Homework 4

CIV-E4080, Material Modeling in Civil Engineering L

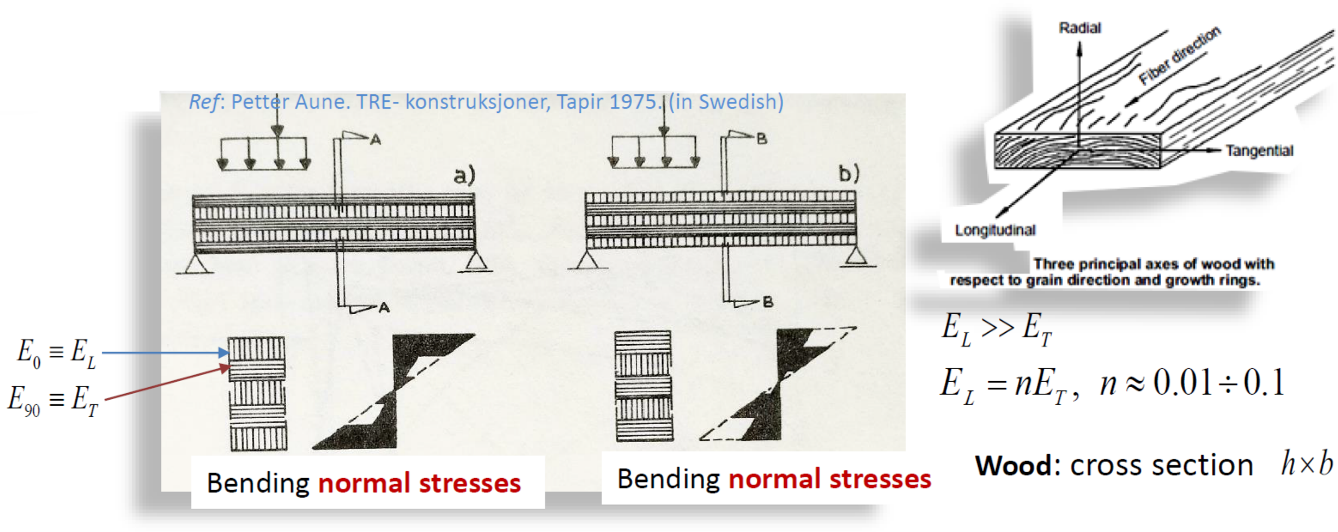
Introduction and Readings

The following content is covered/ needed for doing this homework,

- **Material symmetries** - Degree of symmetry
- **Linear Elasticity** - Matrix Formulation
- **Anisotropy**
- **Isotropy** - Limits on Elasic Parameters Values
- **Orthotropy**
- **Transversal Isotropy** - Limits on Elastic Parameters Values

Problem 1 - Orthotropy

[5 points] **Elastic bending of a Glue Laminated Timber (GLT) Beam**



1. Explain qualitatively and quantitavely (very concisely)
   (a) Why these stresses are as they are ?
   (b) Why is this difference ?

2. For a generic freely supported GL-beam and a constant distributed load, determine the elastic deflection at the mid span for a normal temperature and hygrometry. Account for both bending and shearing contribution for the deflection. (*For material properties refer to table given below).*

Hints

- Determine an effective bending stiffness **'B'** and shear stiffness **'S'** for the composite sections 'a' and 'b'.
- For choosing the correct shear modulus values from tables, think in which material orthotropy plane the corresponding shear strain component occurs.

Some useful tables for wood

Table 4–2. Poisson's ratios for various species at approximately 12% moisture content

Species	μ_{LR}	μ_{LT}	μ_{RT}	μ_{TR}	μ_{RL}	μ_{TL}
Hardwoods						
Ash, white	0.371	0.440	0.684	0.360	0.059	0.051
Aspen, quaking	0.489	0.374	—	0.496	0.054	0.022
Balsa	0.229	0.488	0.665	0.231	0.018	0.009
Basswood	0.364	0.406	0.912	0.346	0.034	0.022
Birch, yellow	0.426	0.451	0.697	0.426	0.043	0.024
Cherry, black	0.392	0.428	0.695	0.282	0.086	0.048
Cottonwood, eastern	0.344	0.420	0.875	0.292	0.043	0.018
Mahogany, African	0.297	0.641	0.604	0.264	0.033	0.032
Mahogany, Honduras	0.314	0.533	0.600	0.326	0.033	0.034
Maple, sugar	0.424	0.476	0.774	0.349	0.065	0.037
Maple, red	0.434	0.509	0.762	0.354	0.063	0.044
Oak, red	0.350	0.448	0.560	0.292	0.064	0.033
Oak, white	0.369	0.428	0.618	0.300	0.074	0.036
Sweet gum	0.325	0.403	0.682	0.309	0.044	0.023
Walnut, black	0.495	0.632	0.718	0.378	0.052	0.035
Yellow-poplar	0.318	0.392	0.703	0.329	0.030	0.019
Softwoods						
Baldcypress	0.338	0.326	0.411	0.356	—	—
Cedar, northern white	0.337	0.340	0.458	0.345	—	—
Cedar, western red	0.378	0.296	0.484	0.403	—	—
Douglas-fir	0.292	0.449	0.390	0.374	0.036	0.029
Fir, subalpine	0.341	0.332	0.437	0.336	—	—
Hemlock, western	0.485	0.423	0.442	0.382	—	—
Larch, western	0.355	0.276	0.389	0.352	—	—
Pine						
Loblolly	0.328	0.292	0.382	0.362	—	—
Lodgepole	0.316	0.347	0.469	0.381	—	—
Longleaf	0.332	0.365	0.384	0.342	—	—
Pond	0.280	0.364	0.389	0.320	—	—
Ponderosa	0.337	0.400	0.426	0.359	—	—
Red	0.347	0.315	0.408	0.308	—	—
Slash	0.392	0.444	0.447	0.387	—	—
Sugar	0.356	0.349	0.428	0.358	—	—
Western white	0.329	0.344	0.410	0.334	—	—
Redwood	0.360	0.346	0.373	0.400	—	—
Spruce, Sitka	0.372	0.467	0.435	0.245	0.040	0.025
Spruce, Engelmann	0.422	0.462	0.530	0.255	0.083	0.058

Chapter 2

Structure of Wood

Regis B. Miller

Table 4–1. Elastic ratios for various species at approximately 12% moisture content[a]

Species	E_T/E_L	E_R/E_L	G_{LR}/E_L	G_{LT}/E_L	G_{RT}/E_L
Hardwoods					
Ash, white	0.080	0.125	0.109	0.077	—
Balsa	0.015	0.046	0.054	0.037	0.005
Basswood	0.027	0.066	0.056	0.046	—
Birch, yellow	0.050	0.078	0.074	0.068	0.017
Cherry, black	0.086	0.197	0.147	0.097	—
Cottonwood, eastern	0.047	0.083	0.076	0.052	—
Mahogany, African	0.050	0.111	0.088	0.059	0.021
Mahogany, Honduras	0.064	0.107	0.066	0.086	0.028
Maple, sugar	0.065	0.132	0.111	0.063	—
Maple, red	0.067	0.140	0.133	0.074	—
Oak, red	0.082	0.154	0.089	0.081	—
Oak, white	0.072	0.163	0.086	—	—
Sweet gum	0.050	0.115	0.089	0.061	0.021
Walnut, black	0.056	0.106	0.085	0.062	0.021
Yellow-poplar	0.043	0.092	0.075	0.069	0.011
Softwoods					
Baldcypress	0.039	0.084	0.063	0.054	0.007
Cedar, northern white	0.081	0.183	0.210	0.187	0.015
Cedar, western red	0.055	0.081	0.087	0.086	0.005
Douglas-fir	0.050	0.068	0.064	0.078	0.007
Fir, subalpine	0.039	0.102	0.070	0.058	0.006
Hemlock, western	0.031	0.058	0.038	0.032	0.003
Larch, western	0.065	0.079	0.063	0.069	0.007
Pine					
Loblolly	0.078	0.113	0.082	0.081	0.013
Lodgepole	0.068	0.102	0.049	0.046	0.005
Longleaf	0.055	0.102	0.071	0.060	0.012
Pond	0.041	0.071	0.050	0.045	0.009
Ponderosa	0.083	0.122	0.138	0.115	0.017
Red	0.044	0.088	0.096	0.081	0.011
Slash	0.045	0.074	0.055	0.053	0.010
Sugar	0.087	0.131	0.124	0.113	0.019
Western white	0.038	0.078	0.052	0.048	0.005
Redwood	0.089	0.087	0.066	0.077	0.011
Spruce, Sitka	0.043	0.078	0.064	0.061	0.003
Spruce, Engelmann	0.059	0.128	0.124	0.120	0.010

[a] E_L may be approximated by increasing modulus of elasticity values in Table 4–3 by 10%.

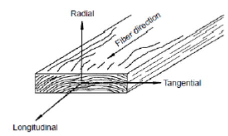


Figure 4–1. Three principal axes of wood with respect to grain direction and growth rings.

$$\frac{\mu_{ij}}{E_i} = \frac{\mu_{ji}}{E_j}, \quad i \neq j \quad i,j = L,R,T$$

Solution

1. Timber being an orthotropic material has different stiffnesses in different directions. Since, GLT is composed of various layers of timber with different grain directions, this causes sudden variation in stresses in different layers.

2. Strain energy of the beam is,

$$U = \frac{1}{2} \int_v \sigma_x \epsilon_x + \tau_{xy} \gamma_{xy} dV \tag{1}$$

Considering the width to be constant,

$$U = \frac{1}{2} b \int_0^l \int_y (\sigma_x \epsilon_x + \tau_{xy} \gamma_{xy}) dy.dx \tag{2}$$

Where, $\sigma_x = E_x \epsilon_x$, $\epsilon_x = y.K_x$, $\tau_{xy} = G_{xy}.\gamma_{xy}$

$$\begin{aligned} U = \frac{1}{2} b \int_0^l [(\int_0^{y1} & E_L.y^2.K_x^2 + G_{TL}.\gamma_{xy}^2) dy \\ & + (\int_{y1}^{y2} E_T.y^2.K_x^2 + G_{LT}.\gamma_{xy}^2) dy \\ & + (\int_{y2}^{y3} E_L.y^2.K_x^2 + G_{TL}.\gamma_{xy}^2) dy] dx \end{aligned} \tag{3}$$

$$U_1 = \frac{1}{2}.2.b \int_0^l (E_L.\frac{y_1^3}{3}.K_x^2 + y_1.G_{TL}.\gamma_{xy}^2) dx \tag{4}$$

$$U_2 = \frac{1}{2}.2.b \int_0^l (E_T.\frac{(y_2 - y_1)^3}{3}.K_x^2 + (y_2 - y_1).G_{LT}.\gamma_{xy}^2) dx \tag{5}$$

$$U_3 = \frac{1}{2}.2.b \int_0^l (E_L.\frac{(y_3 - y_2)^3}{3}.K_x^2 + (y_3 - y_2).G_{TL}.\gamma_{xy}^2) dx \tag{6}$$

Bending strain energy is,

$$U_b = \frac{1}{2}. \int_0^l 2.b[(E_L.\frac{y_1^3}{3} + E_T.\frac{(y_2 - y_1)^3}{3} + E_L.\frac{(y_3 - y_2)^3}{3}].K_x^2 dx \tag{7}$$

Similarly, shear strain energy is,

$$U_s = \frac{1}{2}. \int_0^l 2.b[y_1.G_{TL} + (y_2 - y_1).G_{LT} + (y_3 - y_2).G_{TL}]\gamma_{xy}^2 dx \tag{8}$$

Strain energy of the beam in terms of bending and shear energies is of the form,

$$U = U_b + U_s = \frac{1}{2} \int_0^l EI.K_x^2 dx + \int_0^l GA.\gamma_{xy}^2 dx \tag{9}$$

Hence, the effective stiffnesses are,

$$EI = 2.b[E_L.\frac{y_1^3}{3} + E_T.\frac{(y_2 - y_1)^3}{3} + E_L.\frac{(y_3 - y_2)^3}{3}] \tag{10}$$

$$GA = 2.b[y_1.G_{TL} + (y_2 - y_1).G_{LT} + (y_3 - y_2).G_{TL}] \quad (11)$$

Deflection of a simply supported Timoshenko beam is,

$$u(x) = \frac{q}{24.EI}.(x^4 - 2.l.x^3 + l^3.x) + \frac{q}{2.GA}.((l.x) - x^2)) \quad (12)$$

Therefore, deflection at the mid-span becomes,

$$u(l/2) = \frac{5.q.l^4}{384.EI} + \frac{q.l^2}{8.GA} \quad (13)$$

Problem 2 - Linear Isotropy in 3D elasticity

[5 points] Using Voigt's notation,

1. Write explicitly the stress-strain relation $\sigma = \lambda \mathbf{Tr}(\epsilon)\mathbf{1} + \mathbf{2}\mu\epsilon$ in the matric form $\sigma = \mathbf{D}\epsilon$ using the elasticity constants *E, v* instead of Lamė elastic constants.
   What would be the expression, in matrix form, of the strain energy in terms of strains only?
   What, not trivial result can you conclude for the material stiffness matrix **'D'** from the sign of strain strain energy ?

2. Determine explicitly, from answer of '1', the three dimension compliance matrix ($\sigma_{\mathbf{ij}} = \lambda\epsilon_{\mathbf{kk}}\delta_{\mathbf{ij}} + \mathbf{2}\mu\epsilon_{\mathbf{ij}}$) for an isotropic linear elastic material.

Bonus: What is the physical meaning of material isotropy ? [1 point]

Solution

1.
$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} \tag{14}$$

Invert and compare with:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} \tag{15}$$

and conclude that:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \tag{16}$$

$$\mu = G = \frac{E}{2(1+\nu)} \tag{17}$$

2.
$$\epsilon = C^{-1}\sigma \tag{18}$$

Hence,

$$C = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \tag{19}$$

Problem 3 - Orthotropic Elasticity in Plane

[5 points] Consider a two-dimensional orthotropic material in plane stress state.

1. Determine the constitutive relation $\epsilon = f(\sigma)$ in matrix form for 2-D plane stress. The coefficient matrix is the compliance matrix C.

2. Invert the compliance matrix and deduce the stress strain relation in matrix form. (D = ?)

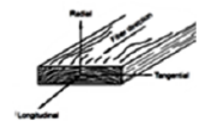
3. Derive the reciprocal relations between Poisson's coefficients v_{ij} and elastic modulus E_i.

Hint: *From which 'basic property' are the reciprocity relations derived ?*

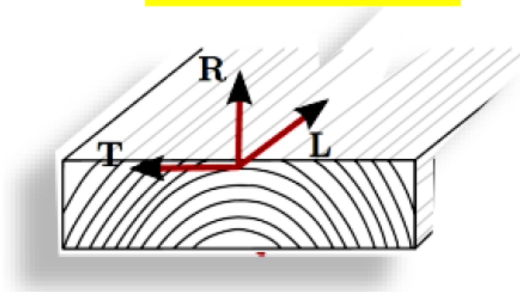
Reciprocal relations:

$$\nu_{12} / E_1 = \nu_{21} / E_2$$
$$\nu_{31} / E_3 = \nu_{13} / E_1$$
$$\nu_{23} / E_2 = \nu_{32} / E_3$$

1, 2, 3 are the material principal directions of orthotropy



L – longitudinal
T – tangential
R – radial



Solution

1.
$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & 0 \\ -\nu_{21}/E_2 & 1/E_2 & 0 \\ 0 & 0 & 1/G_{12} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} \tag{20}$$

2. Using the matrix rule (with A and B both square matrices):

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{bmatrix} \tag{21}$$

Hence, herein:

$$\begin{bmatrix} 1/E_1 & -\nu/E_2 \\ -\nu/E_2 & 1/E_2 \end{bmatrix}^{-1} = \frac{E_1E_2}{(1-\nu_{12}\nu_{21})} \begin{bmatrix} 1/E_2 & -\nu/E_2 \\ -\nu/E_2 & 1/E_1 \end{bmatrix} = \frac{1}{(1-\nu_{12}\nu_{21})} \begin{bmatrix} E_1 & \nu_{21}E_1 \\ \nu_{12}E_2 & E_2 \end{bmatrix} \tag{22}$$

Therefore:
$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} E_1/(1-\nu_{12}\nu_{21}) & \nu_{21}E_1/(1-\nu_{12}\nu_{21}) & 0 \\ \nu_{12}E_2/(1-\nu_{12}\nu_{21}) & E_2/(1-\nu_{12}\nu_{21}) & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \tag{23}$$

$$\sigma_{11} = \frac{E_1}{(1-\nu_{12}\nu_{21})}(\epsilon_{11} + \nu_{21}\epsilon_{22}) \tag{24}$$

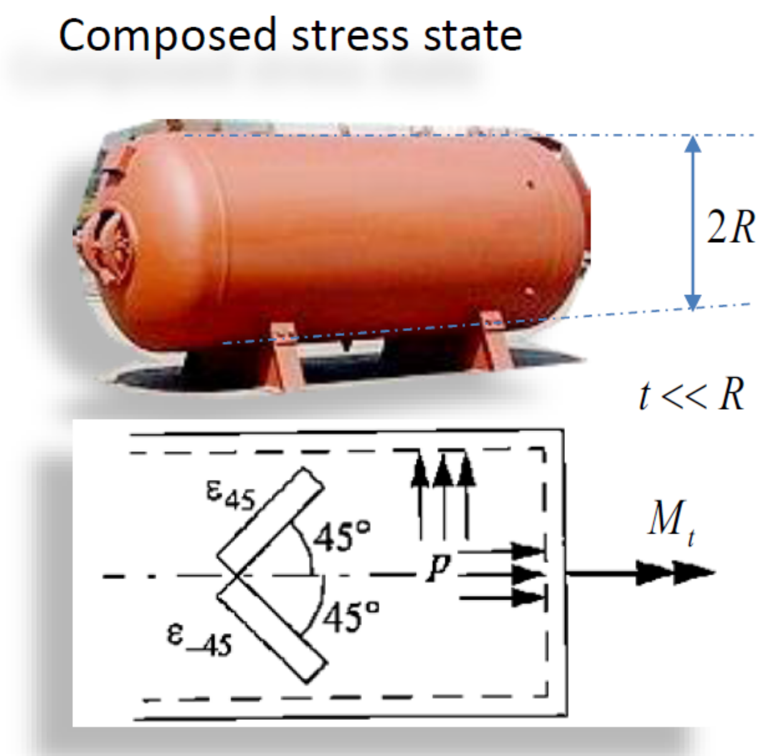
$$\sigma_{22} = \frac{E_2}{(1-\nu_{12}\nu_{21})}(\nu_{12}\epsilon_{11} + \epsilon_{22}) \tag{25}$$

$$\sigma_{12} = G_{12}(2\epsilon_{12}) \tag{26}$$

3. Reciprocal relations can be derived from the fact that the compliance matrix is symmetrical.

Problem 4 - Isotropy in 3D elasticity

[5 points] Consider a thin-walled cylindrical pressure vessel. The wall is very thin, t << R, (t/R < 1/10) as compared to the other dimensions as the radius, and consequently you may assume that the stresses are uniform across the wall thickness 't'. Here we have t = 2mm and D = 2R = 50mm. The internal (over-) pressure p > 0 is uniform. In addition to the gauge pressure 'p' a torque moment $'M_t'$ is applied at the ends of the vessel cylinder.
Two strain gauges mutually perpendicular are perfectly glues on the external surface of the cylinder (as shown in figure). The measure strains are, $\epsilon_{45} = 50\mu m/m$ and $\epsilon_{-45} = -20\mu m/m$
The material is steel and considered *isotropic* and *linear elastic* since the stress state is such that no plastic flow occurs.



$$p \equiv p_{inside} - p_{atmospheric} > 0$$

Oheisen kuvan mukaisen ohutseinämäisen suljetun putken sisällä vallitsee ylipaine p. Putken keskihalkaisija on $d = 50$ mm ja seinämän paksuus $t = 2$ mm. Putkea kuormittaa myös vääntömomentti M_v. Putken ulkopinnalta mitataan kahdella venymäliuskalla venymän arvot $\varepsilon_{45} = 0,00005$ ja $\varepsilon_{-45} = -0,00002$. Määritä paine p ja vääntömomentti M_v.

Determine the torque moment $'M_t'$ and the pressure 'p'.

Hint: Since the geometry and loading is cylindrically symmetric, the stresses are independent of the angular coordinate of the cylindrically coordinate system.

Solution

Material properties are,

$$E = 210GPa, \nu = 0.3, G = E/2(1+\nu) = 80.77GPa$$

Stresses in the cylinder are,

$$\sigma_{11} = \frac{1}{4}\frac{pD}{t} \tag{27}$$

$$\sigma_{22} = \frac{1}{2}\frac{pD}{t} \tag{28}$$

$$\tau_{12} = \frac{2M_t}{\pi t D^2} \tag{29}$$

The strains are,

$$\epsilon_{45} = \frac{1}{2}\epsilon_{11} + \frac{1}{2}\epsilon_{22} + \epsilon_{12} \tag{30}$$

$$\epsilon_{-45} = \frac{1}{2}\epsilon_{11} + \frac{1}{2}\epsilon_{22} - \epsilon_{12} \tag{31}$$

Where,

$$\epsilon_{11} = \frac{\sigma_{11}}{E} - \nu\frac{\sigma_{22}}{E} \tag{32}$$

$$\epsilon_{22} = \frac{\sigma_{22}}{E} - \nu\frac{\sigma_{11}}{E} \tag{33}$$

$$\epsilon_{12} = \frac{\tau_{12}}{2G} = \frac{\epsilon_{45} - \epsilon_{-45}}{2} = 3.5 \times 10^{-5} \tag{34}$$

The torsional moment can be calculated as,

$$\tau_{12} = \frac{2M_t}{t\pi D^2} => M_t = G.\epsilon_{12}.t.\pi.D^2 = 0.044kN.m \tag{35}$$

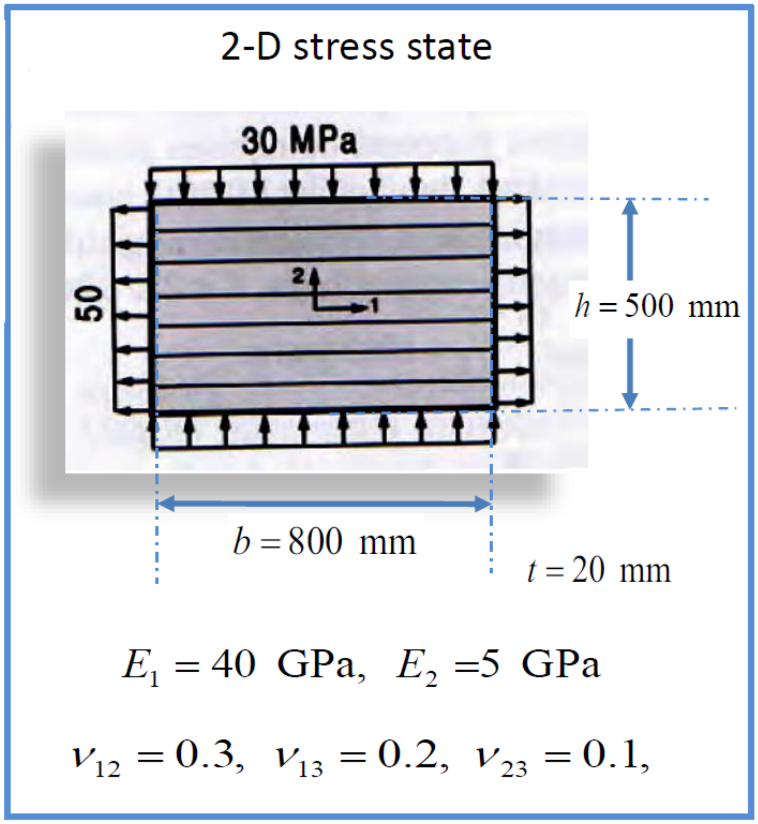
The pressure can be calculated as,

$$\epsilon_{45} + \epsilon_{-45} = \epsilon_{11} + \epsilon_{22} = (1-\nu)(\sigma_{11} + \sigma_{22})/E = \frac{3}{4}(1-\nu)\frac{p.D}{E.t} \tag{36}$$

$$p = 4.E.t(\epsilon_{45} + \epsilon_{-45})/3(1-\nu) = 480kPa \tag{37}$$

Problem 5 - Laminate Plate

[5 points] Consider a laminate plate with principle material directions 1 and 2. The fibers are aligned along direction 1. In the thickness direction of the plate we have transverse isotropy. the thickness is 20mm. (Tehty hiilikuituvahvisteisesta epoksista).
The plate is under stress state shown in figure below,



Determine the length changes in both directions 1 and 2, and in the thickness direction.

Bonus : [5 points]
Compute the solution using FEM software.

Kuitulujitetun levyn paksuus on 20 mm, leveys kuitusuunnassa 1 on 800 mm ja leveys suunnassa 2 on 500 mm. Määritä levyn paksuuden ja leveyden muutokset. Materiaalikertoimet ovat $E_1 = 40$ GPa, $E_2 = 5$ GPa, ja $\nu_{12} = 0,3, \nu_{13} = 0,2$ ja $\nu_{23} = 0,1$.

Solution

Reciprocal relations are,

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1} => \nu_{21} = \frac{E_2}{E_1}\nu_{12} \tag{38}$$

Strains in the plate are,

$$\epsilon_1 = \frac{\sigma_1}{E_1} - \nu_{21}\frac{\sigma_2}{E_2} = \frac{1}{E_1}(\sigma_1 - \nu_{12}\sigma_2) \tag{39}$$

$$\epsilon_2 = -\nu_{12}\frac{\sigma_1}{E_1} + \frac{\sigma_2}{E_2} = \frac{1}{E_2}(\sigma_2 - \nu_{12}\frac{E_2}{E_1}\sigma_1) \tag{40}$$

$$\epsilon_3 = -\nu_{13}\frac{\sigma_1}{E_1} - \frac{\nu_{23}\sigma_2}{E_2} = -\frac{1}{E_2}(\nu_{13}\frac{E_2}{E_1}\sigma_1 + \nu_{23}\sigma_2) \tag{41}$$

Change in length in 1-direction is,

$$\epsilon_1 \times 800mm = \frac{1}{40GPa}(50MPa - 0.3 \times 30MPa) \times 800mm = 1.18mm \tag{42}$$

Change in length in 2-direction is,

$$\epsilon_2 \times 500mm = \frac{1}{5GPa}(30MPa - 0.3 \times \frac{5GPa}{40GPa} \times 50MPa) \times 500mm = -3.1875mm \tag{43}$$

Change in length in 3-direction is,

$$\epsilon_3 \times 20mm = -\frac{1}{5GPa}(0.2 \times \frac{5GPa}{40GPa} \times 50MPa + 0.1 \times 30MPa) \times 20mm = 7 \times 10^{-3}mm \tag{44}$$

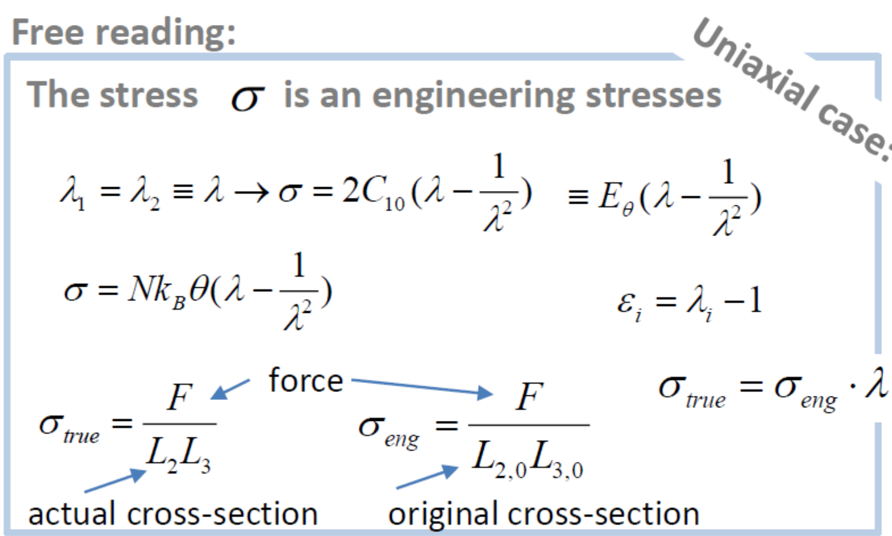
Problem 6 - Hyper Elasticity

[5 points] Consider an ideal rubber in a bi-axial stress state $\sigma_1 \equiv \sigma_x$ and $\sigma_2 \equiv \sigma_y$ at a constant temperature.

1. Starting with the expression for Helmholtz free energy, derive the constitutive law for stresses versus stretches (extension ratio), i.e., find,
$$\sigma_i \equiv \sigma_i(\lambda_1, \lambda_2), i = 1, 2$$
For the ideal ruber, you can use a neo- Hokean model where the Gibbs free energy density is,
$$\psi = C_{10}(I_1 - 3) \equiv 1/2E_\theta(I_1 - 3)$$
With an effective elasticity coefficient E_θ
$$C_{10} = 1/2Nk_B\theta \equiv 1/2E_\theta$$
Account first for incompressibility (deformation occurs at constant volume) the invariant,
$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$
Where, λ_1, λ_2 and λ_3 are Principle stretches

   Partial answers: $\sigma_1 = E_\theta(\lambda_1 - \frac{1}{\lambda_1^3\lambda_2^2})$, $\sigma_2 = E_\theta(\lambda_2 - \frac{1}{\lambda_2^3\lambda_1^2})$.

2. Assume that the elastic modulus is experimentally estimated as, $'E_\theta \approx 3.33$ MPa'. Determine the stresses leading to the stretches $'\lambda_1 = 2, \lambda_2 = 1/2'$.
Model validity: By comparing with experimental results it was found that the model over-estimates the stresses when $1.5 < \lambda_1 < 6$.
What is the quality of your results for stresses ?



Solution

1. The stress in a particular direction in a rubber under deformation is found by taking the partial derivative of the change in Helmholtz free energy with respect to the extension ratio in the direction of interest.

   Also, the deformation happens at constant volume. Thus,

$$\lambda_x \lambda_y \lambda_z = 1 => \lambda_z = 1/\lambda_x \lambda_y \tag{45}$$

$$\psi = \frac{1}{2} E_\theta (I_1 - 3) = 1/2 E_\theta (\lambda_x^2 + \lambda_y^2 + \frac{1}{\lambda_x^2 \lambda_y^2} - 3) \tag{46}$$

   Stress in x-direction is,

$$\frac{\partial \psi}{\partial \lambda_x} = \sigma_x = \frac{1}{2} E_\theta (2\lambda_x - \frac{2}{\lambda_x^3 \lambda_y^2}) = E_\theta (\lambda_x - \frac{1}{\lambda_x^3 \lambda_y^2}) \tag{47}$$

   Stress in y-direction is,

$$\frac{\partial \psi}{\partial \lambda_y} = \sigma_y = \frac{1}{2} E_\theta (2\lambda_y - \frac{2}{\lambda_y^3 \lambda_x^2}) = E_\theta (\lambda_y - \frac{1}{\lambda_y^3 \lambda_x^2}) \tag{48}$$

2.

$$\sigma_x = 3.33(2 - \frac{1}{(2)^3 (1/2)^2}) = 5MPa \tag{49}$$

$$\sigma_y = 3.33(\frac{1}{2} - \frac{1}{(1/2)^3 (2)^2}) = -5MPa \tag{50}$$

Bonus : [5 points] Compute the solution using FEM software.