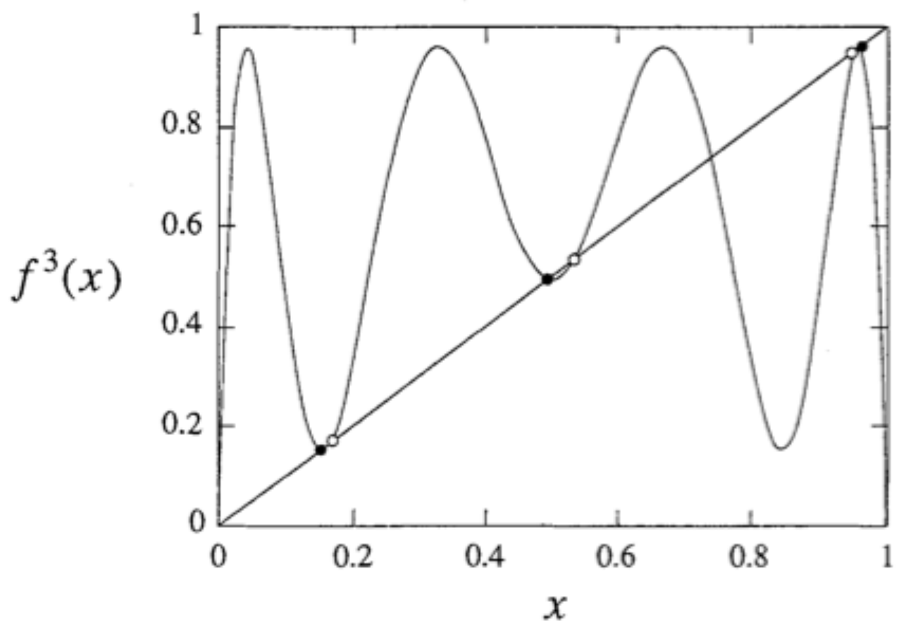
The 3-cycle for $3.8284 \le r \le 3.8415$. $x_{n+3} = f^3(x_n)$.

Key: The third-iterated map $f^3(x)$.

Problem: this is an 8th degree polynomial, so analytical solution is impossible.

Graphically: (r = 3.835) The intersections are the solutions to $f^3(x) = x$. Two solutions are period-1 fixed points, $f(x) = x$, and not interesting.

$r = 3.835$

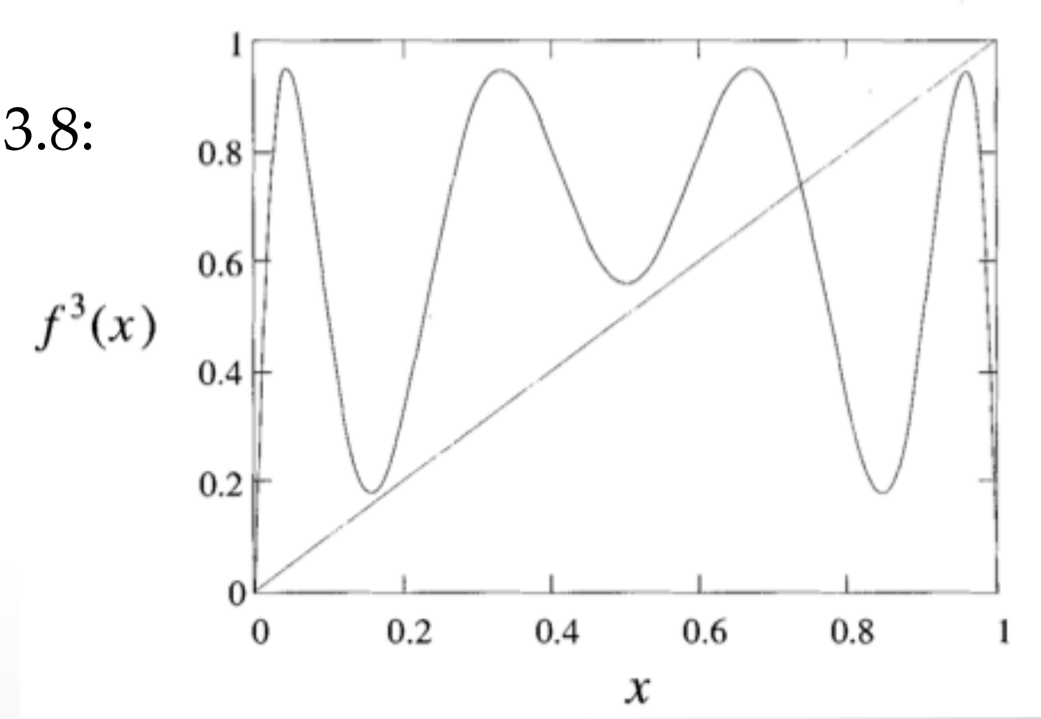


The other six solutions (dots in the figure) are period-3 fixed points: three stable (the slope $|f^{3\prime}(x)| < 1$), three unstable ($|f^{3\prime}(x)| > 1$).

Periodic Windows

Decrease r towards the chaotic regime: The hills move down and the valleys rise. ➔ The intersections vanish. Hence, for some $r \in [3.8, 3.835]$ $f^3(x)$ must have become tangent to the diagonal: the stable and unstable 3-cycles coalesce and annihilate in a **tangent bifurcation**. So, this point ($r = 1 + \sqrt{8} = 3.8284\ldots$) defines the minimum value of r in the periodic window.

$r = 3.8$:

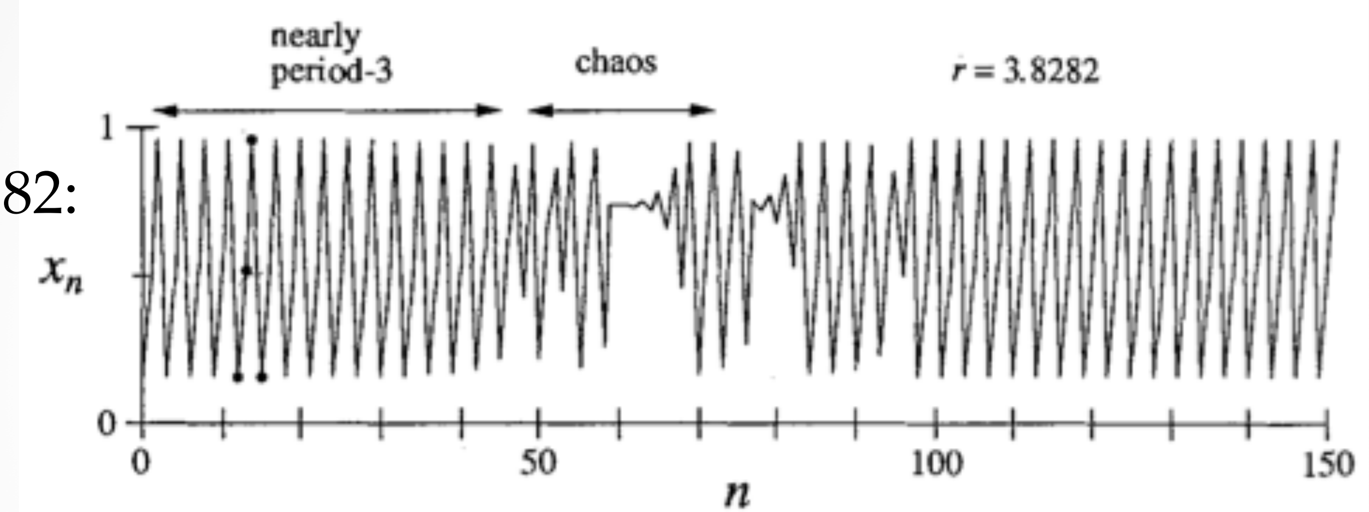


tangent bifurcation = saddle-node bifurcation; (as r increases 3-cycle appears out of blue sky and splits into a stable and unstable 3-cycle)

Intermittency

Interesting behaviour for *r* just below the period-3 window.

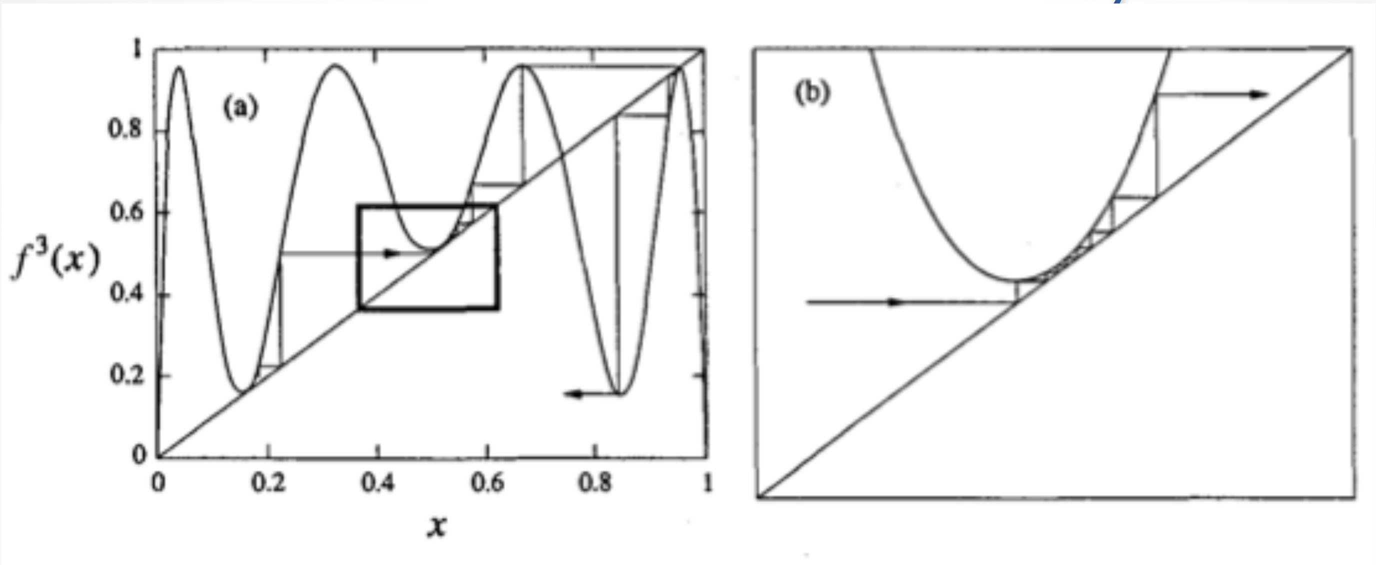
$r = 3.8282$:



Part of the orbit looks like a stable 3-cycle, which alternates with chaotic behavior.

There cannot be a 3-cycle because we are below the tangent bifurcation: it is the ghost of the 3-cycle! No surprise, because the tangent bifurcation is a saddle-node bifurcation.
The orbit returns repeatedly onto the cycle with intermittent bouts of chaos between visits: intermittency.

Intermittency



Cobweb: The system takes a long time to pass through the channels between the diagonal and the curve; here $f^3(x) \sim x \rightarrow$ 3-cycle.

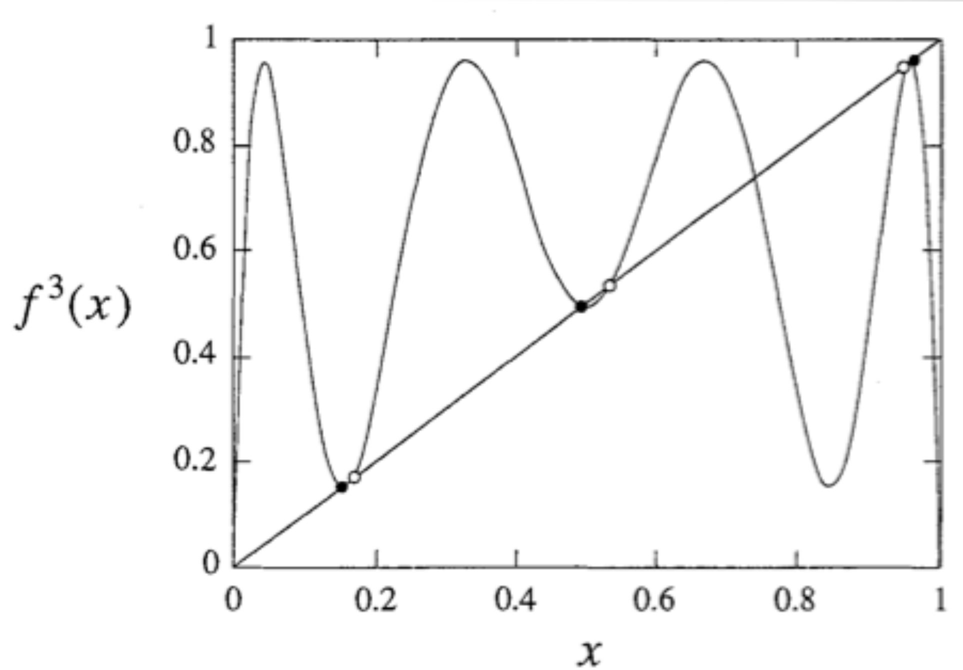
Eventually the system leaves the channel and it moves chaotically until it hits a channel again.

When r moves further away from the periodic window, chaotic behavior is more frequent, periodic behavior becomes increasingly rare and disappears (intermittency route to chaos).

Period-doubling in the window

Just after the 3-cycle is created, the slope of $f^3(x)$ at the black dots (stable cycles) is close to +1.
As r increases, the slope at the black dots goes down and eventually reaches -1: after that the cycle becomes unstable and a flip bifurcation occurs.

At flip bifurcation the black dot "splits in two"(the dot splits in two in the 4th iterate $f^4(x)$): the 3-cycle doubles its period and becomes a 6-cycle.
The same mechanism generates a 12-cycle, a 24-cycle, … $3{\cdot}2^n$ cycles.



All periodic windows have similar period-doubling sequences. This mechanism leads to the "miniature copies" in the orbit diagram.

Liapunov exponent

Aperiodic motion found in logistic map: are we sure it is chaos?

Sensitive dependence on initial conditions?

Initial condition x_0, nearby point $x_0+\delta_0$ with δ_0 very small.

δ_n = separation after n iterates

If $|\delta_n| \sim |\delta_0| e^{n\lambda}$, λ is called the Liapunov exponent.

A positive Liapunov exponent is a signature of chaos.

$$
\begin{aligned}
\lambda \approx \quad & \frac{1}{n} \ln \left| \frac{\delta_n}{\delta_0} \right| \\
= \quad & \frac{1}{n} \ln \left| \frac{f^n(x_0+\delta_0) - f^n(x_0)}{\delta_0} \right| \qquad (\lim_{\delta_0 \to 0}) \\
= \quad & \frac{1}{n} \ln |(f^n)'(x_0)|
\end{aligned}
$$

Liapunov exponent

$$\begin{aligned} \lambda \approx & \quad \frac{1}{n} \ln \left| \frac{\delta_n}{\delta_0} \right| \\ = & \quad \frac{1}{n} \ln \left| \frac{f^n(x_0 + \delta_0) - f^n(x_0)}{\delta_0} \right| \qquad (\lim_{\delta_0 \to 0}) \\ = & \quad \frac{1}{n} \ln |(f^n)'(x_0)| \end{aligned}$$

Iterative linearisations (chain rule; see the first example in "Ruling out limit cycles"):

$$(f^n)'(x_0) = \prod_{i=0}^{n-1} f'(x_i)$$

$$\lambda \approx \frac{1}{n} \ln \left| \prod_{i=0}^{n-1} f'(x_i) \right| = \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|$$

Liapunov exponent

Define $\lambda = \lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=0}^{n-1}\ln|f'(x_i)|\right]$ as the Liapunov exponent.

λ depends on the initial condition x_0, but it is the same for all x_0 in the basin of attraction of a given attractor.

For stable fixed points and cycles λ is negative, for chaotic attractors λ is positive.

Example I

$f(x)$ has a stable p-cycle containing the point x_0. Determine λ.

x_0 is an element of the p-cycle $\rightarrow$ x_0 is a fixed point of $f^p(x)$.

The p-cycle is stable $\rightarrow$ $|(fp)'(x)| < 1 \rightarrow \ln|(f^p)'(x)| < 0$.

$$\lambda = \lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=0}^{n-1}\ln|f'(x_i)|\right] = \frac{1}{p}\sum_{i=0}^{p-1}\ln|f'(x_i)|$$

$$\lambda = \frac{1}{p}\sum_{i=0}^{p-1}\ln|f'(x_i)| = \frac{1}{p}\ln|(f^p)'(x_0)| < 0$$

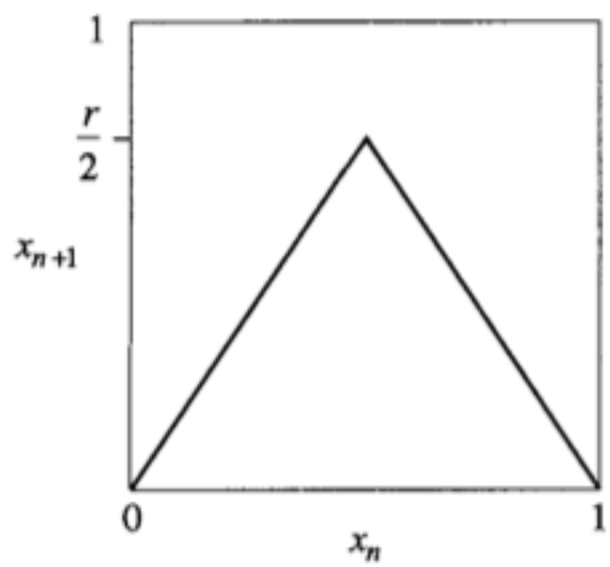
If the p-cycle is superstable, then $|(f^p)'(x_0)| = 0$ and $\lambda = \ln(0)/p = -\infty$.

Example II

The tent map

$$f(x) = \begin{cases} rx, & \text{for } 0 \le x \le \frac{1}{2} \\ r - rx, & \text{for } \frac{1}{2} \le x \le 1 \end{cases}$$

$0 \le r \le 2,\ 0 \le x \le 1.$



Example II

The tent map

$$f(x) = \begin{cases} rx, & \text{for } 0 \le x \le \frac{1}{2} \\ r - rx, & \text{for } \frac{1}{2} \le x \le 1 \end{cases}$$

$0 \le r \le 2,\ 0 \le x \le 1.$

$$f'(x) = \pm r,\ \forall x \quad \rightarrow \quad \lambda = \lim_{n\to\infty} \left[\frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)| \right] = \ln r$$

For $1 < r \le 2$, $\lambda > 0 \rightarrow$ chaotic regime.

Example III

Compute numerically λ for the logistic map $f(x) = rx(1-x)$ in the interval $3 \le r \le 4$.

Procedure:

1) Given a value of r, start from a random initial condition.
2) Iterate the map long enough to let transients decay (300 iterates are usually sufficient).
3) Compute a large number of iterations after that (say 10000).
4) Compute $\ln|f'(x_n)| = \ln|r - 2rxn|$ for the sequence of values.
5) Divide the result by the number of terms (here 10000).
6) Repeat the procedure for another r value, until the desired range is covered.

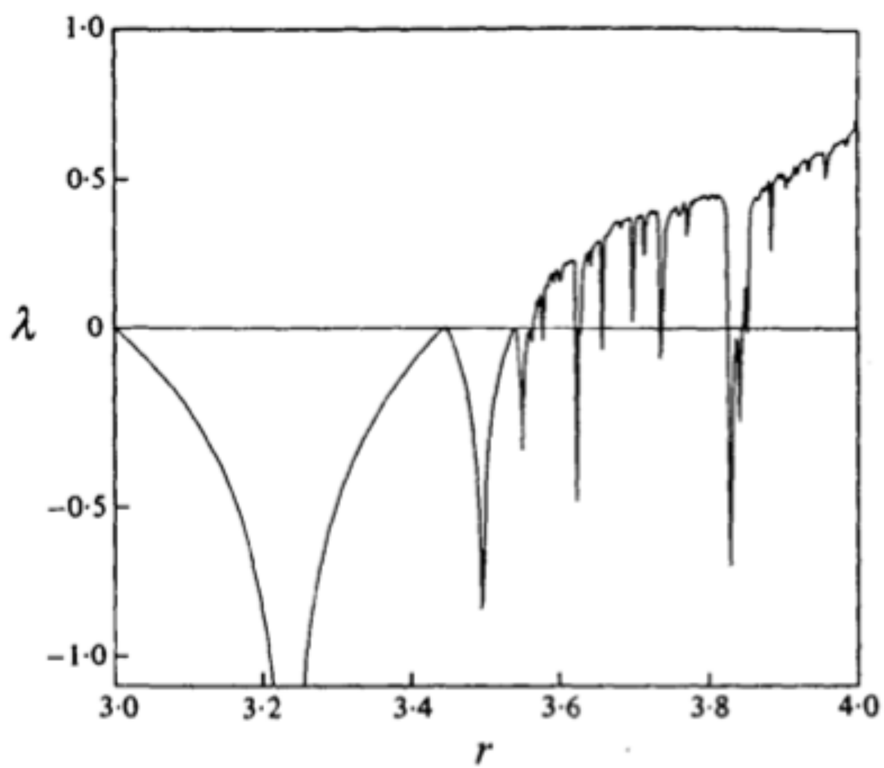
Example III

λ stays negative for all $r < r_\infty \approx 3.57$.

λ is zero at the period-doubling bifurcations.

λ returns negative also for $r > r_\infty$ in some windows (periodic windows).

The dips correspond to superstable cycles (λ = -∞; not seen due to finite resolution).



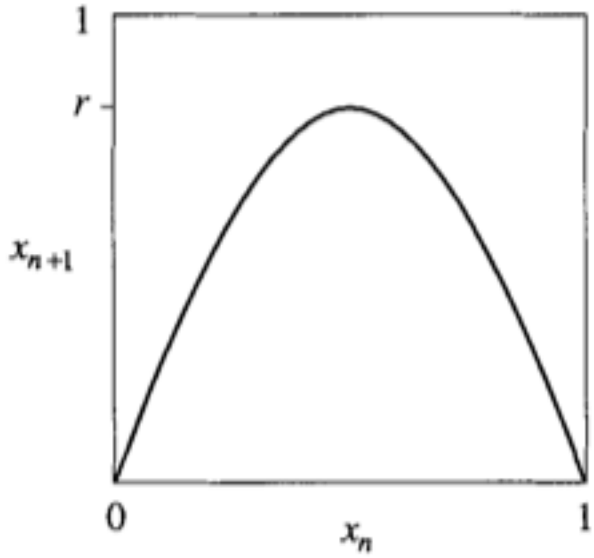
Remark: since each cycle starts with multiplier $f'(x) = 1$ and progressively goes until $f'(x) < -1$, when it becomes unstable, it must cross the point $f'(x) = 0$ (superstability).

Universality and experiments

The sine map

$$x_{n+1} = r \sin \pi x_n$$

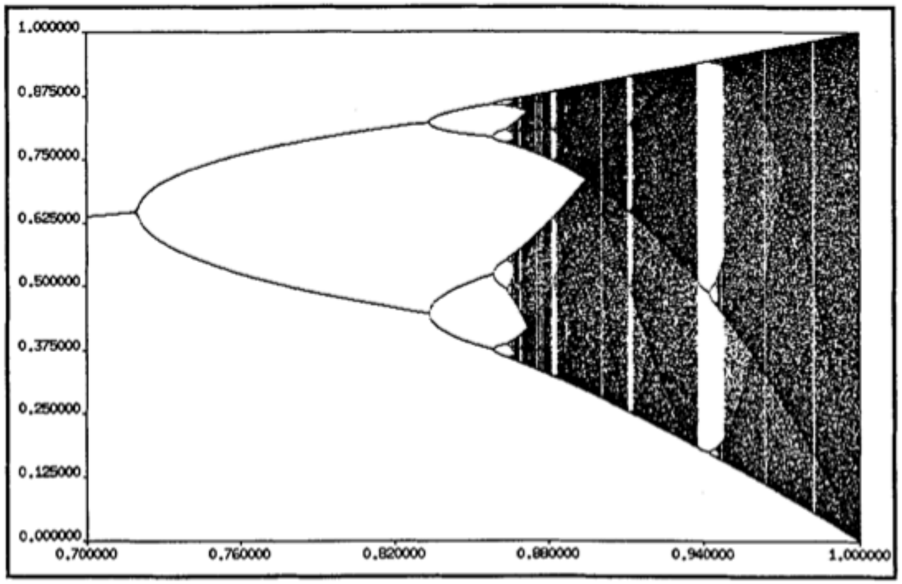
$0 \leq r \leq 1,\ 0 \leq x \leq 1.$



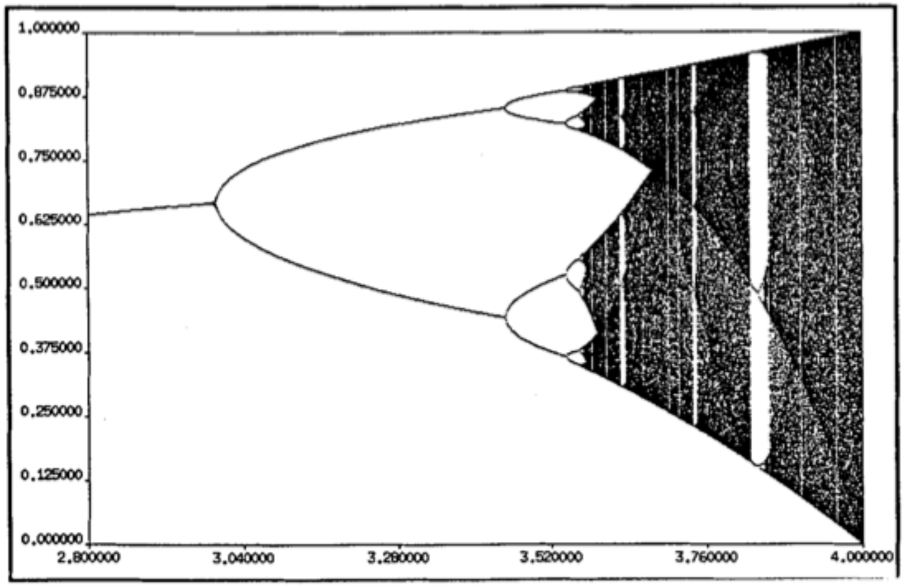
Unimodal map (concave down, single maximum), like the logistic map.

Universality and experiments

Sine map



Logistic map



Universality and experiments

Qualitative dynamics of the two maps are identical: period-doubling routes to chaos, followed by interspersed periodic windows.

Remarkable: periodic windows occur *in the same order* and *same relative sizes.*

Example: period-3 window is the largest and is preceded by two large windows (period-5 and period-6).

Quantitative differences: period doubling bifurcations occur later in the logistic map, and the periodic windows are narrower.

Qualitative universality: the U-sequence

Theorem (Metropolis et al. 1973): for all unimodal maps $x_{n+1} = rf(x_n)$, where $f(0) = f(1) = 0$, stable cycles occur in the same order.

The universal sequence in which periodic attractors occur is called the U-sequence. The algebraic form of $f(x)$ is irrelevant, only the overall shape matters.

U-sequence up to period 6: 1, 2, 2×2, 6, 5, 3, 2×3, 5, 6, 4, 6, 5, 6

The U-sequence has been found in experiments on the Belousov-Zhabotinsky chemical reaction.

The U-sequence is qualitative: it does not say anything about the size of the windows or where they start.

Quantitative universality

Feigenbaum's discovery in 1975

Attempt to find a formula for r_n, i.e. the r-value where a 2^n-cycle first appears.

Numerical checks were done with a handheld calculator.

First observation: the r_n converge geometrically to the onset of chaos r_∞: the size of consecutive windows shrinks by a constant factor 4.669…

Quantitative universality

I spent part of a day trying to fit the convergence rate value, 4.669, to the mathematical constants I knew. The task was fruitless, save for the fact that it made the number memorable.

At this point I was reminded by Paul Stein that period-doubling isn't a unique property of the quadratic map but also occurs, for example, in $x_{n+1} = r \sin \pi x_n$. However my generating function theory rested heavily on the fact that the nonlinearity was simply quadratic and not transcendental. Accordingly, my interest in the problem waned.

Perhaps a month later I decided to compute the r_n's in the transcendental case numerically. This problem was even slower to compute than the quadratic one. Again, it became apparent that the r_n's converged geometrically, and altogether amazingly, the convergence rate was the same 4.669 that I remembered by virtue of my efforts to fit it.

The same constant appears for any unimodal map!

Quantitative universality

$\Delta_n = r_n - r_{n-1}$ = distance between consecutive bifurcation values.

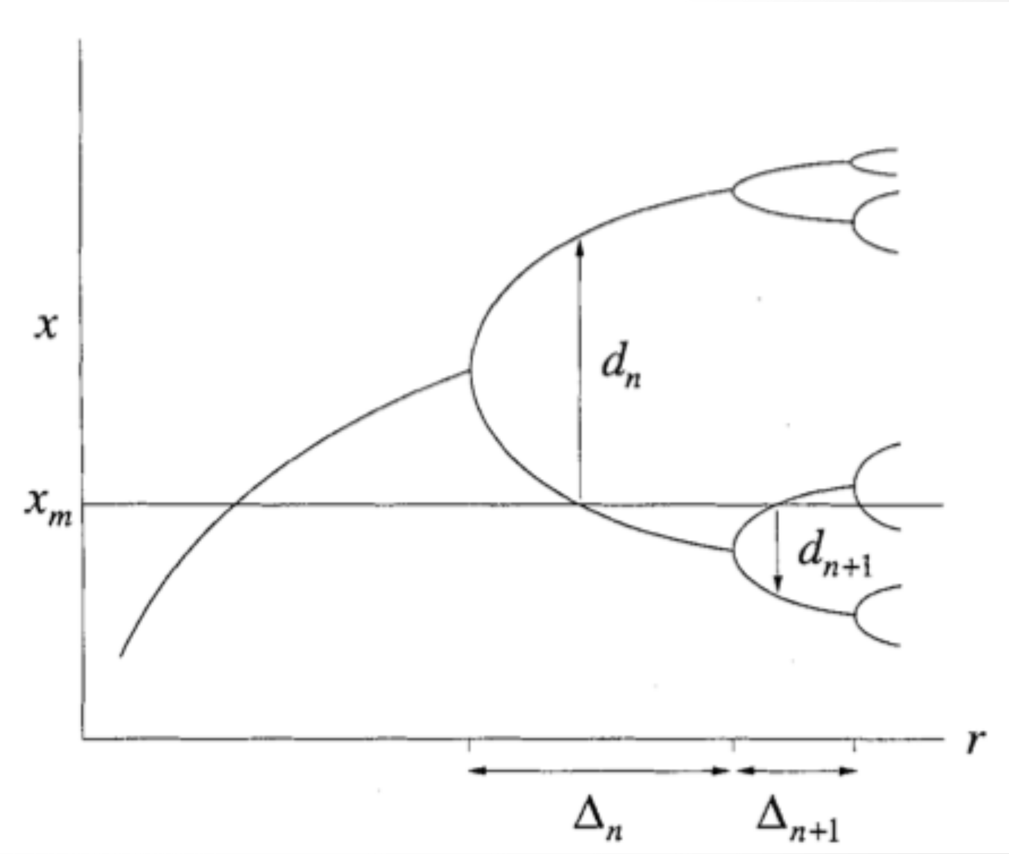
$$\frac{\Delta_n}{\Delta_{n+1}} \to \delta = 4.669\ldots, \quad \text{for } n \to \infty$$

d_n is the smallest distance from the maximum of f, x_m, to the nearest point in a 2^n-cycle.

$$\frac{d_n}{d_{n+1}} \to \alpha \approx -2.5029$$

as $n \to \infty$,

independent of the form of f.



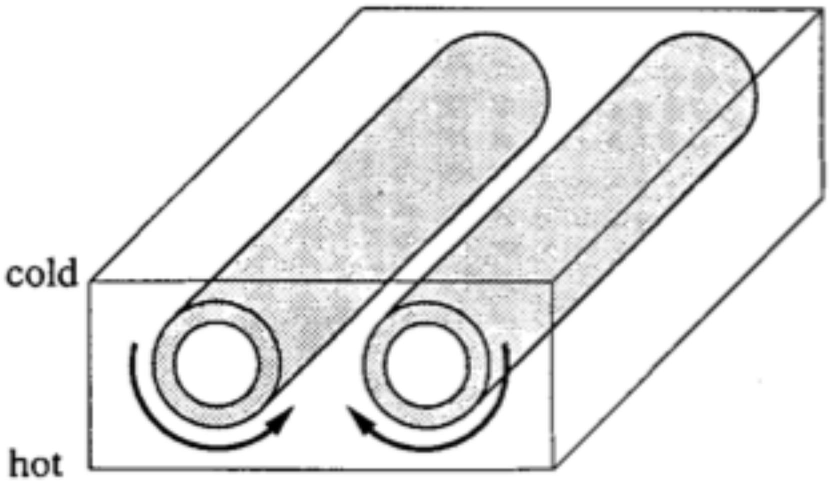
Experimental tests

Convection experiment by Libchaber et al. (1982)

A box of liquid mercury is heated from below.

Control parameter: Rayleigh number R, dimensionless measure of the externally imposed temperature gradient.

1) For $R < R_c$ heat is conducted upward and the liquid remains motionless.
2) For $R > R_c$ convection occurs: hot fluid rises on one side, loses heat and falls on the other side, in cylindrical rolls.



Experimental tests

Convection experiment by Libchaber et al. (1982)

1) If R is just above the threshold, the rolls are straight and the motion is steady, temperature is constant in time at each position.
2) If R is higher, a new instability sets in: a wave propagates back and forth along each roll, and the temperature at a given position oscillates.

Libchaber et al. wanted to stabilize the roll structure by applying a direct current (DC) magnetic field.

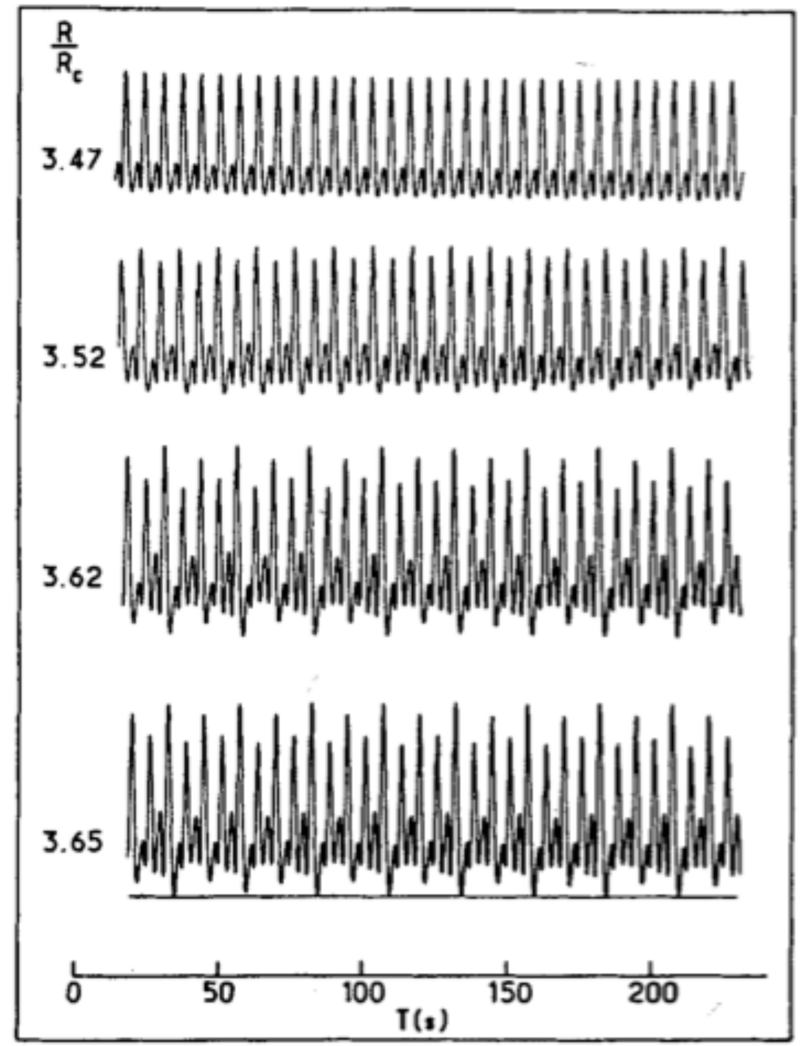
Mercury has a high electrical conductivity → strong tendency of the rolls to align with the field and to remain spatially organized.

Experimental tests

Convection experiment by Libchaber et al. (1982)

Result: the temperature at one point of the fluid undergoes a sequence of period-doublings as the Rayleigh number R increases.

By measuring the R-values at period-doubling bifurcations, Libchaber et al. estimated $\delta = 4.4 \pm 0.1$. (Theoretical $\delta \approx 4.699$.



Experimental tests

Results from experiments in fluid convection and nonlinear electronic circuits

Experiment	Number of period doublings	δ	Authors
Hydrodynamic			
water	4	4.3 (8)	Giglio et al. (1981)
mercury	4	4.4 (1)	Libchaber et al. (1982)
Electronic			
diode	4	4.5 (6)	Linsay (1981)
diode	5	4.3 (1)	Testa et al. (1982)
transistor	4	4.7 (3)	Arecchi and Lisi (1982)
Josephson simul.	3	4.5 (3)	Yeh and Kao (1982)

Agreement between experiments and theory impressive, given the difficulty (and relative errors) of these measurements

What do 1-D maps have to do with science?

Questions:

1) How can the complexity of so many different systems, which involve many degrees of freedom, be captured by a one-dimensional map?
2) Howcome a discrete-time map works so well on continuous-time systems?

Example: the Rössler system (1976)

$$\begin{aligned}\dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c)\end{aligned}$$

Simplest possible continuous-time system with a strange attractor.

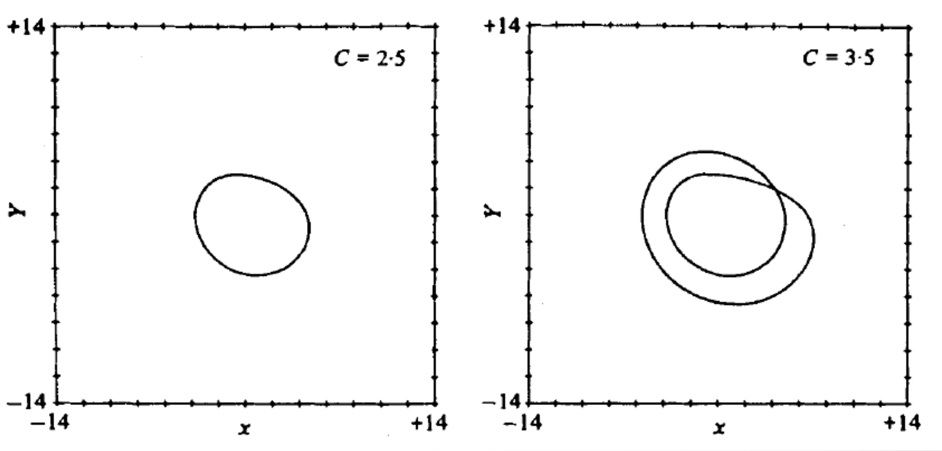
What do 1-D maps have to do with science?

The Rössler system

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c) \end{aligned}$$

$a = b = 0.2$, variable c

1) For $c = 2.5$ the attractor is a simple limit cycle
2) As c is raised to 3.5, the limit cycle winds twice before closing, with an approximately double period than the original one
3) Somewhere in $2.5 < c < 3.5$ a period-doubling bifurcation of cycles occurs (as in 1D maps)



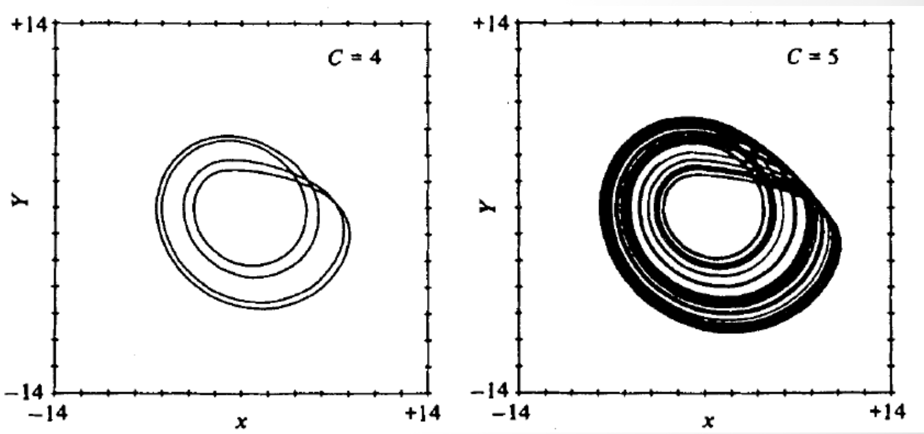
What do 1-D maps have to do with science?

The Rössler system

$$\begin{aligned}\dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c)\end{aligned}$$

$a = b = 0.2$, variable c

4) Another period-doubling bifurcation creates the four-loop cycle shown for c =4
5) After an infinite sequence of period-doublings, one reaches the strange attractor (c=5)



What do 1-D maps have to do with science?

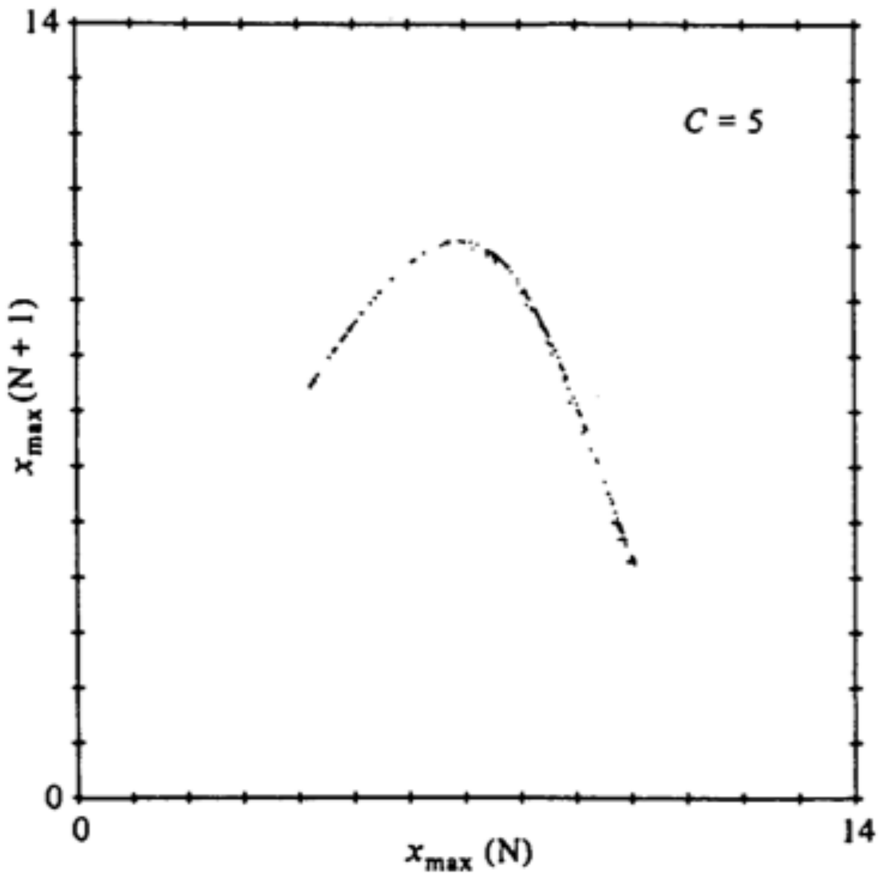
The Rössler system

$$\begin{aligned}\dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c)\end{aligned}$$

Lorenz map: relation between consecutive maxima x_n and x_{n+1} of one coordinate along a trajectory on the strange attractor.

The points fall on a 1D curve → there is a relation between x_n and x_{n+1}.

The curve resembles the logistic map!



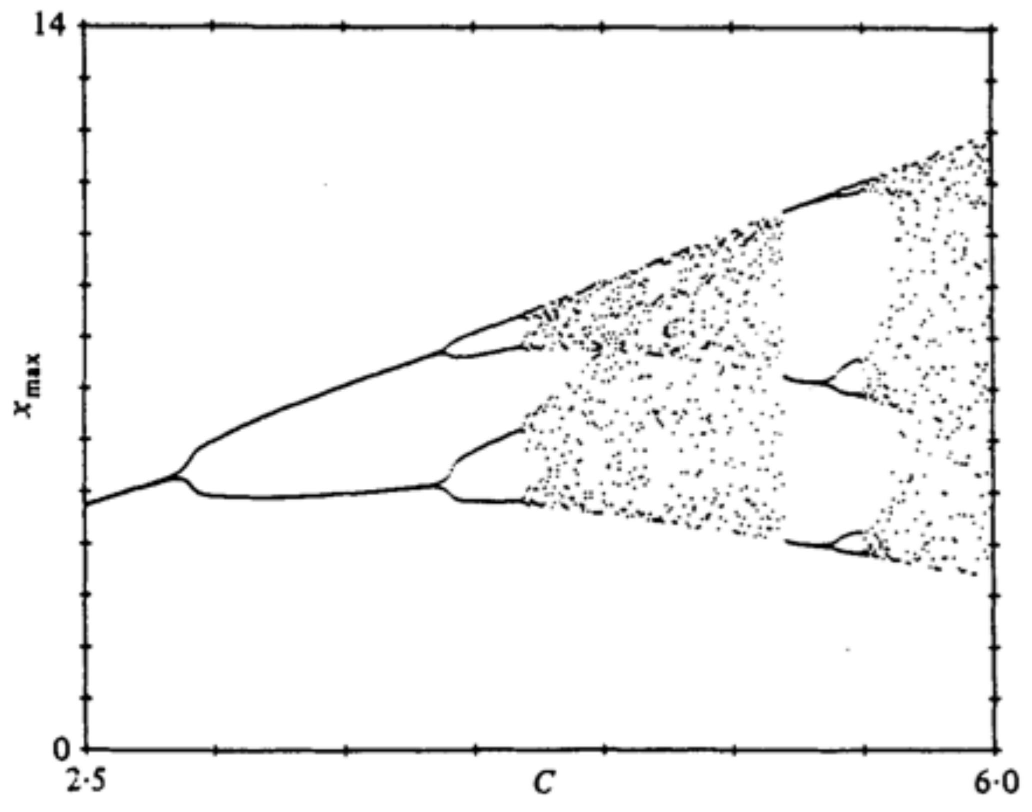
What do 1-D maps have to do with science?

The Rössler system

Orbit diagram for each *c*: position of maximum of *x* (*y* or *z*) on the attractor corresponding to *c*.

Strong similarity with the orbit diagram of the logistic map.

So, in this case, as for the Lorenz equations, Feigenbaum's results hold.



What do 1-D maps have to do with science?

For Rössler and Lorenz systems the map works because the strange attractors are essentially two-dimensional (fractal dimension slightly above 2).

In general Lorenz maps are not one-dimensional and Feigenbaum's theory does not apply.

Examples: fully turbulent fluids, fibrillating hearts.

Renormalization

Let $f(x, r)$ denote a unimodal map (smooth, concave down, single maximum) that undergoes a period-doubling route to chaos as r increases and x_m be the maximum of f. Let r_n denote the value of r at which a 2^n-cycle is born and R_n denote the value of r at which a 2^n-cycle is superstable.

Example: $f(x, r) = r - x^2$

Superstable FP: $x^* = R_0 - (x*)^2$

Superstability condition: $\lambda = (\partial f / \partial x)_{x=x^*} = 0$

$\partial f / \partial x = 2x \Rightarrow x^* = 0$ FP is the maximum of f. $R_0 = 0$

At R_1 there is a 2-cycle. Superstability: $\lambda = (-2p)(-2q) = 0$

$\rightarrow$ point $x = 0$ is one of the points in the 2-cycle.

Renormalization

Point $x = 0$ is one of the points in the 2-cycle.

→ period-2 condition $f^2(0, R_1) = 0 \;\Rightarrow R_1 - (R_1)^2 = 0$

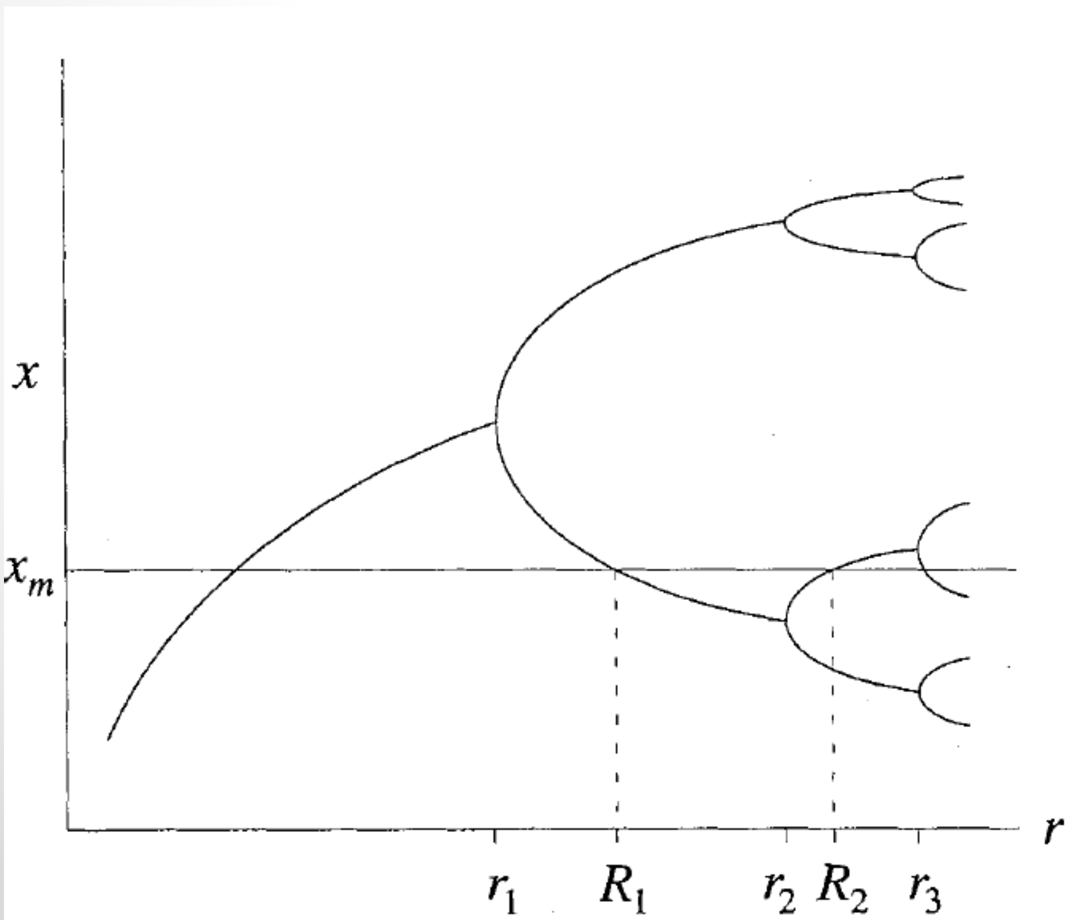
$\Rightarrow R_1 = 1$ (2-cycle)

General rule: A superstable cycle of a unimodal map always contains x_m as one of its points.

→ **Graphical way to locate R_n**: Draw a horizontal line at height x_m. Then R_n occurs where this line intersects the *figtree (= Feigenbaum)* portion of the orbit diagram.

Renormalization

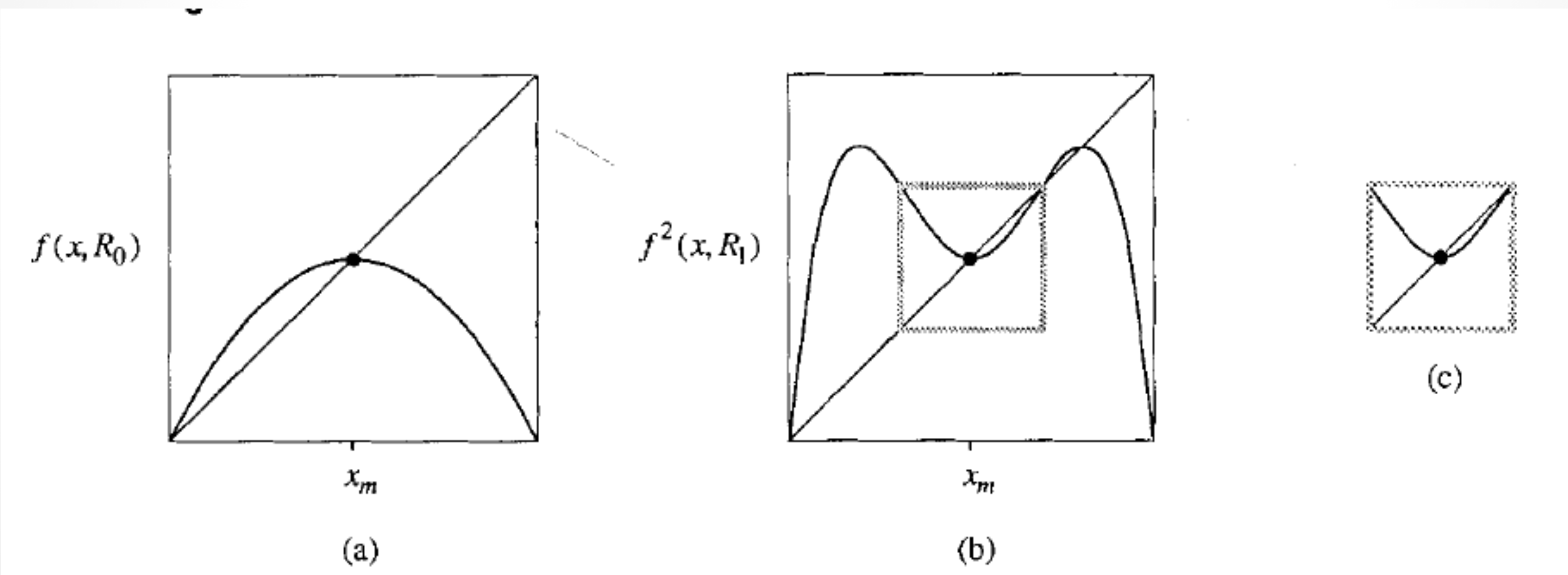
$$r_n < R_n < r_{n+1}$$



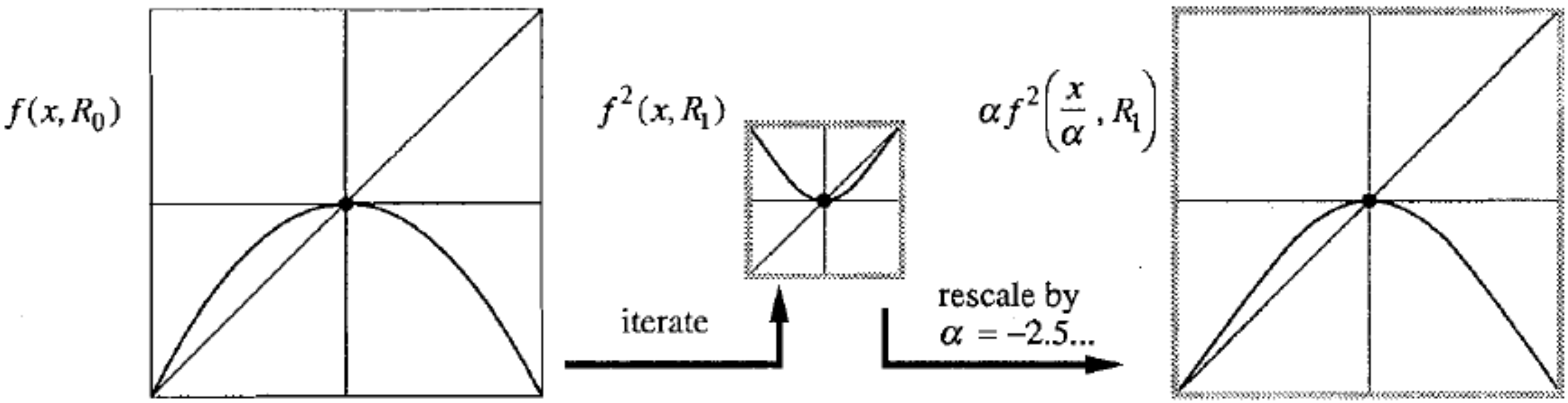
Numerical experiments: The spacing between successive R_n shrinks by the universal factor $\delta \approx 4.669$.

Renormalization

The renormalization theory is based on the **self-similarity** of the figtree: twigs look like the earlier branches, they are only scaled down in both the x and r directions. Mathematically, compare f with its second iterate f^2 at corresponding values of r and then renormalize one map into the other.



Renormalization



Renormalization of *f*: $$f(x, R_0) \approx \alpha f^2\left(\frac{x}{\alpha}, R_1\right)$$

Continue: $$f^2\left(\frac{x}{\alpha}, R_1\right) \approx \alpha^2 f^2\left(\frac{x}{\alpha^2}, R_2\right)$$

$$f(x, R_0) \approx \alpha^n f^{2^n}\left(\frac{x}{\alpha^n}, R_n\right)$$

Renormalization

Feigenbaum found numerically that

$$\lim_{n\to\infty} \alpha^n f^{(2^n)}\left(\frac{x}{\alpha^n}, R_n\right) = g_0(x),$$

where $g_0(x)$ is a ***universal function*** with a superstable fixed point. The limiting function exists only if α is chosen correctly,

$$\alpha = -2.5029 \ldots$$

"Universal": $g_0(x)$ is independent of *f*. Compare the qualitative similarity of orbit diagrams for different *f* in unimodal mapping.

Self-similarity → fractals; next time.