Homework 5

CIV-E4080, Material Modeling in Civil Engineering L

Engineering viscoelasticity

Introduction

The purpose of this homework is to study classical viscoelasticity concepts through solving examples in order to consolidate what you've learn till now from textbooks and during lectures. The content of this homework are the constitutive models of linear viscoelasticity listed below:

- Maxwell model
- Kelvin-Voight model
- Standard linear solid model
- Generalized Maxwell model
- Kelvin chain model

There are total 7 problems in this home work but ONLY 4 are compulsory. Problem 6 is obligatory. Other problems can be solved for extra marks.

Readings

Chapter 4.3: **Lemaitre and Chaboche**, *Mechanics of Solid Materials.*
Also see references at the end of this homework set.

Reminder - Viscoelasticity, in short & in words

Viscoelastic materials have i) a time dependent response to constant loading as for instance, to force, temperature and strain, and ii) they exhibit also rate depend responses. iii) depending on the duration of the excitation (fast as in shock problems or slow as in long-term stability problems), the same material may exhibit some of solid or fluid behavior or both of them at the same time.

They have the ability to creep, recover partially or fully, undergo stress relaxation and absorb energy. ***They possess*** *some of the combined mechanical properties of fluid and solids as related to the stress-strain response.*

Problem 1

[5 points] A material can be described by In a **Kelvin-Voigt** (KV) model. Consider creep response of such a material for constant stress σ_o. Such response is modeled by the following expression,

$$\epsilon(t) = \frac{\sigma_0}{E}[1 - e^{-t/\tau_c}], \qquad (1)$$

where, $\tau_c \equiv \eta/E$ - retardation time and η - viscosity of the dashpot

1. Show the above result for creep response of the material. [3 points]

2. **Consider the following creep test:** A material having E = 600 MPa is initially loaded with a constant stress σ_0. The constitutive behavior (behavior law) of the material can be described by Kelvin-Voigt (KV) model.

   Half an hour (t = 30 min) after applying stress, the measured strain is 0.111. Another hour later (t = 90 min) , the strain becomes 0.264.

   Determine the strain after three hours of loading. [1 point]

   After what time the strain reaches back to 0.001 ? [1 point]

Solution

1. We know the expression,

$$\epsilon(t) = \frac{\sigma_0}{E}[1 - e^{-t/\tau_c}], \tag{2}$$

   Inputting values from two measurements,

$$0.111 = \frac{\sigma_0}{600MPa}[1 - e^{-0.5hrs/\tau_c}], \tag{3}$$

$$0.264 = \frac{\sigma_0}{600MPa}[1 - e^{-1.5hrs/\tau_c}], \tag{4}$$

   Dividing equation (3) by (4) and solving for τ_c,

$$\tau_c = 1.97hrs \tag{5}$$

   We can now find σ_0 from equation (3) or (4),

$$\sigma_0 = 297.2MPa \tag{6}$$

2. Expression for creep response of the material becomes,

$$\epsilon(t) = \frac{297.2MPa}{600MPa}[1 - e^{-t/1.97hrs}] \tag{7}$$

   Strain after 3 hours is,

$$\epsilon(3hrs) = \frac{297.2MPa}{600MPa}[1 - e^{-3hrs/1.97hrs}] = 0.39 \tag{8}$$

   When the strain is 0.001, the time (t) is found to be,

$$\epsilon(t) = 0.001 = \frac{297.2MPa}{600MPa}[1 - e^{-t/1.97hrs}] => t = 14.3s \tag{9}$$

Problem 2

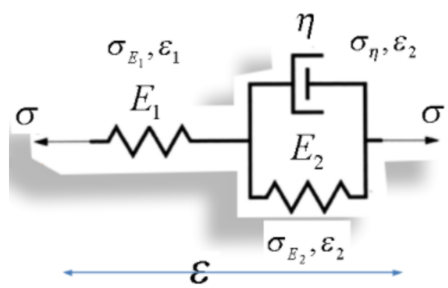
1. Describe concisely the difference between a **Creep Test** and **Stress Relaxation Test**. [2 points]

2. **The Standard Linear Solid Model**

   (a) Consider a **standard linear solid (SLS)**. Derive the constitutive equation relating the overall stress, stress rate, strain and strain rate. The model parameters of the system are, E_1, E_2 and η.

   (b) For the standard linear solid discussed earlier, determine the expression of the total strain $\epsilon(t)$ in terms of E_1, E_2 and η for the case of constant stress. What is the creep function?

   (c) As you may remember, retardation time is defined as $\tau = \eta/E$ [s]. Consider the following situation: immediately after applying stress, the strain is 0.002 (instantaneous strain), after 1000 seconds the strain grows to 0.004 and approaches asymptotically 0.006 after a very long time the strain. Determine the retardation time τ ?



A Standard Linear Solid Model.

Solution

1. In a creep test, constant stress is applied and strain is measured, whereas, in a relaxation test, a constant strain is applied and stress is measured.

2. (a)

$$\sigma = \sigma_1 = \sigma_2 \tag{10}$$

$$\sigma = E_1.\epsilon_1 = E_2.\epsilon_2 + \eta\dot{\epsilon_2} \tag{11}$$

$$\dot{\epsilon} = \dot{\epsilon_1} + \dot{\epsilon_2} = \frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta} - \frac{E_2\epsilon_2}{\eta} \tag{12}$$

$$(\epsilon_2 = \epsilon - \epsilon_1 = \epsilon - \frac{\sigma}{E_1})$$

$$\dot{\epsilon} = \frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta} - \frac{E_2}{\eta}(\epsilon - \frac{\sigma}{E_1}) \tag{13}$$

$$\dot{\epsilon} + \frac{E_2}{\eta}\epsilon = \frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta}(1 + \frac{E_2}{E_1}) \tag{14}$$

$$\dot{\epsilon} + \frac{E_2}{\eta}\epsilon = \frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta}(\frac{E_1 + E_2}{E_1}) \quad (15)$$

(b) For the case of constant stress,

$$\eta\dot{\epsilon} + E_2\epsilon = (1 + \frac{E_2}{E_1})\sigma \quad (16)$$

Solving the differential equation of type a*y + b*y = c.

$$\epsilon(t) = K.exp(\frac{-E_2 t}{\eta}) + \frac{(E_1 + E_2)}{E_2 E_1}\sigma \quad (17)$$

Where K is an integration constant. We know that immediately after applying the stress, the strain will be entirely from the lone spring ($\epsilon_1 = 0$) and so,

$$\epsilon(t = 0) = \frac{\sigma}{E_1} \quad (18)$$

$$\frac{\sigma}{E_1} = K\frac{(E_2 + E_1)}{E_1 E_2}\sigma => K = \frac{-\sigma}{E_2} \quad (19)$$

$$\epsilon(t) = \frac{-\sigma}{E_2}.exp(\frac{-E_2 t}{\eta}) + \frac{(E_1 + E_2)}{E_2 E_1}\sigma = \frac{\sigma}{E_2}.[1 + \frac{E_2}{E_1} - exp(\frac{-E_2 t}{\eta})] \quad (20)$$

(c)

$$\epsilon(0) = \frac{\sigma}{E_1} = 0.002 \quad (21)$$

$$\epsilon(\infty) = \frac{\sigma}{E_2} + \frac{\sigma}{E_1} = 0.006 => \frac{\sigma}{E_2} + 0.002 = 0.006 => \frac{\sigma}{E_2} = 0.004 \quad (22)$$

$$\epsilon(1000) = \frac{\sigma}{E_2} + \frac{\sigma}{E_1} - \frac{\sigma}{E_2}[exp(\frac{-1000}{\tau})] = 0.004 + 0.002 - 0.004[exp(\frac{-1000}{\tau})] \quad (23)$$

$$\epsilon(1000) = 0.006 - 0.004[exp(\frac{-1000}{\tau})] = 0.004 \quad (24)$$

$$=> 0.004[exp(\frac{-1000}{\tau})] = 0.002 \quad (25)$$

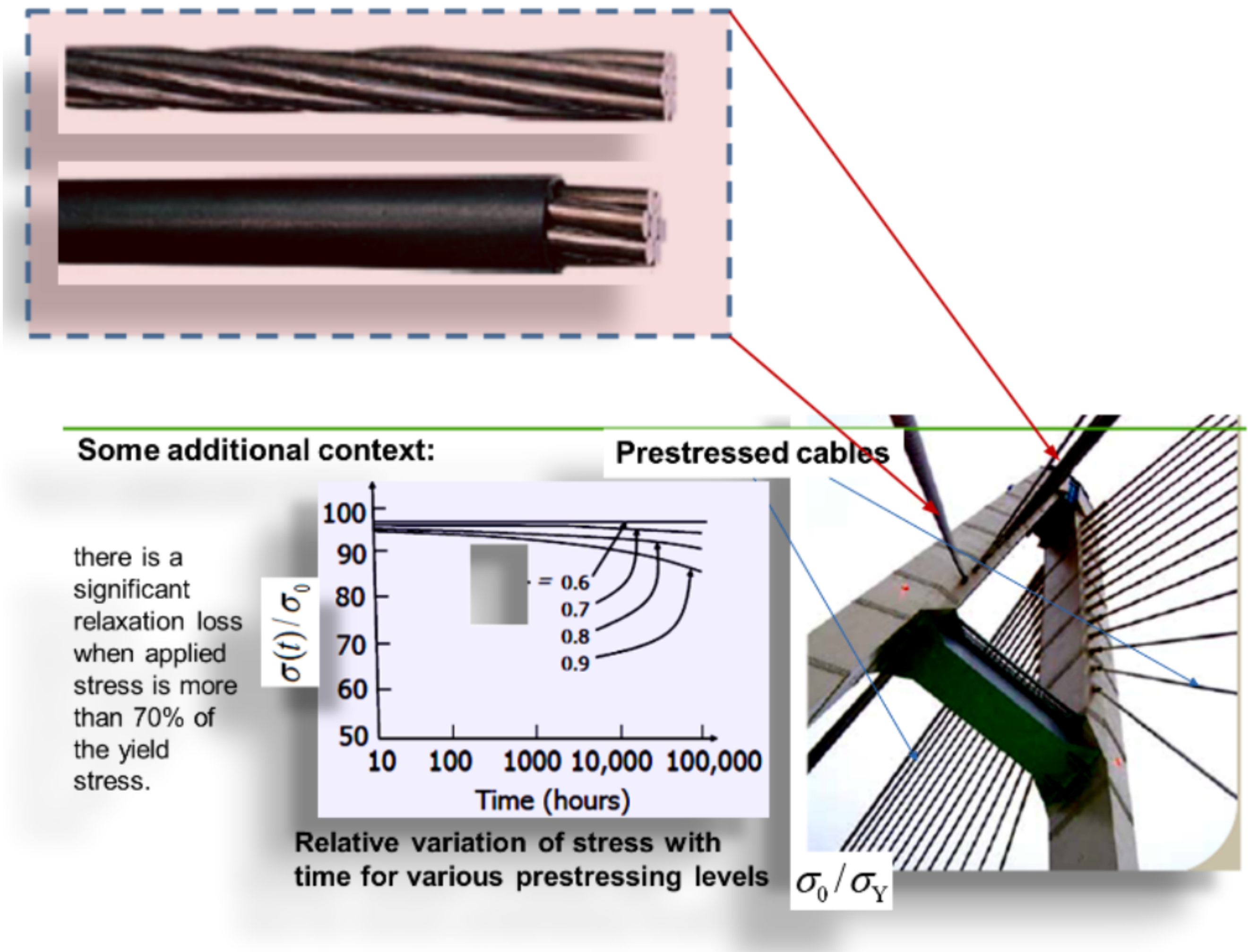
$$=> \tau = 1443 sec \quad (26)$$

Problem 3

[5 points] **Relaxation experiment for the material : (observation)**
After 2 weeks, a loss of 2MPa is observed in a cable while the initial stress was 100MPa.

1. Derive relaxation function (modulus) - Use simple Maxwell model.
2. Determine the characteristic relaxation time from relaxation experiment.
3. What should be the initial pre-stress level in order to keep over 150MPa stress after a year ?

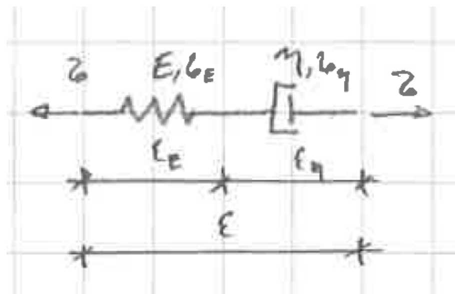
Assume a constant operating temperature of $\approx 20^o$ C.



Maxwell model relaxation function - The simplest model: $\sigma(t) = \epsilon_0.Ee^{-\frac{E}{\eta}t} \equiv \sigma_0 e^{-\frac{t}{\tau_R}}$
τ_R - The characteristic relaxation time of the material.

Solution

1. Maxwell model is given below,



The constitutive models for individual components are,

Spring,

$$\sigma = E\epsilon \tag{27}$$

Dashpot,

$$\sigma = \eta\dot{\epsilon} \tag{28}$$

Equilibrium conidtions for the model are,

$$\sigma = \sigma_E = \sigma_\eta \tag{29}$$

Compatibility conditions are,

$$\epsilon = \epsilon_E + \epsilon_\eta \tag{30}$$

$$\dot{\epsilon} = \dot{\epsilon_E} + \dot{\epsilon_\eta} \tag{31}$$

The constitutive equation can be obtained as,

$$\dot{\epsilon} = \dot{\epsilon_E} + \dot{\epsilon_\eta} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta} \tag{32}$$

$$\dot{\sigma} + \frac{E}{\eta}\sigma = E\dot{\epsilon} \tag{33}$$

Applying the known boundary condition for relaxation test, i.e., $\dot{\epsilon} = 0$, one obtains,

$$\dot{\sigma} + \frac{E}{\eta}\sigma = \frac{d\sigma}{dt} + \frac{E}{\eta}\sigma = 0 \tag{34}$$

$$=> \frac{d\sigma}{dt} = -\frac{E}{\eta}\sigma \tag{35}$$

$$=> \frac{d\sigma}{\sigma} = -\frac{E}{\eta}dt \tag{36}$$

Integrating both sides for 0 to t,

$$\int_0^t \frac{1}{\sigma} d\sigma = -\int_0^t \frac{E}{\eta} dt \tag{37}$$

$$=> \frac{\sigma(t)}{\sigma(0)} = e^{\frac{-E}{\eta}t} \tag{38}$$

Where, $\sigma(0) = \sigma_0$

$$=> \sigma(t) = \sigma_0.e^{\frac{-E}{\eta}t} \tag{39}$$

We know that the initial stress, $\sigma_0 = \epsilon_0.E$

$$=> \sigma(t) = \epsilon_0.E.e^{\frac{-E}{\eta}t} \tag{40}$$

The relaxation function is defined as,

$$G(t) = \frac{\sigma(t)}{\epsilon_0} \tag{41}$$

$$=> G(t) = E.e^{\frac{-E}{\eta}t} \tag{42}$$

We know that, $\frac{\eta}{E} = \tau_R$,

$$=> G(t) = E.e^{\frac{-t}{\tau_R}} \tag{43}$$

2. Characteristic relaxation time can be obtained as,

$$\sigma(2w) = \sigma_0.e^{t/\tau_R} = (100MPa).e^{-2w/\tau_R} = 98MPa \tag{44}$$

$$\tau_R = 98,997weeks \tag{45}$$

3. To keep the stress over 150MPa for over a year (52 weeks),

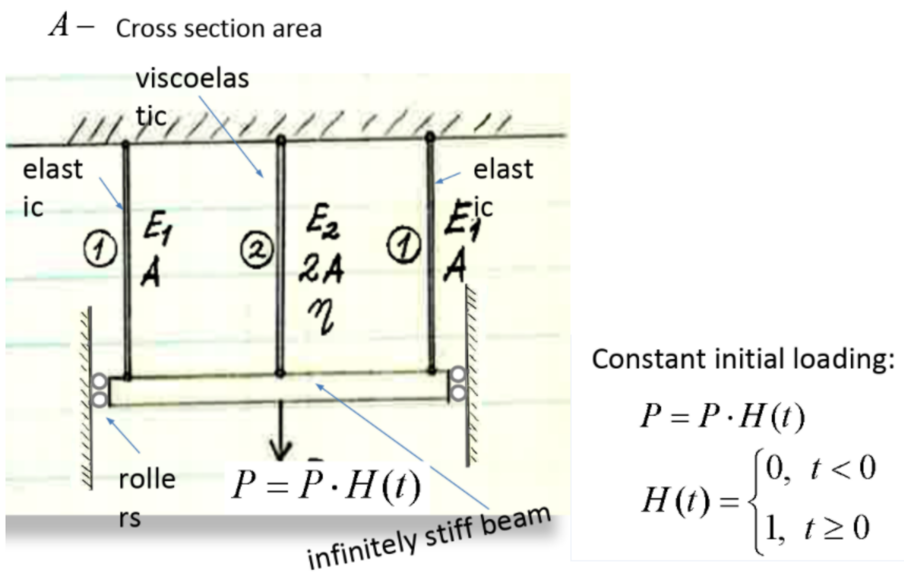
$$\sigma(52w) \geq 150MPa \tag{46}$$

$$\sigma_0.e^{52w/\tau_R} \geq 150MPa \tag{47}$$

$$=> \sigma_0 \geq 254MPa \tag{48}$$

Problem 4

[5 points] Consider the mechanical system formed by 3 vertical bars 1, 2 and 3 in tension. The horizontal beam is infinitely stiff and remains horizontal during the motion. The constitutive equations of materials are given below,



Determine the time dependent forces when loaded quasi-statically by a constant force P = P.H(t). (Inertia terms are ignored)

The material behavior (constitutive law)

Member 1 : $\epsilon = \sigma/E_1$ (Hooke element)

Member 2 : $\epsilon = \sigma/E_2 + \sigma/\eta$ (Maxwell element)

Solution

Equilibrium of Forces is,

$$2S_1 + S_2 = P \tag{49}$$

Also,

$$2\dot{S}_1 + \dot{S}_2 = 0 => \dot{S}_2 = -2\dot{S}_1 \tag{50}$$

For bars 1 and 3,

$$\epsilon = S_1/A.E_1 \tag{51}$$

For bar 2,

$$\dot{\epsilon} = \dot{S}_2/2A.E_2 + S_2/2.A.\eta \tag{52}$$

Compatibility condition is,

$$\epsilon = \epsilon_1 = \epsilon_2 \tag{53}$$

Hence,

$$\dot{S}_1/A.E_1 = \dot{S}_2/2A.E_2 + S_2/2.A.\eta \tag{54}$$

$$=> -\dot{S}_2/E_1 = \dot{S}_2/E_2 + S_2/\eta \tag{55}$$

$$=> \dot{S}_2(\frac{1}{E_1} + \frac{1}{E_2}) + \frac{S_2}{\eta} = 0 \tag{56}$$

If $\alpha = \frac{1}{\eta}(\frac{1}{E_1} + \frac{1}{E_2})$

$$\dot{S}_2 + \alpha S_2 = 0 \tag{57}$$

Integrating with respect to t,

$$S_2 = Ce^{-\alpha t} \tag{58}$$

At t = 0, when a Force (P) is applied, all the deformations are elastic, i.e.,

$$\dot{\epsilon} = \frac{\dot{S}_1}{A.E_1} = \frac{\dot{S}_2}{2.A.E_2} \tag{59}$$

Also,

$$2\dot{S}_1 + \dot{S}_2 = P => (2.\frac{E_1}{2.E_2} + 1)\dot{S}_2 = P \tag{60}$$

For t = 0, $C = \dot{S}_2$

$$C = \dot{S}_2 = P/(1 + \frac{E_1}{E_2}) \tag{61}$$

$$=> \underline{S_2} = \frac{P}{1 + \frac{E_1}{E_2}} e^{-t/\eta(1/E_1 + 1/E_2)} \tag{62}$$

Similarly,

$$=> \underline{S_1} = \frac{1}{2}(P - S_2) \tag{63}$$

Problem 5

Consider a short reinforced concrete column concentrically loaded by a constant compressive force $P = P \cdot H(t)$, where $H(t)$ being the Heaviside unit-step function.

The material behavior (constitutive laws).

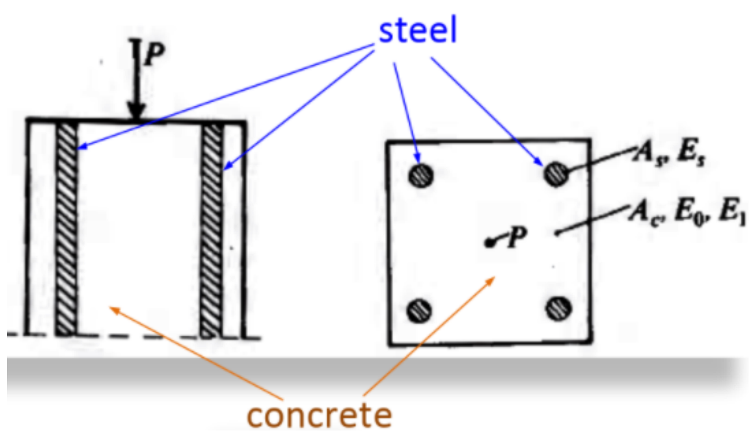
- **Steel:** Considered elastic (as compared to concrete for the time durations considered here).

$$\sigma_s = E_s \epsilon_s \tag{64}$$

- **Concrete:** Viscoelastic having obeying constitutive law of a Standard Linear Solid (SLS) in the form

$$\sigma_c + \frac{E_2}{\eta}\sigma_c = E_0 \epsilon_c + \frac{E_1 E_2}{\eta}\epsilon_c, \tag{65}$$

  where $E_0 = E_1 + E_2$ being the the initial modulus.



Assume the steel reinforcement is perfectly bonded to the concrete, $\epsilon_s = \epsilon_c$.

Question:

Determine the stresses in concrete and steel (separately).

Hints:

Compatibility and equilibrium.....
Cross-section area, A_s + A_c = A => A_s , $A_c \approx$ A.
Steel ratio, n $\equiv$ A_s / A.

Solution

The material properties are;
Steel,

$$\sigma_s = E_s\epsilon_s, \dot{\sigma_s} = E_s\dot{\epsilon_s} \tag{66}$$

Concrete,

$$\dot{\sigma_c} + \frac{E_2}{\eta}\sigma_c = \frac{E_1E_2}{\eta}\epsilon_c + E_0\dot{\epsilon_c} \tag{67}$$

Where, $E_0 = E_1 + E_2$
The applied Force can be written as,

$$P_0 = A_s.\sigma_s + A_c.\sigma_c \tag{68}$$

Compatibility equation,

$$\epsilon = \epsilon_s = \epsilon_c \tag{69}$$

We can write from strains,

$$\epsilon = \frac{\sigma_s}{E_s} = \frac{1}{E_s.A_s}(P_0 - A_c.\sigma_c) \tag{70}$$

For t > 0,

$$\dot{\sigma_c} + \frac{E_2}{\eta}\sigma_c = \frac{E_1E_2}{\eta}\frac{1}{E_sA_s}(P_0 - A_c\sigma_c) - \frac{E_0A_c}{E_sA_s}\dot{\sigma_c} \tag{71}$$

$$=> (1 + \frac{E_0A_c}{E_sA_s})\dot{\sigma_c} + \frac{E_2}{\eta}(1 + \frac{E_1A_c}{E_sA_s})\sigma_c = \frac{1}{\eta}\frac{E_1E_2}{E_sA_s}P_0 \tag{72}$$

$$=> \dot{\sigma_c} + \lambda\sigma_c = C_0 \tag{73}$$

Where, $\lambda = \frac{E_2}{\eta}\frac{E_sA_s+E_1A_c}{E_sA_s+E_0A_c}$, $C_0 = \frac{1}{\eta}\frac{E_1E_2}{E_sA_s+E_0A_c}P_0$
When t=0,

$$\sigma_c(0) = E_0\epsilon(0) = \frac{E_0}{E_sA_s}[P_0 - A_c\sigma_c(0)] \tag{74}$$

$$=> \sigma_c(0) = \frac{E_0P_0}{E_sA_s + E_0A_c} \tag{75}$$

For a constant Force, P_0

$$A = \frac{C_0}{\lambda} \tag{76}$$

Homogeneous part of the general solution is,

$$\sigma_{ch} = C_1.exp(-\lambda t) \tag{77}$$

$$\sigma_c = C_1.exp(-\lambda t) + \frac{C_0}{\lambda} \tag{78}$$

The initial condition, $\sigma_c(0) = \sigma_{c0}$

$$\sigma_{c0} = C_1 + \frac{C_0}{\lambda} => C_1 = \sigma_{c0} - \frac{C_0}{\lambda} \tag{79}$$

$$=> \sigma_c = \sigma_{c0}.exp(-\lambda t) + \frac{C_0}{\lambda}(1 - exp(\lambda t)) \tag{80}$$

Stress in steel is, $\sigma_s = E_s\epsilon = \frac{P_0}{A_s} - \frac{A_c}{A_s}\sigma_c$

$$\sigma_s = \frac{P_0}{A_s} - \frac{A_c}{A_s}\frac{E_0P_0}{E_sA_s + E_0A_c} = \frac{E_sP_0}{E_sA_s + E_0A_c} \tag{81}$$

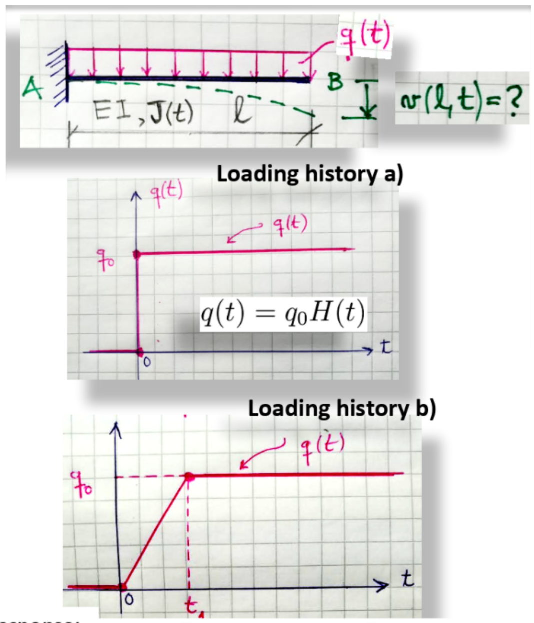
For t $- > \infty$

$$\sigma_c(t- > \infty) = \frac{C_0}{\lambda} = \frac{E - 1P_0}{E_sA_s + E_1A_c} \tag{82}$$

$$\sigma_s(t- > \infty) = \frac{P_0}{A_s} - \frac{A_c}{A_s}\frac{E_1P_0}{E_sA_s + E_1A_c} = \frac{E_sP_0}{E_sA_s + E_1A_c} \tag{83}$$

Problem 6

[5(KV) + 5(SLS) points] Consider the cantilever AB loaded as shown. The material is visco-elastic.



Creep response:

$$\varepsilon(t) = \sigma_0\left[\frac{1}{E_1} + \frac{1}{E_2}[1 - e^{-\frac{E_2}{\eta}t}]\right] \equiv \sigma_0 J(t)$$

$$\Rightarrow \varepsilon(t) = \frac{\sigma_0}{E}[1 - e^{-\frac{E}{\eta}t}]$$

Consider two separate cases: 1) Kelvin - Voigt (KV) and 2) Standard Linear Solid (SLS) visco-elastic materials.

Our aim is to determine the tip displacement as a function of time for cases 1) and 2) under two different loading history hypotheses a) and b).

- **Case a)** The loading is, $q(t) = q_0 H(t)$. Where, $H(t)$ is Heaviside unit-step function.
- **Case b)** The loading history is shown in figure; it grows linearly till t_1 and is then kept constant.

Question:

Determine the history of tip displacement for the two materials (KV) and (SLS) for both loading histories.

Extra 10 points

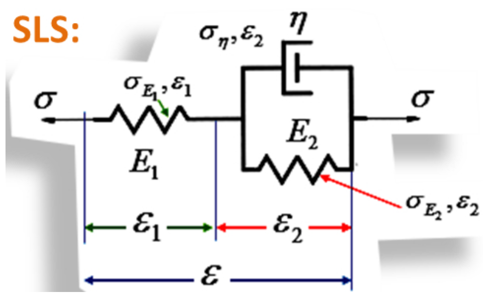
Using FEM verify your results. Use required numerical values for the material constants, when needed.

Problem 7 - Visco-elasticity: integrating the ODEs

This is only one exercise from 5...6. The student should solve at least four exercises.

1 The Standard Linear Solid

Assume a Linear Standard Solid. Derive the ordinary differential equation below:



$$\sigma + \tau\dot{\sigma} = G_\infty \epsilon + \tau G_0 \dot{\epsilon} \tag{84}$$

$$G_\infty = \frac{E_1 E_2}{E_1 + E_2}, G_0 = E_1, \tau = \frac{\eta}{E_1 + E_2} \tag{85}$$

$$\dot{\sigma} = \frac{G_\infty}{\tau}\epsilon + G_0\dot{\epsilon} - \frac{\sigma}{\tau} \tag{86}$$

$$\dot{\sigma} = f(\epsilon, \sigma) \tag{87}$$

The differential equation (84) or (87) can be numerically integrated using appropriate initial conditions for any known history of the deformations or of the stresses, respectively.

In order to determine some material properties of the visco-elastic material the cyclic strain history below is imposed and the corresponding stress history was recorded.

$$\epsilon(t) = \epsilon_0 \sin(\omega t), \tag{88}$$

where $\omega = 1$ (1/s), $\tau = 1$ (s) , $\epsilon_0 = 0.008, G_\infty = 550$ MPa, $G_0 = 1.5$ GPa. The initial conditions are $t = 0, \epsilon(0) = 0, \sigma(0) = 0$.

2 The Problem

The idea is to play around a response of a visco-elastic material (SLS) and obtain some elements of understanding of the mechanical behaviour of such class of materials.

Solve[1] So, solve the stresses by integrating analytically [5 points] and numerically [5 points].

1. determine the time-series of the response in term of stress $\sigma(t)$ for the given periodic excitation $\epsilon(t)$

[1]Hint! There is a similar solved problem in the course supporting material. You can use the Matlab-scripts, two *m*-files, I put for the time-integration of the stresses (numerical integration of the ODE).

2. draw the graphs of $\sigma(t)$ and $\epsilon(t)$
3. draw the graph $\sigma - \epsilon$ for few cycles in order to observe the hysteresis loop
4. estimate from the delay (the lag) $\Delta t = \delta/\omega$ of the two time-serie; the excitation $\epsilon(t)$ and the response $\sigma(t)$ (scale adequately these graphs to draw them on the same axes to estimate, graphically Δt)
5. give an estimation for the storage E' and loss modulus E''

Estimate from the delay (the lag) $\Delta t = \delta/\omega$ of the two time-serie; the exitation $\epsilon(t)$ and the response $\sigma(t)$ Give an estimation for the storage E' and loss modulus E''.

For the numerical time-integration you can use and edit the two m-scripts I put in MyCourses: `Main-SLS-distribute.m` and `Main-SLS-distribute.m`.

Additional Reading

Not exclusive Examples for reading

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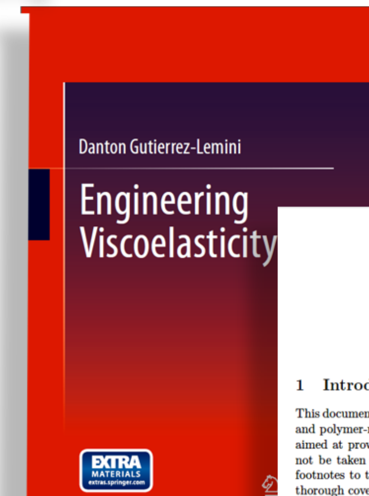

Reading: A concise course from MIT:

ENGINEERING VISCOELASTICITY

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October 24, 2001

1Cours__MIT_visco.pdf

1 Introduction

This document is intended to outline an important aspect of the mechanical response of polymers and polymer-matrix composites: the field of *linear viscoelasticity*. The topics included here are aimed at providing an instructional introduction to this large and elegant subject, and should not be taken as a thorough or comprehensive treatment. The references appearing either as footnotes to the text or listed separately at the end of the notes should be consulted for more thorough coverage.

Viscoelastic response is often used as a probe in polymer science, since it is sensitive to the material's chemistry and microstructure. The concepts and techniques presented here are important for this purpose, but the principal objective of this document is to demonstrate how linear viscoelasticity can be incorporated into the general theory of mechanics of materials, so that structures containing viscoelastic components can be designed and analyzed.

While not all polymers are viscoelastic to any important practical extent, and even fewer are *linearly* viscoelastic[1], this theory provides a usable engineering approximation for many applications in polymer and composites engineering. Even in instances requiring more elaborate treatments, the linear viscoelastic theory is a useful starting point.

2 Molecular Mechanisms

When subjected to an applied stress, polymers may deform by either or both of two fundamentally different atomistic mechanisms. The lengths and angles of the chemical bonds connecting the atoms may distort, moving the atoms to new positions of greater internal energy. This is a small motion and occurs very quickly, requiring only $\approx 10^{-12}$ seconds.

If the polymer has sufficient molecular mobility, larger-scale rearrangements of the atoms may also be possible. For instance, the relatively facile rotation around backbone carbon-carbon single bonds can produce large changes in the conformation of the molecule. Depending on the mobility, a polymer molecule can extend itself in the direction of the applied stress, which decreases its conformational entropy (the molecule is less "disordered"). Elastomers — rubber — respond almost wholly by this entropic mechanism, with little distortion of their covalent bonds or change in their internal energy.

[1] For an overview of *nonlinear* viscoelastic theory, see for instance W.N. Findley et al., *Creep and Relaxation of Nonlinear Viscoelastic Materials*, Dover Publications, New York, 1989.

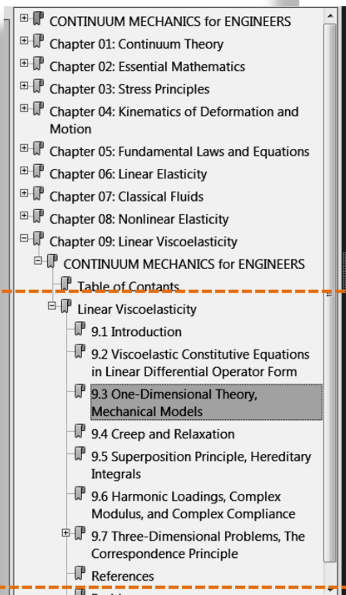
Have a look:

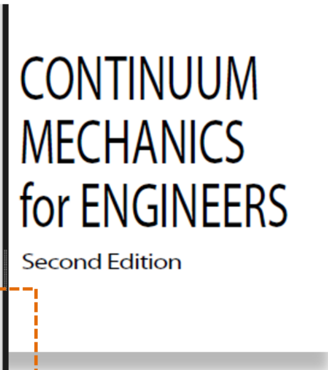
https://ocw.mit.edu/courses/

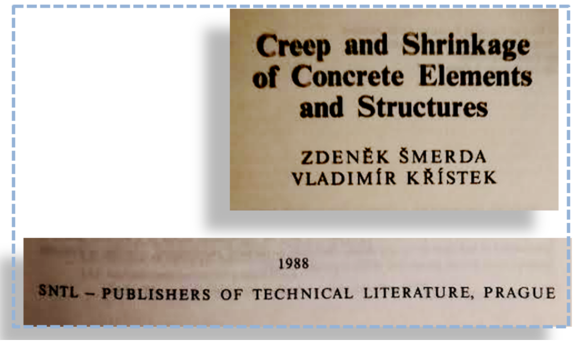
http://web.mit.edu/course/3/3.11/www/modules/visco.pdf (30.11.2016)

Not exclusive Examples for reading:

Have a look:







Viscoelasticity

In general, almost all the materials exhibit less or more several inelastic properties some of which will be covered in this short course.

- ✓ In general, stress in such materials depends on strain and the history of strain
- ✓ Such properties can be modeled by the theory of viscoelasticity

Content

- experimental observations: evidence of viscoelastic behavior
- stress relaxation at constant strain
- creep at constant stress
- strain-rate dependence
- constitutive models in the rate form:
  - Maxwell model
  - Kelvin-Voight model
  - Standard linear solid model
  - Generalized Maxwell model
  - Kelvin chain model

Viscoelasticity, in short:

Viscoelastic materials have i) a time dependent response to constant loading as for instance, to force, temperature and strain, and ii) they exhibit also rate depend responses. iii) depending on the duration of the excitation (fast as in shock problems or slow as in long-term stability problems), the same material may exhibit some of solid or fluid behavior or both of them at the same time.

They have the ability to creep, recover partially or fully, undergo stress relaxation and absorb energy. They possess some of the combined mechanical properties of fluid and solids as related to the stress-strain response.

Why the constitutive equation is needed*? $\mathcal{F}(\sigma,\dot{\sigma},\varepsilon,\dot{\varepsilon},T,\dot{T})=0$

The structure of the problem

Example problem : think of the need of solving the *equation of motion* or the *quasi-static equilibrium* in order to **determine the field of displacements** within a body having **a rate dependent mechanical response**.

- **¿The unknown field of displacements?**

$$\vec{u}(\vec{x},t)=\begin{bmatrix}u\\v\\w\end{bmatrix}$$

- **The equation of motion** or also of equilibrium when $\rho\dot{\vec{u}}=\vec{0}$

$$\operatorname{div}\boldsymbol{\sigma}+\rho\vec{f}=\rho\dot{\vec{u}},\quad \text{in } \Omega$$

- **The constitutive equation or the behavior law:** $\mathcal{F}(\sigma,\dot{\sigma},\varepsilon,\dot{\varepsilon},T,\dot{T})=0$

- **The kinematic relation:** $\boldsymbol{\varepsilon}=\nabla^{\text{sym}}\begin{bmatrix}u\\v\\w\end{bmatrix}$

- **The boundary conditions:**

- **The initial condition at** $t=0,\ in\ \Omega$

  The initial solution is given

$$\vec{u}(\vec{x},t=0)=\vec{u}_0(\vec{x})$$

In this short course, explicit forms for $\mathcal{F}(\boldsymbol{\sigma},\dot{\boldsymbol{\sigma}},\boldsymbol{\epsilon},\dot{\boldsymbol{\epsilon}})=0$ will be derived for some classical viscoelastic mechanical response

* ... to mathematically close the problem