# Homework # 2(3)

Small project [15 points (obligatory) + 15 points (bonus) ]

Elasticity

Elastic orthotropy

Bending of plates

Reading: Chapter 7.2 from:

I put the necessary ‘equations’ in the end slide

7.2 ORTHOTROPIC AND STIFFENED PLATES

**Thin Plates and Shells**

Theory, Analysis, and Applications

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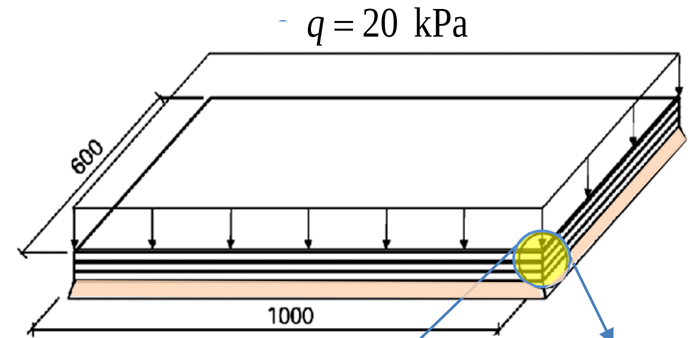
Due date: 30.4.2017

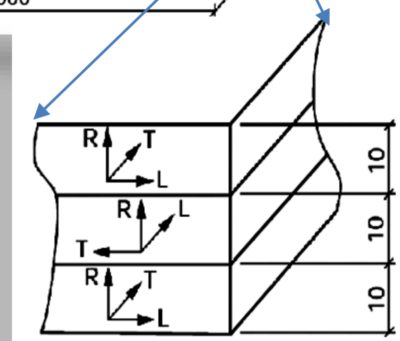
# Problem 6: Orthotropic plate or laminate

**Physical problem:** Consider the laminate plate (Glue Laminated Timber, GLT) formed by three perfectly bonded layers having respective principle material directions L, T and R (*cf*. figure).

The plate is under a transversal loading. The plate can freely rotate along the four support lines (freely supported, hinged on all sides). The individual layers should be modeled as a linear elastic orthotropic material.




L – longitudinal
T – tangential
R – radial




Dimensions are in mm.

**Problem** **[BONUS, elective <u>15 pnts</u>]**: Using Abaqus, <u>Comsol</u> (easy to use) or any other FE-software you need.

1. Determine the displacement along the horizontal center lines
2. The stress distributions at a section at the center and close to the supports.
3. Analyze your results

**Problem [obligatory, <u>15 pnts</u>]**: You know well the thin plate theory of orthotropic plates (*cf.* textbooks, the pdf-version in your previous master course of *plate and shells*) and you are a clever and responsible engineer who wants somehow to check his FE-results. For this purpose you want to obtain an analytical (or semi-analytical) solution for comparison with FE-results. Do you have any idea how to proceed? If yes, then how? Do it.
**Hint**: one can find an *equivalent* <u>one layer orthotropic plate</u> having effective bending and torsional rigidities integrated from the 3-layer plate *in order to conserve strain energy (cf. the mentioned textbook). Other ways toward the solution exist and are allowed.*

*Ref*: this homework, except the bonus, was adapted from the course:
**13-02-0003-vl Werkstoffmechanik**
Technische Universität Darmstadt
**Lehrende:** Prof. Dr.-Ing. Michael Vormwald; Dipl.-Ing. Melanie Fiedler

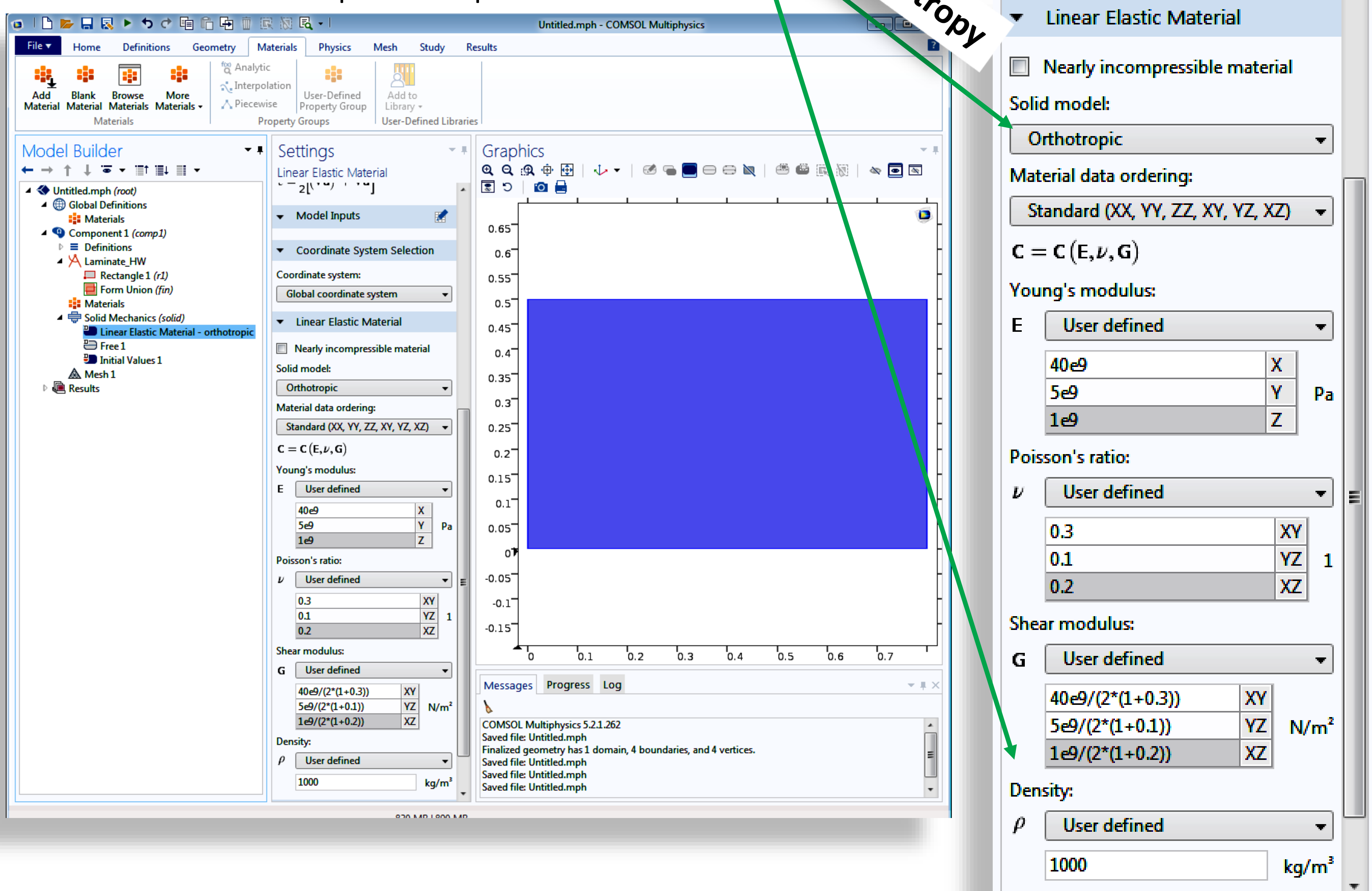
Eine Platte aus Brettschichtholz wird allseitig gelenkig gelagert und mit einer konstanten Flächenlast $q$ = 0.02 MPa belastet. Die einzelnen Schichten sollen als orthotroper Werkstoff modelliert werden. Die Werkstoffkonstanten in einem L-R-T-Koordinatensystem betragen:

| $E_L$[MPa] | $E_R$[MPa] | $E_T$[MPa] | $G_{LR}$[MPa] | $G_{LT}$[MPa] | $G_{RT}$[MPa] | $\nu_{LT}$ | $\nu_{LR}$ | $\nu_{RT}$ |
|---|---|---|---|---|---|---|---|---|
| 11990 | 820 | 420 | 620 | 740 | 240 | 0.7749 | 0.6071 | 0.6031 |

Due date: 30.4.2017

See how simple, in this example, if one uses *Comsol*

*NB*. This is not the example of our problem



# Some useful tables for wood

**Table 4–2. Poisson's ratios for various species at approximately 12% moisture content**

| Species | $\mu_{LR}$ | $\mu_{LT}$ | $\mu_{RT}$ | $\mu_{TR}$ | $\mu_{RL}$ | $\mu_{TL}$ |
|---|---|---|---|---|---|---|
| **Hardwoods** | | | | | | |
| Ash, white | 0.371 | 0.440 | 0.684 | 0.360 | 0.059 | 0.051 |
| Aspen, quaking | 0.489 | 0.374 | — | 0.496 | 0.054 | 0.022 |
| Balsa | 0.229 | 0.488 | 0.665 | 0.231 | 0.018 | 0.009 |
| Basswood | 0.364 | 0.406 | 0.912 | 0.346 | 0.034 | 0.022 |
| Birch, yellow | 0.426 | 0.451 | 0.697 | 0.426 | 0.043 | 0.024 |
| Cherry, black | 0.392 | 0.428 | 0.695 | 0.282 | 0.086 | 0.048 |
| Cottonwood, eastern | 0.344 | 0.420 | 0.875 | 0.292 | 0.043 | 0.018 |
| Mahogany, African | 0.297 | 0.641 | 0.604 | 0.264 | 0.033 | 0.032 |
| Mahogany, Honduras | 0.314 | 0.533 | 0.600 | 0.326 | 0.033 | 0.034 |
| Maple, sugar | 0.424 | 0.476 | 0.774 | 0.349 | 0.065 | 0.037 |
| Maple, red | 0.434 | 0.509 | 0.762 | 0.354 | 0.063 | 0.044 |
| Oak, red | 0.350 | 0.448 | 0.560 | 0.292 | 0.064 | 0.033 |
| Oak, white | 0.369 | 0.428 | 0.618 | 0.300 | 0.074 | 0.036 |
| Sweet gum | 0.325 | 0.403 | 0.682 | 0.309 | 0.044 | 0.023 |
| Walnut, black | 0.495 | 0.632 | 0.718 | 0.378 | 0.052 | 0.035 |
| Yellow-poplar | 0.318 | 0.392 | 0.703 | 0.329 | 0.030 | 0.019 |
| **Softwoods** | | | | | | |
| Baldcypress | 0.338 | 0.326 | 0.411 | 0.356 | — | — |
| Cedar, northern white | 0.337 | 0.340 | 0.458 | 0.345 | — | — |
| Cedar, western red | 0.378 | 0.296 | 0.484 | 0.403 | — | — |
| Douglas-fir | 0.292 | 0.449 | 0.390 | 0.374 | 0.036 | 0.029 |
| Fir, subalpine | 0.341 | 0.332 | 0.437 | 0.336 | — | — |
| Hemlock, western | 0.485 | 0.423 | 0.442 | 0.382 | — | — |
| Larch, western | 0.355 | 0.276 | 0.389 | 0.352 | — | — |
| Pine | | | | | | |
| Loblolly | 0.328 | 0.292 | 0.382 | 0.362 | — | — |
| Lodgepole | 0.316 | 0.347 | 0.469 | 0.381 | — | — |
| Longleaf | 0.332 | 0.365 | 0.384 | 0.342 | — | — |
| Pond | 0.280 | 0.364 | 0.389 | 0.320 | — | — |
| Ponderosa | 0.337 | 0.400 | 0.426 | 0.359 | — | — |
| Red | 0.347 | 0.315 | 0.408 | 0.308 | — | — |
| Slash | 0.392 | 0.444 | 0.447 | 0.387 | — | — |
| Sugar | 0.356 | 0.349 | 0.428 | 0.358 | — | — |
| Western white | 0.329 | 0.344 | 0.410 | 0.334 | — | — |
| Redwood | 0.360 | 0.346 | 0.373 | 0.400 | — | — |
| Spruce, Sitka | 0.372 | 0.467 | 0.435 | 0.245 | 0.040 | 0.025 |
| Spruce, Engelmann | 0.422 | 0.462 | 0.530 | 0.255 | 0.083 | 0.058 |

Chapter 2

## Structure of Wood

Regis B. Miller

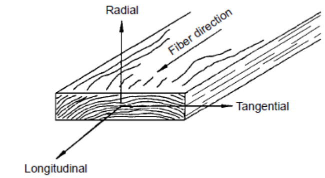


Figure 4–1. Three principal axes of wood with respect to grain direction and growth rings.

**Table 4–1. Elastic ratios for various species at approximately 12% moisture content[a]**

| Species | $E_T/E_L$ | $E_R/E_L$ | $G_{LR}/E_L$ | $G_{LT}/E_L$ | $G_{RT}/E_L$ |
|---|---|---|---|---|---|
| **Hardwoods** | | | | | |
| Ash, white | 0.080 | 0.125 | 0.109 | 0.077 | — |
| Balsa | 0.015 | 0.046 | 0.054 | 0.037 | 0.005 |
| Basswood | 0.027 | 0.066 | 0.056 | 0.046 | — |
| Birch, yellow | 0.050 | 0.078 | 0.074 | 0.068 | 0.017 |
| Cherry, black | 0.086 | 0.197 | 0.147 | 0.097 | — |
| Cottonwood, eastern | 0.047 | 0.083 | 0.076 | 0.052 | — |
| Mahogany, African | 0.050 | 0.111 | 0.088 | 0.059 | 0.021 |
| Mahogany, Honduras | 0.064 | 0.107 | 0.066 | 0.086 | 0.028 |
| Maple, sugar | 0.065 | 0.132 | 0.111 | 0.063 | — |
| Maple, red | 0.067 | 0.140 | 0.133 | 0.074 | — |
| Oak, red | 0.082 | 0.154 | 0.089 | 0.081 | — |
| Oak, white | 0.072 | 0.163 | 0.086 | — | — |
| Sweet gum | 0.050 | 0.115 | 0.089 | 0.061 | 0.021 |
| Walnut, black | 0.056 | 0.106 | 0.085 | 0.062 | 0.021 |
| Yellow-poplar | 0.043 | 0.092 | 0.075 | 0.069 | 0.011 |
| **Softwoods** | | | | | |
| Baldcypress | 0.039 | 0.084 | 0.063 | 0.054 | 0.007 |
| Cedar, northern white | 0.081 | 0.183 | 0.210 | 0.187 | 0.015 |
| Cedar, western red | 0.055 | 0.081 | 0.087 | 0.086 | 0.005 |
| Douglas-fir | 0.050 | 0.068 | 0.064 | 0.078 | 0.007 |
| Fir, subalpine | 0.039 | 0.102 | 0.070 | 0.058 | 0.006 |
| Hemlock, western | 0.031 | 0.058 | 0.038 | 0.032 | 0.003 |
| Larch, western | 0.065 | 0.079 | 0.063 | 0.069 | 0.007 |
| Pine | | | | | |
| Loblolly | 0.078 | 0.113 | 0.082 | 0.081 | 0.013 |
| Lodgepole | 0.068 | 0.102 | 0.049 | 0.046 | 0.005 |
| Longleaf | 0.055 | 0.102 | 0.071 | 0.060 | 0.012 |
| Pond | 0.041 | 0.071 | 0.050 | 0.045 | 0.009 |
| Ponderosa | 0.083 | 0.122 | 0.138 | 0.115 | 0.017 |
| Red | 0.044 | 0.088 | 0.096 | 0.081 | 0.011 |
| Slash | 0.045 | 0.074 | 0.055 | 0.053 | 0.010 |
| Sugar | 0.087 | 0.131 | 0.124 | 0.113 | 0.019 |
| Western white | 0.038 | 0.078 | 0.052 | 0.048 | 0.005 |
| Redwood | 0.089 | 0.087 | 0.066 | 0.077 | 0.011 |
| Spruce, Sitka | 0.043 | 0.078 | 0.064 | 0.061 | 0.003 |
| Spruce, Engelmann | 0.059 | 0.128 | 0.124 | 0.120 | 0.010 |

[a] $E_L$ may be approximated by increasing modulus of elasticity values in Table 4–3 by 10%.

$$\frac{\mu_{ij}}{E_i} = \frac{\mu_{ji}}{E_j}, \quad i \neq j \quad i,j = L,R,T$$

must obtain a new set of stress–strain relations that reflects the orthotropic properties of a material of the plate. Such a set of relations is shown below [3]:

$$\varepsilon_x = \frac{\sigma_x}{E_x} - \nu_y \frac{\sigma_y}{E_y}; \quad \varepsilon_y = \frac{\sigma_y}{E_y} - \nu_x \frac{\sigma_x}{E_x}; \quad \gamma_{xy} = \frac{\tau_{xy}}{G}, \tag{7.22}$$

where $E_x$, $E_y$, $\nu_x$, $\nu_y$, and $G$ are assumed to be elastic constants of an orthotropic material, i.e., $E_x$, $E_y$, and $\nu_x$, $\nu_y$ are the moduli of elasticity and Poisson's ratios in the $x$ and $y$ directions, respectively. They are independent of one another. $G$ is the shear modulus, which is the same for both isotropic and orthotropic materials. It can be expressed in terms of $E_x$ and $E_y$ as follows:

$$G \approx \frac{\sqrt{E_x E_y}}{2\left(1 + \sqrt{\nu_x \nu_y}\right)}. \tag{7.23}$$

The following relationship exists between independent elastic constants introduced above:

$$\frac{\nu_x}{E_x} = \frac{\nu_y}{E_y}. \tag{7.24}$$

This equality directly results from Betti's reciprocal theorem. Solving Eqs (7.22) for the stress components and taking into account (7.24), we obtain

$$\sigma_x = \frac{E_x}{1 - \nu_x \nu_y}(\varepsilon_x + \nu_y \varepsilon_y); \quad \sigma_y = \frac{E_y}{1 - \nu_x \nu_y}(\varepsilon_y + \nu_x \varepsilon_x); \quad \tau_{xy} = G\gamma_{xy}. \tag{7.25}$$

The derivation of the governing differential equation of bending of an orthotropic plate is based on the general hypotheses introduced in Sec. 1.3. The strain-deflection relations (2.6) hold for orthotropic plates also. So, substituting the relations (2.6) into Eqs (7.25) gives the following:

$$\sigma_x = -\frac{E_x}{1 - \nu_x \nu_y}\left(\frac{\partial^2 w}{\partial x^2} + \nu_y \frac{\partial^2 w}{\partial y^2}\right)z; \quad \sigma_y = -\frac{E_y}{1 - \nu_x \nu_y}\left(\frac{\partial^2 w}{\partial y^2} + \nu_x \frac{\partial^2 w}{\partial x^2}\right)z; \quad \tau_{xy} = -2Gz\frac{\partial^2 w}{\partial x \partial y}. \tag{7.26}$$

Substituting the above into Eqs (2.11) and integrating over the plate thickness, yields the following bending and twisting moments deflection relations for orthotropic plates:

where $D_x$, $D_y$, $D_{xy}$, $D_{yx}$, and $D_s$ are the flexural and torsional rigidities of an orthotropic plate, respectively, and are given as

$$D_x = \frac{E_x}{1 - \nu_x \nu_y}\frac{h^3}{12}; \quad D_y = \frac{E_y}{1 - \nu_x \nu_y}\frac{h^3}{12}; \quad D_{xy} = \frac{E_x \nu_y}{1 - \nu_x \nu_y}\frac{h^3}{12}; \quad D_{yx} = \frac{E_y \nu_x}{1 - \nu_x \nu_y}\frac{h^3}{12}; \quad D_s = \frac{Gh^3}{12}. \tag{7.28}$$

In view of the expressions (7.24), one can conclude that $D_{xy} = D_{yx}$. The shear force expressions (2.22) become

$$Q_x = -\frac{\partial}{\partial x}\left(D_x \frac{\partial^2 w}{\partial x^2} + H\frac{\partial^2 w}{\partial y^2}\right); \quad Q_y = -\frac{\partial}{\partial y}\left(H\frac{\partial^2 w}{\partial x^2} + D_y \frac{\partial^2 w}{\partial y^2}\right), \tag{7.29}$$

where

$$H = D_{xy} + 2D_s. \tag{7.30}$$

The governing differential equation (2.24) for orthotropic plates becomes

$$D_x \frac{\partial^4 w}{\partial x^4} + 2H\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = p(x, y). \tag{7.31}$$

We give below the expression for the potential energy of bending for orthotropic plates, which follows from Eqs (2.52) and (7.26):

$$U = \frac{1}{2}\int\int_A \left[D_x\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_{xy}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + D_y\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4D_s\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\mathrm{d}A. \tag{7.32}$$

**Thin Plates and Shells**

Theory, Analysis, and Applications

Eduard Ventsel
Theodor Krauthammer

*The Pennsylvania State University*
*University Park, Pennsylvania*

$$U = \frac{1}{2}\int\int_A \left[D_x\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_{xy}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + D_y\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4D_s\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\mathrm{d}A.$$

In the assignment, a plate consisting of three orthotropic layers were to be analyzed using a) Abaqus software and b) analytically for comparison. The geometry of the plate is shown in the Figure 1 and the material properies of the layers are given in the Table 1. The plate is loaded with uniform load $q$ = 20 kN/m$^2$.

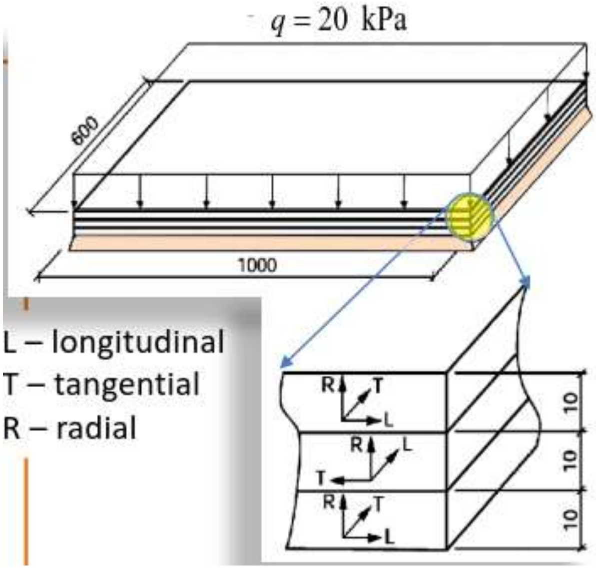



*Figure 1. The plate given in the assignment.*

*Table 1. Material properties of the material.*

| $E_L$ [MPa] | $E_R$ [MPa] | $E_T$ [MPa] | $G_{LR}$ [MPa] | $G_{LT}$ [MPa] | $G_{RT}$ [MPa] | $\nu_{LT}$ | $\nu_{LR}$ | $\nu_{RT}$ |
|---|---|---|---|---|---|---|---|---|
| 11990 | 820 | 420 | 620 | 740 | 240 | 0.7749 | 0.6071 | 0.6013 |

## a) Modelling the plate in Abaqus software

*Model*

The plate was modelled in Abaqus as a shell using 4-node shell elements (S4). The stiffness properties of the plate were given by defining composite layup for the plate, which Abaqus uses to calculate the stiffness of the plate for the calculation. All the layers were given same orthotropic material properties and the direction of the layers were defined in the composite layup. The material coordinate system was chosen so that; 1-axis refers to L-direction, 2-axis refers to T-direction and 3-axis refers to R-direction, which leads to properties below:

$E_1 = E_L$ = 11990 MPa

$E_2 = E_T$ = 420 MPa

$E_3 = E_R$ = 820 MPa

$G_{12} = G_{LT}$ = 740 MPa

$G_{13} = G_{LR}$ = 620 MPa

$G_{23} = G_{RT}$ = 240 MPa

$v_{12} = v_{LT} = 0.7749$

$v_{13} = v_{LR} = 0.6071$

$v_{23} = v_{TR} = v_{RT}E_T / E_R = 0.3089$.

The model is shown in the Figure 2. Top and bottom layers are orientated in x-direction and the middle layer in y-direction. Boundary conditions on all the edges are pinned ($u_1 = u_2 = u_3 = 0$). Material properties are shown in the Table 2. Abaqus takes transverse shear deformations automatically into account, so the analysis was done twice; in the case a) using given material properties and in the case b) using 1000x higher transverse shear moduli to neglect the effect of the transverse shear deformation.

*Table 2. Material properties in the analyses. In the case a) properties all the properties of the material as given is used and in the case b) transverse shear moduli are increased 1000x to neglect effects of the transverse shear deformation in the analysis.*

| | $E_1$ [MPa] | $E_2$ [MPa] | $G_{12}$ [MPa] | $G_{13}$ [MPa] | $G_{23}$ [MPa] | $v_{12}$ |
|---|---|---|---|---|---|---|
| Case a) | 11990 | 420 | 740 | 620 | 240 | 0.7749 |
| Case b) | 11990 | 420 | 740 | $620 \times 10^3$ | $240 \times 10^3$ | 0.7749 |

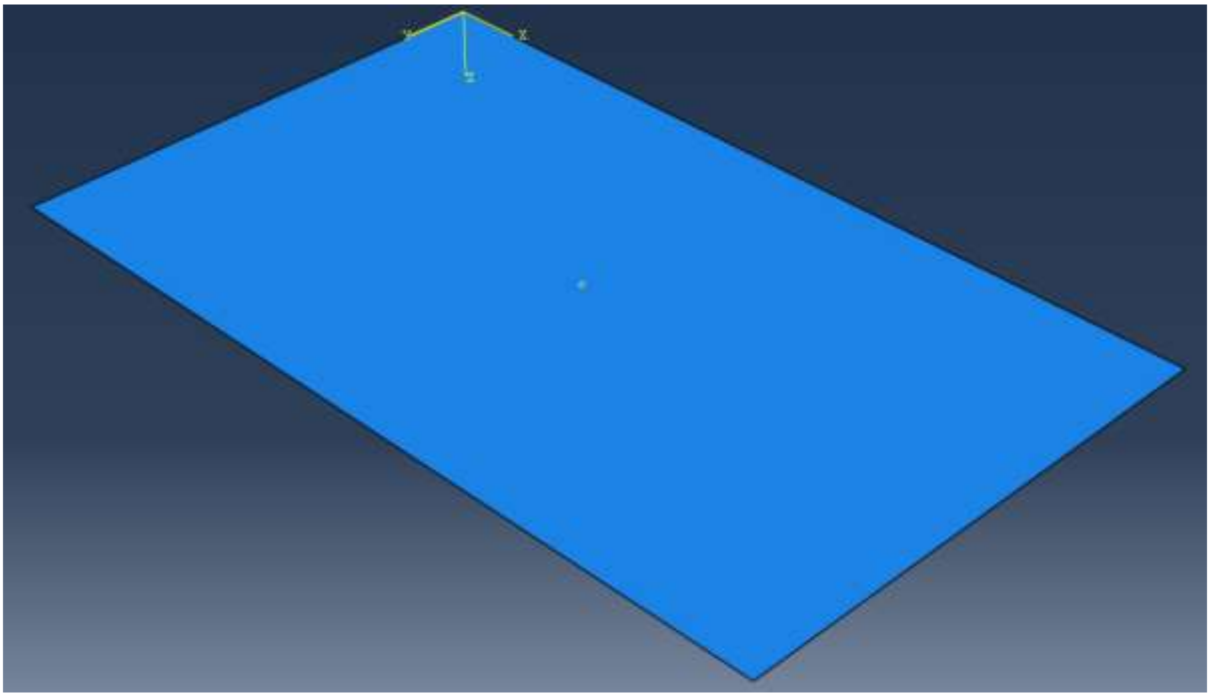


*Figure 2. The model of the plate in Abaqus. All boundaries are pinned and uniform load $q = 20$ kN/m$^2$, acting downwards, is applied on the plate.*

## Results

Deflection of the centerlines of the plate along x- and y-directions are illustrated in the Figure 3 and Figure 4. The maximum deflection is found at the mid-point of the plate and it was in the case a) $w_{max} = 5.42$ mm and in the case b) $w_{max} = 5.07$ mm.

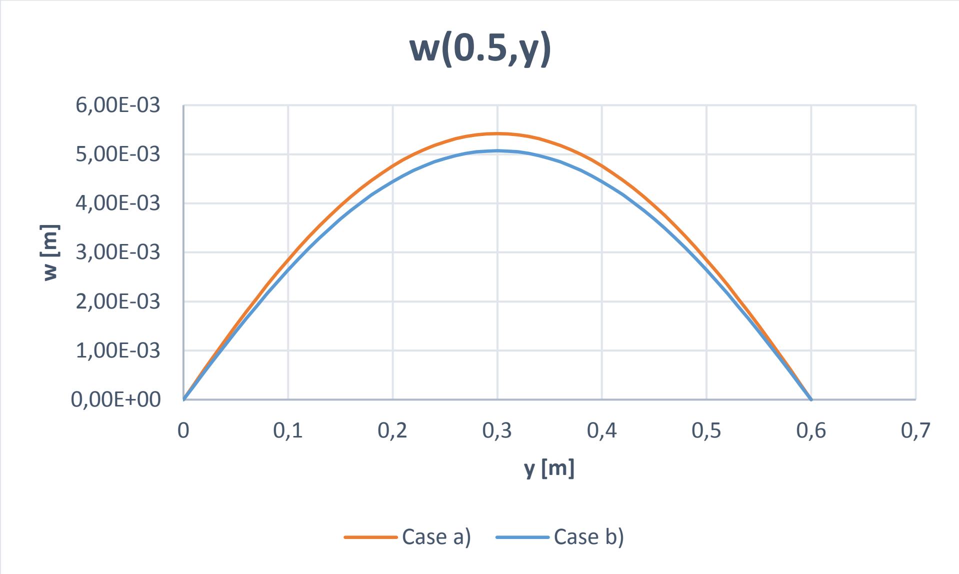



*Figure 3. Deflection along the centerline in x-direction.*

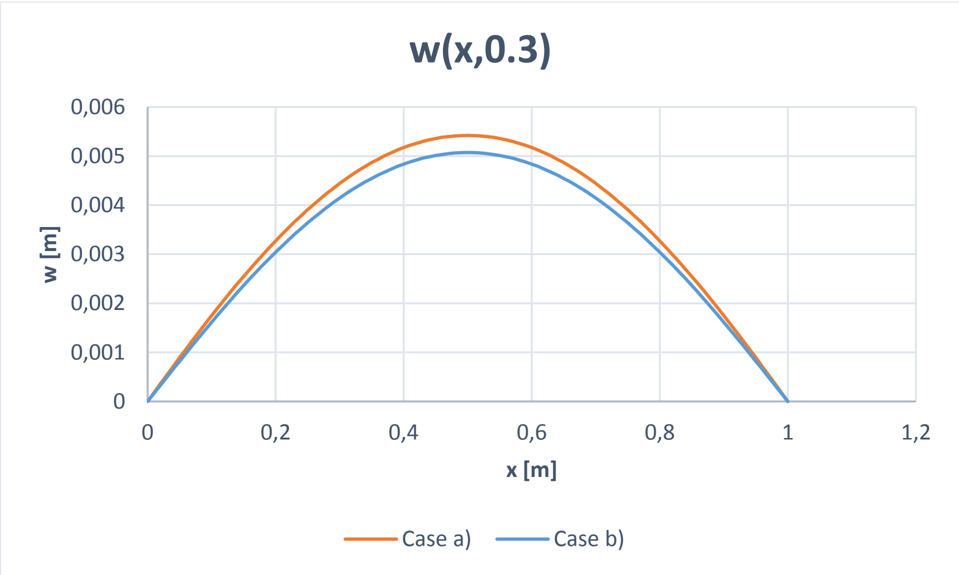



*Figure 4. Deflection along the centerline in y-direction.*

The stress distributions at mid-point of the plate ($x = 0.5$, $y = 0.3$) and and near the corner ($x = 0.1$, $y = 0.06$) over the sections are plotted in the Figure 5 and Figure 6, respectively. The stresses had only small differences between the cases a) and b), so only stresses from the case b) are illustrated.

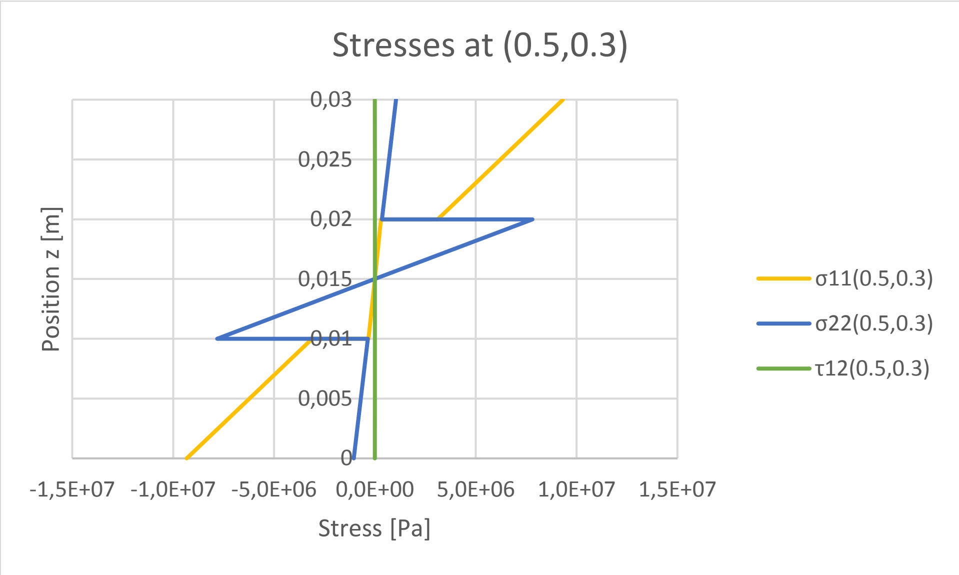



*Figure 5. Stresses over the section at mid-point of the plate (x = 0.5 m, y = 0.3 m).*

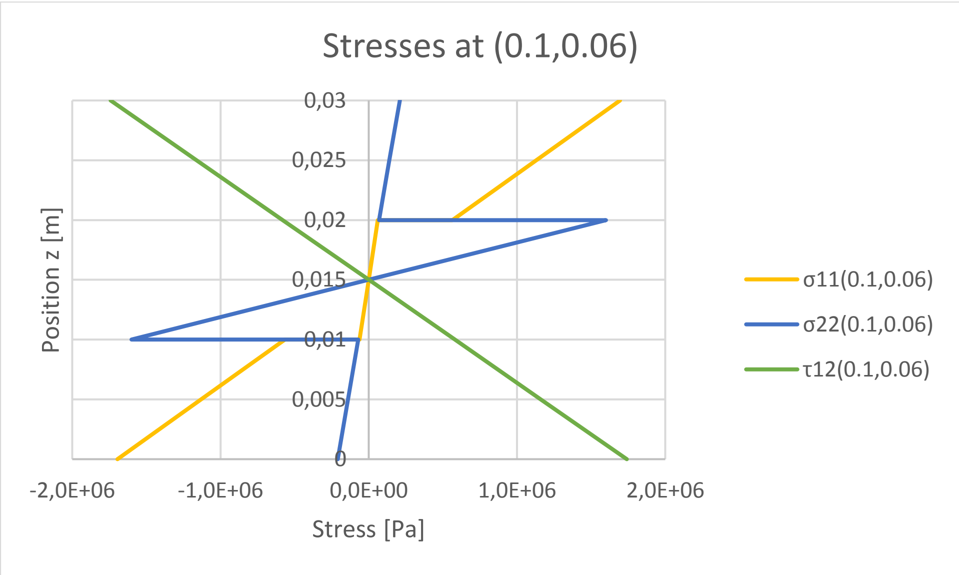



*Figure 6. Stresses over the section near the corner of the plate (x = 0.1 m, y = 0.06 m).*

## Analysis of the results

The deflections from the model are as expected, including transverse shear deformations in the analysis produces larger deflection. In this case the effect in quite limited, 7% increase in deflection, due to low thickness to span ratio of the plate.

The variation of the stresses in the section at both points, mid-point and near the corner point, is as expected. The layers with higher elastic modulus in certain direction display higher stresses in the corresponding direction. In the Figure 5 (mid-point) torsional stresses are close to zero, as they should be theoretically, since torsional moment should be zero there.

In the Figure 6 it can be seen that torsional stresses are high. This also in line with the theory, since the highest torsional moment should exist near the corners, leading to high torsional stresses.

**b) Analytical modelling of the plate**

Given plate can be homogenized to a plate with orthotropic properties by equation strain energies of the actual composite plate with strain energy of the orthotropic homogenous plate. Strain energy of orthotropic plate can be given as

$$U = \frac{1}{2}\iint\limits_A \left[ D_x \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_{xy}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + D_y\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4D_s\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right] dA.$$

In the composite case, planar stress-strain relationship in each layer is given as

$$\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{pmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\upsilon_{12}}{E_2} & 0 \\ -\frac{\upsilon_{21}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{pmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{pmatrix}$$

The stresses are obtained by inverting the stiffness matrix, giving

$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{pmatrix} = \begin{bmatrix} \frac{E_1}{1-\upsilon_{12}\upsilon_{21}} & \frac{\upsilon_{21}E_1}{1-\upsilon_{12}\upsilon_{21}} & 0 \\ \frac{\upsilon_{12}E_2}{1-\upsilon_{12}\upsilon_{21}} & \frac{E_2}{1-\upsilon_{12}\upsilon_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{pmatrix}.$$

Assuming linear strain distribution over the section, the stresses over the section are given by

$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{pmatrix}(z) = -\begin{bmatrix} \frac{E_1}{1-\upsilon_{12}\upsilon_{21}} & \frac{\upsilon_{21}E_1}{1-\upsilon_{12}\upsilon_{21}} & 0 \\ \frac{\upsilon_{12}E_2}{1-\upsilon_{12}\upsilon_{21}} & \frac{E_2}{1-\upsilon_{12}\upsilon_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \begin{pmatrix}\frac{\partial^2 w}{\partial x^2}\\ \frac{\partial^2 w}{\partial y^2}\\ 2\frac{\partial^2 w}{\partial x \partial y}\end{pmatrix} z.$$

The strain energy of the plate is given by

$$U = \frac{1}{2}\iiint\limits_V \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\, dV$$

and by substituting strains and stresses gives (in Kirchhoff-Love plate theory)

$$U = \frac{1}{2}\iiint_V \left[\frac{E_1}{1-\upsilon_{12}\upsilon_{21}}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \frac{\upsilon_{21}E_1+\upsilon_{12}E_2}{1-\upsilon_{12}\upsilon_{21}}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \frac{E_2}{1-\upsilon_{12}\upsilon_{21}}\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4G_{12}\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right] z^2 dV.$$

Due to symmetry in our case, the neutral plane is located in the mid-plane of the plate. Given that the distances of the layer surfaces from the mid-plane are given $z_0$, $z_1$, $z_2$ and $z_3$ and noting that the material properties are piecewise contantm one can write strain energy as

$$U = \frac{1}{2}\iint_A \left\{\left[\frac{1}{3}\sum_{i=1}^{3}\frac{E_{1,i}}{1-\upsilon_{12,i}\upsilon_{21,i}}(z_i^3 - z_{i-1}^3)\right]\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left[\frac{1}{3}\sum_{i=1}^{3}\frac{\upsilon_{21,i}E_{1,i}+\upsilon_{12,i}E_{2,i}}{1-\upsilon_{12,i}\upsilon_{21,i}}(z_i^3 - z_{i-1}^3)\right]\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \left[\frac{1}{3}\sum_{i=1}^{3}\frac{E_{2,i}}{1-\upsilon_{12,i}\upsilon_{21,i}}(z_i^3 - z_{i-1}^3)\right]\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4\left[\frac{1}{3}\sum_{i=1}^{3}G_{12,i}\,(z_i^3 - z_{i-1}^3)\right]\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right\}$$

Comparing the strain energies of the orthotropic plate and composite plate, it may be seen that effective stiffnesses of the composite plate are given as

$$D_x = \frac{1}{3}\sum_{i=1}^{3}\frac{E_{1,i}}{1-\upsilon_{12,i}\upsilon_{21,i}}(z_i^3 - z_{i-1}^3)$$

$$D_{xy} = \frac{1}{6}\sum_{i=1}^{3}\frac{\upsilon_{21,i}E_{1,i}+\upsilon_{12,i}E_{2,i}}{1-\upsilon_{12,i}\upsilon_{21,i}}(z_i^3 - z_{i-1}^3)$$

$$D_y = \frac{1}{3}\sum_{i=1}^{3}\frac{E_{2,i}}{1-\upsilon_{12,i}\upsilon_{21,i}}(z_i^3 - z_{i-1}^3)$$

$$D_s = \frac{1}{3}\sum_{i=1}^{3}G_{12,i}\,(z_i^3 - z_{i-1}^3)$$

$$H = D_{xy} + 2D_s.$$

Substituting the given material properties $E_L$, $E_T$, $G_{LT}$ and $\nu_{LT}$ for each layer as well as using $t$ for the thickness of a layer gives

$$D_x = \frac{1}{12}\frac{t^3(26E_L + E_T)}{(1-\upsilon_{LT}\upsilon_{TL})}$$

$$D_{xy} = \frac{9}{8}\frac{t^3(\upsilon_{TL}E_L + \upsilon_{LT}E_T)}{(1-\upsilon_{LT}\upsilon_{TL})}$$

$$D_y = \frac{1}{12}\frac{t^3(E_L + 26E_T)}{(1-\upsilon_{LT}\upsilon_{TL})}$$

$$D_s = \frac{9}{4}t^3 G_{LT}$$

Deflection of simply supported uniformly loaded plate is obtained from []

$$w = \frac{16p_0}{\pi^6}\sum_{m=1,3,\ldots}^{\infty}\sum_{n=1,3,\ldots}^{\infty}\frac{\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{mn\left[D_x\left(\frac{m^4}{a^4}\right)+2H\left(\frac{m^2n^2}{a^2b^2}\right)+D_y\left(\frac{n^4}{b^4}\right)\right]},$$

where *a* is the span in x-direction, *b* is the span in y-direction and $p_0$ is the unifrom load on the plate.

Calculating the deflection in the mid-span taking into account 3x3 terms gives $w_{max}$ = 5.07 mm, which is equal to deflection obtained from the Abaqus model without the shear deformations (case b).

Equal deflection confirms that used Abaqus model is reliable in the case where transverse shear deformations are neglected. The stresses can be simply derived from

$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{pmatrix}(z) = -\begin{bmatrix}\frac{E_1}{1-\upsilon_{12}\upsilon_{21}} & \frac{\upsilon_{21}E_1}{1-\upsilon_{12}\upsilon_{21}} & 0\\ \frac{\upsilon_{12}E_2}{1-\upsilon_{12}\upsilon_{21}} & \frac{E_2}{1-\upsilon_{12}\upsilon_{21}} & 0\\ 0 & 0 & G_{12}\end{bmatrix}\begin{pmatrix}\frac{\partial^2 w}{\partial x^2}\\ \frac{\partial^2 w}{\partial y^2}\\ 2\frac{\partial^2 w}{\partial x\partial y}\end{pmatrix}z.$$

where the curvature are obtained by differentation as

$$\frac{\partial^2 w}{\partial x^2} = -\frac{16p_0}{\pi^4 a^2}\sum_{m=1,3,\ldots}^{\infty}\sum_{n=1,3,\ldots}^{\infty}\frac{m\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{n\left[D_x\left(\frac{m^4}{a^4}\right)+2H\left(\frac{m^2n^2}{a^2b^2}\right)+D_y\left(\frac{n^4}{b^4}\right)\right]},$$

$$\frac{\partial^2 w}{\partial y^2} = -\frac{16p_0}{\pi^4 b^2}\sum_{m=1,3,\ldots}^{\infty}\sum_{n=1,3,\ldots}^{\infty}\frac{n\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{m\left[D_x\left(\frac{m^4}{a^4}\right)+2H\left(\frac{m^2n^2}{a^2b^2}\right)+D_y\left(\frac{n^4}{b^4}\right)\right]},$$

$$\frac{\partial^2 w}{\partial x\partial y} = \frac{16p_0}{\pi^4 ab}\sum_{m=1,3,\ldots}^{\infty}\sum_{n=1,3,\ldots}^{\infty}\frac{\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}}{\left[D_x\left(\frac{m^4}{a^4}\right)+2H\left(\frac{m^2n^2}{a^2b^2}\right)+D_y\left(\frac{n^4}{b^4}\right)\right]}.$$

Finally the stresses from Abaqus and analytical model were plotten in the same figures. In the Figure 7 it may be seen that the stresses from the analytical model are almost equal to stresses obtained from the Abaqus model. In the Figure 8 stresses near the corner of the plate are compared. There exists some differences between the Abaqus and analytical model. Partly the reason is that measuring the stresses from exact coordinate was not possible in Abaqus, and the stresses, which had large gradient in the region, were only approximations. Still the stresses in the corner are clearly comparable between the models and may be considered as validation of the used FE model.

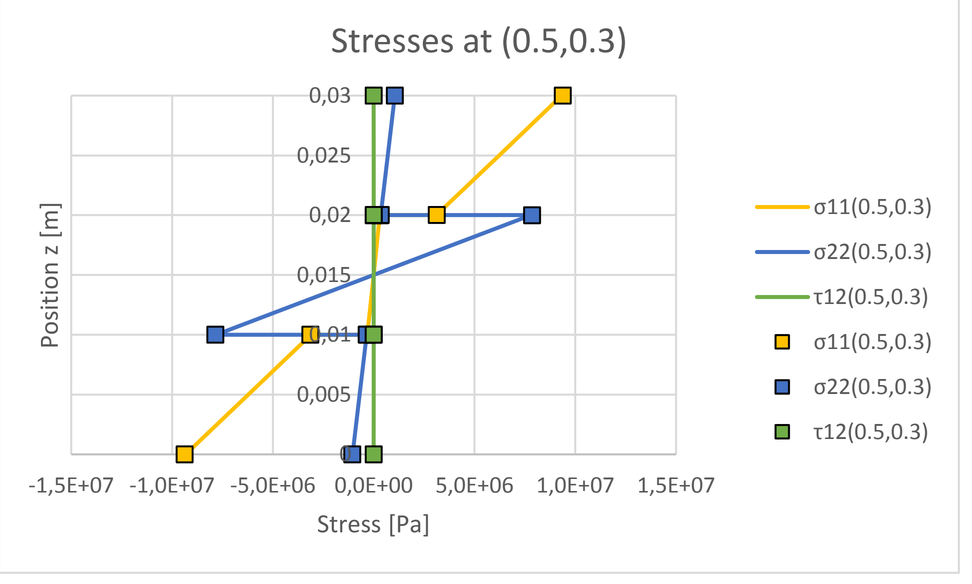



*Figure 7. Stresses from the analytical model (squares) compared to Abaqus results (lines) on the mid-point of the plate.*

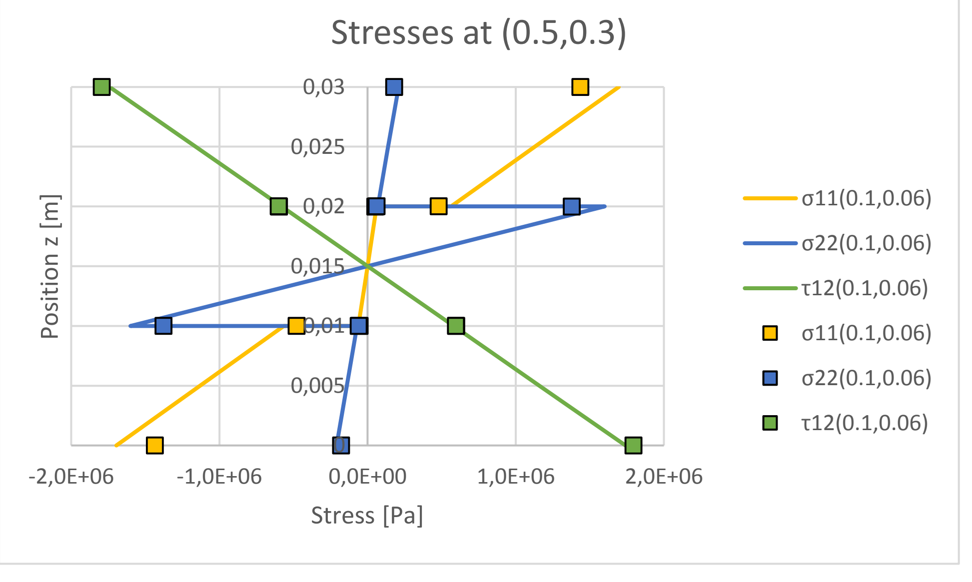



*Figure 8. Stresses from the analytical model (squares) compared to Abaqus results (lines) near the corner of the plate.*