Homework 2

CIV-E4080, Material Modeling in Civil Engineering L

Introduction

Recall of a necessary minimum of some basic concepts and equations of solid mechanics necessary to follow this course.

Readings

1. Chapter 1 from: **D. Gross and T. Seelig**, *Fracture Mechanics: With an Introduction to Micromechanics.*
2. **Reddy**; Chapters 3 (*kinematics of continua*) and 4 (*stress measures*)
3. Also refer to the Appendix I added to MyCourses. *Recall: Stress invariants.*

Problem 1 - Stress Invariants

[5 points] Consider the plane stress state below,

$$\sigma = \begin{bmatrix} \sigma_o & 0 & 0 \\ 0 & \alpha\sigma_o & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1}$$

where, $\alpha \in$ [-1, 1], $\sigma_0 > 0$ and x_1, x_2, x_3, coordinates in the basis $\vec{i_1}, \vec{j_2}, \vec{k_3}$.

Experimental setting giving rise to such bi-axial stress state is often used to determine parameters of material models.
Determine as a function of the parameter α,

1. The mean stress α_m, called also the hydrostatic pressure - p.
2. The equivalent, von Mises, stress ($\sigma_e = \sqrt{3J_2}$).
3. Lode angle $\theta \equiv 1/3 \arccos(3\sqrt{3}/2 \cdot J_3/J_2^{3/2})$
4. The maximum shear stress τ_{max}.
5. The normal of the maximum shear stress plane

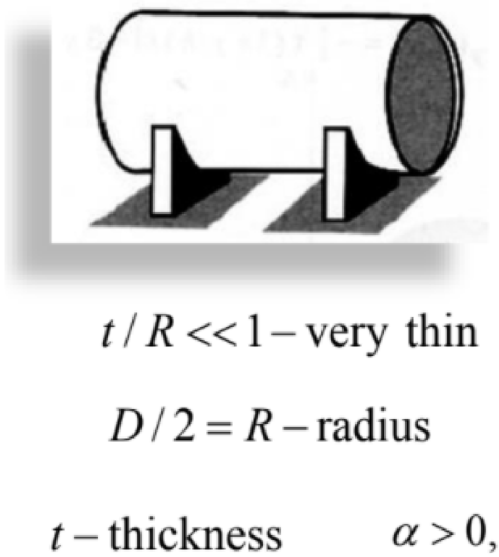
What is the stress state corresponding to $\alpha = -1$?

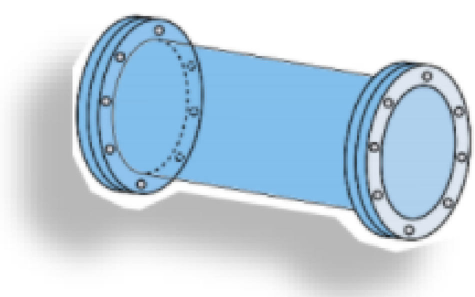
Solution

1. Hydrostatic Pressure / Mean Stress, $\alpha_m = (\sigma_o + \alpha\sigma_o + 0)/3 = \sigma(1+\alpha)/3$
2. $J_2 = \frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] => \frac{1}{3}\sigma_o^2(1 - \alpha + \alpha^2)$
   $\sigma_e = \sqrt{3J_2} => \sqrt{(1 - \alpha + \alpha^2)}\sigma_o$
3. $J_3 = dets => -\frac{1}{9}(2 - \alpha)(2\alpha - 1)(1 + \alpha)\sigma_o^3$
   $\theta \equiv 1/3 \arccos(3\sqrt{3}/2 \cdot J_3/J_2^{3/2})$
   $=> 1/3 \arccos(3\sqrt{3}/2 \cdot (-\frac{1}{9}(2 - \alpha)(2\alpha - 1)(1 + \alpha)\sigma_o^3)/(\frac{1}{3}\sigma_o^2(1 - \alpha + \alpha^2))^{3/2})$
4. $\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3) => \frac{1}{2}(\sigma_o)$
5. The maximum shear shear stresses are in planes forming angles of 45 degrees with Principal Planes.

Problem 2 - Pressure Vessel

[5 points] Consider a thin-walled metallic cylindrical pressure vessel shown in the figure below.





The pressure vessel (the thin walled tube) is loaded with an internal pressure p_o and a torsion moment $M_t = \alpha p_0 t D^2$ resulting in a bi-axial membrane stress state. Assume a membrane stress state for the tube with a cylindrical symmetry.

1. Determine the stress state in the thin-wall of the cylinder (membrane stress state with a cylindrical symmetry).

2. Plot the stress state in the deviatoric plane for $\alpha >$... i.e., plot a 'stress point' $P(\sigma)$ using the coordinates (ρ, θ)

Hints

$J_2 = \frac{1}{2} s : s$, $J_3 = \det s$, Deviatoric radius: $\rho = \sqrt{2J_2}$ and Lode angle: $\cos 3\theta = (\frac{3\sqrt{3}}{2} \frac{J_3}{J_2^{3/2}})$.

Solution

1. Since, the thickness to diameter ratio is very small, t/D << 1, we can assume to have uniform stress distribution in the shell and a plane stress state. The respective stress equations are,

   $\sigma_x = \frac{1}{2}p_o\frac{D}{t}$, $\sigma_y = \frac{1}{4}p_o\frac{D}{t}$, $\tau_{xy} = M_x/W_v = \alpha p_o t D^2/\frac{1}{4}\pi D^2 t = (4\alpha/\pi)p_o$

   The Stress Matrix is,

$$\sigma = \begin{bmatrix} \frac{1}{2}\frac{D}{t} & (4\alpha/\pi) & 0 \\ (4\alpha/\pi) & \frac{1}{4}\frac{D}{t} & 0 \\ 0 & 0 & 0 \end{bmatrix} p_o$$

$$\sigma = \begin{bmatrix} \frac{1}{2} & \beta & 0 \\ \beta & \frac{1}{4} & 0 \\ 0 & 0 & 0 \end{bmatrix} p_o D/t$$

   Where, $\beta = 4\alpha t/\pi D$

2. The mean stress is,

$$\sigma_m = \frac{1}{3}tr\sigma => (1/4)p_o D/t$$

   The deviatoric stress becomes,

$$s = \begin{bmatrix} \frac{1}{4} & \beta & 0 \\ \beta & 0 & 0 \\ 0 & 0 & -\frac{1}{4} \end{bmatrix} p_o D/t$$

   $J_2 = \frac{1}{2}s_{ij}sji = \frac{1}{2}(2\frac{1}{16} + 2\beta^2)p_o^2 D^2/t^2 => (\frac{1}{16} + \beta^2)p_o^2 D^2/t^2$

   $J_3 = dets => \frac{1}{4}\beta^2(p_o D/t)^3$

   Devitoric Radius, $\rho = \sqrt{2J_2} => \sqrt{\frac{1}{8} + 2\beta^2 p_o D/t}$

   Lode Angle, $\cos 3\theta = \frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}} => \frac{3\sqrt{3}}{8}\frac{\beta^2}{(\frac{1}{16}+\beta^2)^{3/2}}$

   The stress point on deviatoric plane can be plotted by polar coordinates ρ and θ as a functions of β, which is directly proportional to α.