Homework 2 — Sample solutions *7 March 2019*

Return solution by 20:00, 13 March 2019 — electronically in MyCourses

2. (a) Here we explore some basic concept of the reciprocal basis vectors. Take three complex vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ where

$$\mathbf{a} = \mathbf{u}_x + \mathbf{u}_y \tag{1}$$

$$\mathbf{b} = \mathbf{u}_y + 2\mathrm{j}\mathbf{u}_z \tag{2}$$

$$\mathbf{c} = \mathrm{j}\mathbf{u}_x - 3\mathbf{u}_z \tag{3}$$

i. Compute the reciprocal vectors $\mathbf{a}', \mathbf{b}', \mathbf{c}'$

ii. Compute the dyadic $\mathbf{aa}' + \mathbf{bb}' + \mathbf{cc}'$

iii. Expand the vector $\mathbf{u}_x = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$

(b) Dyadics: Show, by expanding in the cartesian components that

$$(\mathbf{a} \times \mathbf{b}) \times \overline{\overline{\mathsf{I}}} = \mathbf{ba} - \mathbf{ab} \tag{4}$$

You can use as test examples the vectors $\mathbf{a} = a_x\mathbf{u}_x$ and $\mathbf{b} = b_x\mathbf{u}_x + b_y\mathbf{u}_y + b_z\mathbf{u}_z$. Similarly, is the expression

$$\left(\mathbf{b} \times \overline{\overline{\mathsf{I}}}\right) \times \mathbf{a} = \mathbf{ab} - (\mathbf{a} \cdot \mathbf{b})\overline{\overline{\mathsf{I}}} \tag{5}$$

valid or not?

(c) Dyadic: Assuming the unit dyadic $\overline{\overline{\mathsf{I}}}$ and

$$\overline{\overline{\mathsf{B}}} = \overline{\overline{\mathsf{I}}} + 2\left(\mathbf{u}_x\mathbf{u}_y + \mathbf{u}_y\mathbf{u}_x\right) \tag{6}$$

answer/compute the following:

i. Is $\overline{\overline{\mathsf{B}}}$ symmetric?

ii. $\mathrm{tr}\overline{\overline{\mathsf{B}}} \quad \left(= \overline{\overline{\mathsf{B}}} : \overline{\overline{\mathsf{I}}}\right)$,

iii. $\mathrm{spm}\overline{\overline{\mathsf{B}}} \quad \left(= \frac{1}{2}\overline{\overline{\mathsf{B}}} {}^{\times}_{\times} \overline{\overline{\mathsf{B}}} : \overline{\overline{\mathsf{I}}}\right)$,

iv. $\det\overline{\overline{\mathsf{B}}} \quad \left(= \frac{1}{6}\overline{\overline{\mathsf{B}}} {}^{\times}_{\times} \overline{\overline{\mathsf{B}}} : \overline{\overline{\mathsf{B}}}\right)$, and

v. its inverse $\quad \left(= \overline{\overline{\mathsf{B}}}^{-1}\right)$.

vi. Check and show that $\overline{\overline{\mathsf{B}}} \cdot \overline{\overline{\mathsf{B}}}^{-1} = \overline{\overline{\mathsf{I}}}$ and $\overline{\overline{\mathsf{B}}}^{-1} \cdot \overline{\overline{\mathsf{B}}} = \overline{\overline{\mathsf{I}}}$.

Sample solutions

2. (a) Using results in Section 1.6 of the book, we find (note that $\mathbf{a}\times\mathbf{b}\cdot\mathbf{c} = -5$)

$$\mathbf{a}' = \frac{1}{5}(3\mathbf{u}_x + 2\mathbf{u}_y + \mathrm{j}\mathbf{u}_z), \quad \mathbf{b}' = \frac{1}{5}(-3\mathbf{u}_x + 3\mathbf{u}_y - \mathrm{j}\mathbf{u}_z), \quad \mathbf{c}' = \frac{1}{5}(-2\mathrm{j}\mathbf{u}_x + 2\mathrm{j}\mathbf{u}_y - \mathbf{u}_z)$$

Noting that

$$\mathbf{aa}' = \frac{1}{5}(3\mathbf{u}_x\mathbf{u}_x + 2\mathbf{u}_x\mathbf{u}_y + \mathrm{j}\mathbf{u}_x\mathbf{u}_z + 3\mathbf{u}_y\mathbf{u}_x + 2\mathbf{u}_y\mathbf{u}_y + \mathrm{j}\mathbf{u}_y\mathbf{u}_z)$$

$$\mathbf{bb}' = \frac{1}{5}(-3\mathbf{u}_y\mathbf{u}_x + 3\mathbf{u}_y\mathbf{u}_y - \mathrm{j}\mathbf{u}_y\mathbf{u}_z - 6\mathrm{j}\mathbf{u}_z\mathbf{u}_x + 6\mathrm{j}\mathbf{u}_z\mathbf{u}_y + 2\mathbf{u}_z\mathbf{u}_z)$$

$$\mathbf{cc}' = \frac{1}{5}(2\mathbf{u}_x\mathbf{u}_x - 2\mathbf{u}_x\mathbf{u}_y - \mathrm{j}\mathbf{u}_x\mathbf{u}_z + 6\mathrm{j}\mathbf{u}_z\mathbf{u}_x - 6\mathrm{j}\mathbf{u}_z\mathbf{u}_y + 3\mathbf{u}_z\mathbf{u}_z)$$

their sum becomes the unit dyadic.
The expansion comes easily using the previous result:

$$\mathbf{u}_x = \overline{\overline{\mathsf{I}}}\cdot\mathbf{u}_x = (\mathbf{aa}' + \mathbf{bb}' + \mathbf{cc}')\cdot\mathbf{u}_x = \frac{3}{5}\mathbf{a} - \frac{3}{5}\mathbf{b} - \mathrm{j}\frac{2}{5}\mathbf{c}$$

(b)

$$\mathbf{a}\times\mathbf{b} = a_x b_y \mathbf{u}_z - a_x b_z \mathbf{u}_y$$

$$(\mathbf{a}\times\mathbf{b})\times\overline{\overline{\mathsf{I}}} = a_x b_y \mathbf{u}_y\mathbf{u}_x - a_x b_y \mathbf{u}_x\mathbf{u}_y - a_x b_z \mathbf{u}_x\mathbf{u}_z + a_x b_z \mathbf{u}_z\mathbf{u}_x$$

On the other hand,

$$\mathbf{ba} = a_x b_x \mathbf{u}_x\mathbf{u}_x + a_x b_y \mathbf{u}_y\mathbf{u}_x + a_x b_z \mathbf{u}_z\mathbf{u}_x \quad \text{and} \quad \mathbf{ab} = a_x b_x \mathbf{u}_x\mathbf{u}_x + a_x b_y \mathbf{u}_x\mathbf{u}_y + a_x b_z \mathbf{u}_x\mathbf{u}_z$$

whence their difference equals $(\mathbf{a}\times\mathbf{b})\times\overline{\overline{\mathsf{I}}}$.
Likewise the component expansion proves that

$$\left(\mathbf{b}\times\overline{\overline{\mathsf{I}}}\right)\times\mathbf{a} = \mathbf{ab} - (\mathbf{a}\cdot\mathbf{b})\overline{\overline{\mathsf{I}}}$$

which can be seen also from

$$\left(\mathbf{b}\times\overline{\overline{\mathsf{I}}}\right)\times\mathbf{a} = \mathbf{b}\times\overline{\overline{\mathsf{I}}}\cdot\mathbf{a}\times\overline{\overline{\mathsf{I}}} = \mathbf{b}\times\left(\mathbf{a}\times\overline{\overline{\mathsf{I}}}\right) = \mathbf{ab} - (\mathbf{a}\cdot\mathbf{b})\overline{\overline{\mathsf{I}}}$$

where in the last one we used the "bac-cab" rule.

(c)
i. Yes.
ii. $\mathrm{tr}\overline{\overline{\mathsf{B}}} = 3$
iii. $(1/2)\overline{\overline{\mathsf{B}}}{}^{\times}_{\times}\overline{\overline{\mathsf{B}}} = \mathbf{u}_x\mathbf{u}_x + \mathbf{u}_y\mathbf{u}_y - 3\mathbf{u}_z\mathbf{u}_z - 2(\mathbf{u}_x\mathbf{u}_y + \mathbf{u}_y\mathbf{u}_x)$ which means that $\mathrm{spm}\overline{\overline{\mathsf{B}}} = -1$.
iv. $\det\overline{\overline{\mathsf{B}}} = -3$
v. $\overline{\overline{\mathsf{B}}}^{-1} = (1/2)\left(\overline{\overline{\mathsf{B}}}{}^{\times}_{\times}\overline{\overline{\mathsf{B}}}\right)^T/\det\overline{\overline{\mathsf{B}}} = (1/3)\left(-\mathbf{u}_x\mathbf{u}_x - \mathbf{u}_y\mathbf{u}_y + 3\mathbf{u}_z\mathbf{u}_z + 2(\mathbf{u}_x\mathbf{u}_y + \mathbf{u}_y\mathbf{u}_x)\right)$
vi. Straightforward computation.