LECTURES ON THE CALCULATION OF THE AUTOCORRELATION AND AUTOCOVARIANCE FUNCTIONS OF STATIONARY AR- AND MA-PROCESSES

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Time Series Analysis

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MA(1) Moving Average of order 1

1

$$Y_t = \mu + \theta u_{t-1} + u_t$$

where $u_t \sim WN(0, \delta^2)$
white Noise
indep.
uncorr

Expected value

$$E(Y_t) = E(\mu + \theta u_{t-1} + u_t)$$
$$= E\mu + E(\theta u_{t-1}) + E u_t$$
$$= \mu + \theta \underbrace{E u_{t-1}}_{=0} + \underbrace{E u_t}_{=0}) \quad = \mu$$

Variance

$$\delta_0 = Var(Y_t) = Var(\mu + \theta u_{t-1} + u_t)$$
$$= Var\,\mu + Var(\theta u_{t-1}) + Var\,u_t$$
$$= 0 + \theta^2 \underbrace{Var\,u_{t-1}}_{\delta^2} + \underbrace{Var\,u_t}_{\delta^2} = \boxed{(\theta^2+1)\delta^2}$$

Autocovariance

$$\delta_1 = Cov(Y_t, Y_{t-1})$$
$$= Cov(\mu + \theta u_{t-1} + u_t, \mu + \theta u_{t-2} + u_{t-1})$$
$$= \cancel{Cov(\mu,\mu)} + \cancel{Cov(\mu, \theta u_{t-2})} + \cancel{Cov(\mu, u_{t-1})}$$
$$+ \cancel{Cov(\theta u_{t-1}, \mu)} + \cancel{Cov(\theta u_{t-1}, \theta u_{t-2})} + Cov(\theta u_{t-1}, u_{t-1})$$
$$+ \cancel{Cov(u_t, \mu)} + \cancel{Cov(u_t, \theta u_{t-2})} + \cancel{Cov(u_t, u_{t-1})}$$
$$= \cancel{\theta^2 Cov(u_{t-1}, u_{t-2})} + \theta\, Cov(u_{t-1}, u_{t-1})$$
$$+ \cancel{\theta\, Cov(u_t, u_{t-2})} + \cancel{Cov(u_t, u_{t-1})}$$
$$= \theta\, Var(u_{t-1}) = \boxed{\theta \delta^2}$$

$\gamma_2 = Cov(Y_t, Y_{t-2})$

$= Cov(\mu + \theta u_{t-1} + u_t, \mu + \theta u_{t-3} + u_{t-2})$

$= Cov(\theta u_{t-1}, \theta u_{t-3}) + Cov(\theta u_{t-1}, u_{t-2})$

$+ Cov(u_t, \theta u_{t-3}) + Cov(u_t, u_{t-2}) = 0$

Similarly $\gamma_3 = 0$, $\gamma_4 = 0, \dots$

So $\gamma_k = 0 \quad k \geq 2$

<u>Autocorrelation</u>

$$\tau_k = \frac{\gamma_k}{\gamma_0}$$

$$\tau_1 = \frac{\gamma_1}{\gamma_0} = \frac{\theta \sigma^2}{(\theta^2+1)\sigma^2} = \frac{\theta}{\theta^2+1}$$

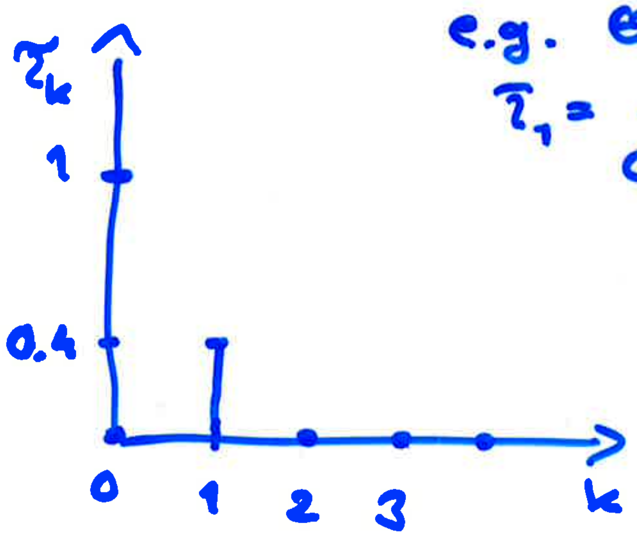
$$\tau_2 = \frac{\gamma_2}{\gamma_0} = 0$$

$\tau_3 = 0$

$\tau_k = 0, \quad k \geq 2$

Autocorrelation function

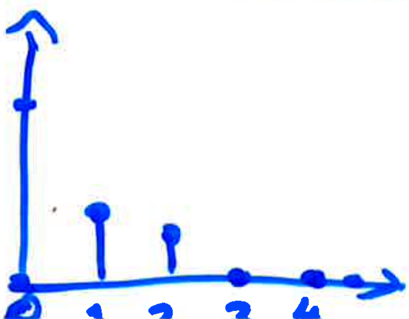
ACF



e.g. $\theta = .5$

$$\tau_1 = \frac{0.5}{0.25+1} = \underline{0.4}$$

MA(2): $Y_t = \mu + \theta_1 u_{t-1} + \theta_2 u_{t-2} + u_t$



<u>NOTE:</u> MA(q) process is <u>always</u> (weakly) stationary!

<u>AR(1)</u> $\quad y_t = \mu + \phi_1 y_{t-1} + u_t$

$$u_t \sim WN(0, \sigma^2)$$

Simplify by assuming $\mu = 0$.

Assume $\phi = 1/2$

$$y_t = \frac{1}{2} y_{t-1} + u_t$$

$$y_{t-1} = \frac{1}{2} y_{t-2} + u_{t-1}$$

$$\Rightarrow y_t = \frac{1}{4} y_{t-2} + \frac{1}{2} u_{t-1} + u_t$$

$$y_{t-2} = \frac{1}{2} y_{t-3} + u_{t-2}$$

$$\Rightarrow y_t = \frac{1}{8} y_{t-3} + \frac{1}{4} u_{t-2} + \frac{1}{2} u_{t-1} + u_t$$

$$y_{t-3} = \ldots$$

$$\vdots$$

$$\Rightarrow y_t = \underline{\left(\frac{1}{2}\right)^k y_{t-k}} + \left(\frac{1}{2}\right)^{k-1} u_{t-(k-1)} + \ldots + u_t$$

$$\longrightarrow \quad 0 + u_t + \frac{1}{2} u_{t-1} + \frac{1}{4} u_{t-2} + \ldots$$

when $k \to \infty$

<u>Conclusion</u>: Assuming $|\theta| < 1$ stationary AR(1) process can be given as an infinite MA(∞) process

Same is true for any stationary AR(p) process (assumption on coefficients $\theta_1, \theta_2, \ldots \theta_p$ will be given later)

Lag operator notation (L)

AR(1): $y_t = \phi_1 y_{t-1} + u_t$

$$L y_t = y_{t-1}$$

$$L^2 = L L y_t = y_{t-2}$$

$$\vdots$$

$$L^i y_t = y_{t-i}$$

AR(1): $y_t = \mu + \phi_1 L y_t + u_t$

Rearranging gives

$$y_t - \phi_1 L y_t = \mu + u_t$$

$$(1 - \phi_1 L) y_t = \mu + u_t \quad \Big| (1-\phi_1 L)^{-1}$$

Going back

$$y_t = (1-\phi_1 L)^{-1} (\mu + u_t)$$

$$= \frac{1}{1-\phi_1 L} (\mu + u_t) = \frac{\mu + u_t}{1-\phi_1 L}$$

Geometric series sum $\Rightarrow$ $q = \phi_1 L$

$$y_t = (1 + \phi_1 L + \phi_1^2 L^2 + \dots)(\underline{\mu} + u_t)$$

Simplify by setting $\mu = 0$ (does not affect variance or covariance)

$$y_t = (1 + \phi_1 L + \phi_1^2 L^2 + \dots)\, u_t$$
$$= u_t + \phi_1 u_{t-1} + \phi_1^2 u_{t-2} + \phi_1^3 u_{t-3} + \dots$$

MA(∞)-representation of AR(1)

AR(1): $E(y_t) = \mu = 0$ $\quad u_t \sim WN(0, \sigma^2)$

$$\boxed{Var(y_t)} = Var(u_t + \phi_1 u_{t-1} + \phi_1^2 u_{t-2} + \dots)$$
$$= \underbrace{Var(u_t)}_{\sigma^2} + \phi_1^2 \underbrace{Var(u_{t-1})}_{\sigma^2} + \phi_1^4 \underbrace{Var(u_{t-2})}_{\sigma^2} + \dots$$
$$= \sigma^2 (1 + \phi_1^2 + \phi_1^4 + \dots)$$
$$= \boxed{\frac{\sigma^2}{1-\phi_1^2} = \gamma_0}$$

Has to be $\phi_1^2 < 1$

$\boxed{|\phi_1| < 1}$

$$\gamma_1 = Cov(y_t, y_{t-1})$$
$$= Cov(u_t + \phi_1 u_{t-1} + \phi_1^2 u_{t-2} + \dots,\ u_{t-1} + \phi_1 u_{t-2} + \phi_1^2 u_{t-3} + \dots)$$
$$= Cov(\phi_1 u_{t-1}, u_{t-1}) + Cov(\phi_1^2 u_{t-2}, \phi_1 u_{t-2}) + \dots$$
$$= \phi_1 \sigma^2 + \phi_1^3 \sigma^2 + \phi_1^5 \sigma^2 + \dots$$
$$= \sigma^2 (\phi_1 + \phi_1^3 + \phi_1^5 + \dots) = \boxed{\frac{\sigma^2 \phi_1}{1-\phi_1^2} = \gamma_1}$$

$\boxed{|\phi_1| < 1}$

$\vdots$

$$\gamma_2 = \dots \frac{\sigma^2 \phi_1^2}{1-\phi_1^2}, \qquad \gamma_s = \frac{\sigma^2 \phi_1^s}{1-\phi_1^2}$$

stationarity condition

STATIONARITY OF AR(p) PROCESS

Process AR(p) is stationary if and only if all the roots of the characteristic equation z_i, $i = 1, \dots p$ are $|z_i| > 1$

("characteric roots are outside of the unit circle")

Ex. 1. $y_t = \frac{1}{2} y_{t-1} + u_t$ AR(1)

$$y_t - \underbrace{\tfrac{1}{2} y_{t-1}}_{\frac{1}{2} L y_t} = u_t$$

$$\underbrace{(1 - 0{,}5 L)}\, y_t = u_t$$

Characteristic equation $1 - 0{,}5z = 0$

Characteristic root $\boxed{z = 2} > 1$

$\Rightarrow$ $\boxed{\text{stationary}}$

Ex. 2 $y_t = \frac{5}{6} y_{t-1} - \frac{1}{6} y_{t-2} + u_t$ AR(2)

$$y_t - \tfrac{5}{6} y_{t-1} - \tfrac{1}{6} y_{t-2} = u_t$$

$$(1 - \tfrac{5}{6} L + \tfrac{1}{6} L^2) y_t = u_t$$

Char. eq. $1 - \frac{5}{6} z + \frac{1}{6} z^2 = 0$

$az^2 + bz + c = 0$

$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$z^2 - 5z + 6 = 0$$

$$z = \frac{5 \pm \sqrt{25 - 4 \cdot 6}}{2} = \frac{5 \pm 1}{2} = \begin{matrix} 3 \\ 2 \end{matrix} \Big\} > 1 \Rightarrow \boxed{\text{stationary}}$$

Ex. 3 $y_t = 3 y_{t-1} - 0.5\, y_{t-2} + u_t$

$$y_t - 3 y_{t-1} + 0.5\, y_{t-2} = u_t$$

$$(1 - 3L + 0.5 L^2) = u_t$$

Char. eq. $1 - 3z + 0.5 z^2 = 0$

$$0.5 z^2 - 3z + 1 = 0$$

$$z = \frac{3 \pm \sqrt{9 - \cancel{4} 4 \cdot 0.5 \cdot 1}}{1} = 3 \pm \sqrt{7}$$

$$3 + \sqrt{7} > 1$$

$$0 < 3 - \sqrt{7} < 1 \Rightarrow \boxed{y_t \text{ is non-stationary}}$$

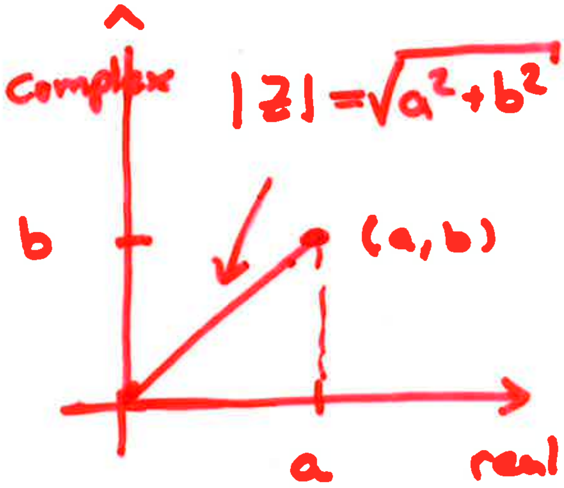
Ex. 4 Complex numbers

$i = \sqrt{-1}$ $\quad i^2 = -1$

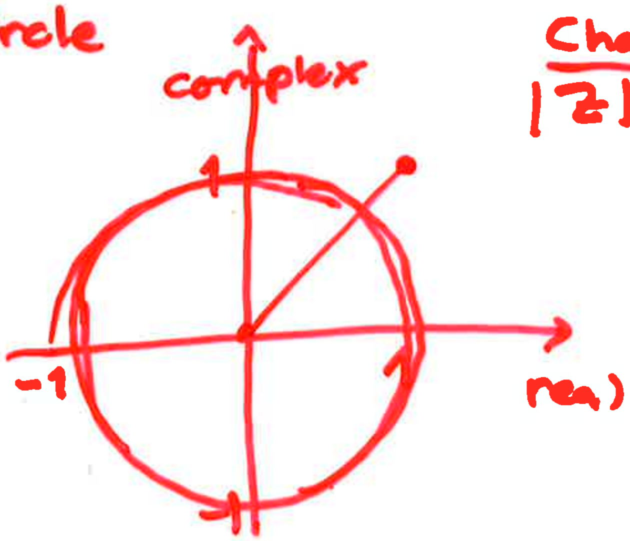
Complex number

$z = a + bi$

real part $\quad$ complex part



Unit circle



Characteristic roots

$|z| > 1 \Rightarrow$ stationary process

$|z| \leq 1 \Rightarrow$ non-stationary process

Ex. 4 $y_t = 0.5\, y_{t-1} - 0.5\, y_{t-2} + u_t$

$$y_t - 0.5\, y_{t-1} + 0.5\, y_{t-2} = u_t$$

$$(1 - 0.5 L + 0.5 L^2)\, y_t = u_t$$

Char. eq. $\underline{1 - 0.5 z + 0.5 z^2 = 0}$

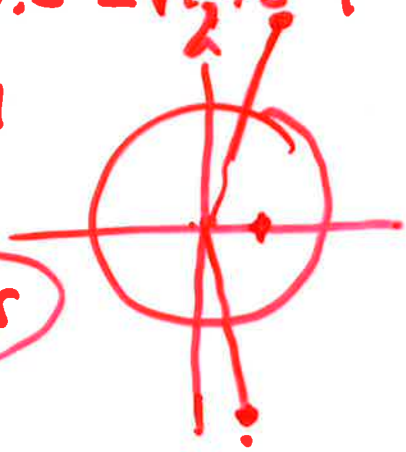
$$z = \frac{0.5 \pm \sqrt{0.5^2 - 4 \cdot 0.5 \cdot 1}}{1}$$

$$= 0.5 \pm \sqrt{-1.75} = 0.5 \pm \sqrt{1.75}\, i$$

$$|z|^2 = 0.5^2 + 1.75 = 2 > 1$$

for both roots

$|z| = \sqrt{2}$



$\Rightarrow$ stationary process