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Time Series Analysis

Yule Walker Equations

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Yule-Walker Equations

Yule-Walker Equations can be used to calculate auto-covariance and auto-correlation function (theoretically) if processes are *stationary*

Example: AR(2) process

$$\begin{aligned} Y_t &= \mu + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + u_t \\ \gamma_k &= \text{Cov}(Y_t, Y_{t-k}) = \text{Cov}(\mu + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + u_t, Y_{t-k}) \\ &= \text{Cov}(\phi_1 Y_{t-1}, Y_{t-k}) + \text{Cov}(\phi_2 Y_{t-2}, Y_{t-k}) + \text{Cov}(u_t, Y_{t-k}) \\ &= \phi_1 \underbrace{\text{Cov}(Y_{t-1}, Y_{t-k})}_{\gamma_{k-1}} + \phi_2 \underbrace{\text{Cov}(Y_{t-2}, Y_{t-k})}_{\gamma_{k-2}} + \underbrace{\text{Cov}(u_t, Y_{t-k})}_{= \begin{cases} \sigma^2 & \text{if } k = 0 \\ 0, & k \geq 1 \end{cases}} \end{aligned}$$

$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} + \text{Cov}(u_t, Y_{t-k})$$

Yule-Walker Equations

$$\begin{aligned} k = 0: \text{Cov}(u_t, Y_{t-k}) &= \text{Cov}(u_t, \mu + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + u_t) \\ &= \underbrace{\text{Cov}(u_t, \phi_1 Y_{t-1})}_{= 0} + \underbrace{\text{Cov}(u_t, \phi_2 Y_{t-2})}_{= 0} + \underbrace{\text{Cov}(u_t, u_t)}_{\text{Var}_u u_t = \sigma^2} \end{aligned}$$

Note: $\gamma_{-k} = \gamma_k$

$$k = 0: \gamma_0 = \phi_1 \gamma_1 + \phi_2 \gamma_2 + \sigma^2 \quad (0)$$

$$k = 1: \gamma_1 = \phi_1 \gamma_0 + \phi_2 \gamma_1 \quad (1)$$

$$k = 2: \gamma_2 = \phi_1 \gamma_1 + \phi_2 \gamma_0 \quad (2)$$

$$k \geq 3: \gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} \quad (3)$$

Solving equations (0), (1) and (2), we can find γ_0 , γ_1 and γ_2 . γ_k , $k \geq 3$ can be then obtained from equation (3)

Yule-Walker Equations

Autocorrelations:

$$\tau_k = \frac{\gamma_k}{\gamma_0}$$

$$\tau_0 = 1$$

Divide (0), (1), (2), (3) by γ_0

$$k = 0: \tau_0 = 1 = \phi_1 \tau_1 + \phi_2 \tau_2 + \frac{\sigma^2}{\gamma_0}$$

$$k = 1: \tau_1 = \phi_1 + \phi_2 \tau_1$$

$$k = 2: \tau_2 = \phi_1 \tau_1 + \phi_2$$

$$k \geq 3: \tau_k = \phi_1 \tau_{k-1} + \phi_2 \tau_{k-2}$$

Solve τ_1 and τ_2 : $\tau_1 - \phi_2 \tau_1 = \phi_1 \rightarrow (1 - \phi_2) \tau_1 = \phi_1 \rightarrow \tau_1 = \frac{\phi_1}{1 - \phi_2}$

$$\tau_2 = \frac{\phi_1^2}{1 - \phi_2} + \phi_2 = \frac{\phi_1^2 - (1 - \phi_2)\phi_2}{1 - \phi_2} = \frac{\phi_1^2 - \phi_2 + \phi_2^2}{1 - \phi_2}$$

$$\tau_3 = \phi_1 \tau_2 + \phi_2 \tau_1$$

Yule-Walker Equations

Yule-Walker matrix form equations:

$$\underbrace{\begin{bmatrix} \tau_1 \\ \tau_2 \\ \dots \\ \tau_3 \end{bmatrix}}_{\tau} = \underbrace{\begin{bmatrix} 1 & \tau_1 & \tau_2 & \dots & \tau_{p-1} \\ \tau_1 & 1 & \tau_1 & \dots & \tau_{p-2} \\ \dots & \dots & \dots & \dots & \dots \\ \tau_{p-1} & \tau_{p-2} & \dots & \dots & 1 \end{bmatrix}}_R \underbrace{\begin{bmatrix} \phi_1 \\ \phi_2 \\ \dots \\ \phi_3 \end{bmatrix}}_{\phi}$$

R
Autocorrelation matrix

$$\boxed{\tau = R\phi}$$

Yule-Walker Equations

How to calculate partial correlation

1. Always $\tau_{11} = \tau_1$

E.g. AR(1) $Y_t = \mu + \phi_1 Y_{t-1}$

$$\tau_1 = \phi_1 = \tau_{11}$$

For AR(p) model

2. $Y_t = \mu + \phi_1 Y_{t-1} + \dots + \phi_p Y_{t-p} + u_t$

$$\tau_{pp} = \phi_p$$

3. $\tau_{kk} = 0$ for $k \geq p$ AR (p)

4. τ_{kk} when $2 \leq k < p$

τ_{kk} is calculated using YULE-WALKER EQUATIONS assuming we know

$\tau_1, \tau_2, \dots, \tau_k$

Yule-Walker Equations

4. τ_{kk} when $2 \leq k < p$

τ_{kk} is calculated using YULE-WALKER EQUATIONS assuming we know $\tau_1, \tau_2, \dots, \tau_k$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_k \end{bmatrix} = \begin{bmatrix} 1 & \tau_1 & \tau_2 & \dots & \tau_{k-1} \\ \tau_1 & 1 & \tau_1 & \dots & \tau_{k-2} \\ \dots & \dots & \dots & \dots & \dots \\ \tau_{k-1} & \tau_{k-2} & \dots & \dots & 1 \end{bmatrix} \begin{bmatrix} \tau_{k1} \\ \tau_{k2} \\ \vdots \\ \tau_{kk} \end{bmatrix}$$

← This is solved

Example:

$$\text{AR}(3): Y_t = \mu + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \phi_3 Y_{t-3} + u_t$$

$$\tau_{11} = \tau_1$$

$$\tau_{33} = \phi_3$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} 1 & \tau_1 \\ \tau_1 & 1 \end{bmatrix} \begin{bmatrix} \tau_{21} \\ \tau_{22} \end{bmatrix}$$

← This is solved