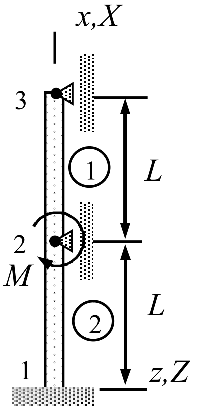
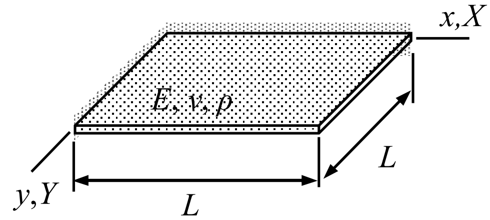
MEC-E8001 Finite Element Analysis, Exam 10.04.2019

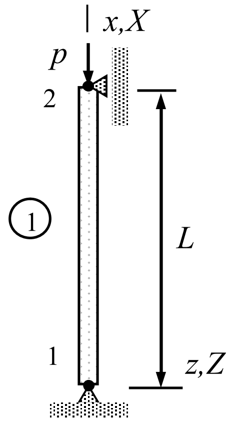
1. Beam structure of the figure is loaded by a point moment acting on node 2. Determine the rotations θ_{Y2} and θ_{Y3} by using two beam elements. Displacements are confined to the XZ-plane. The cross-section properties of the beam A, I and Young's modulus of the material E are constants.



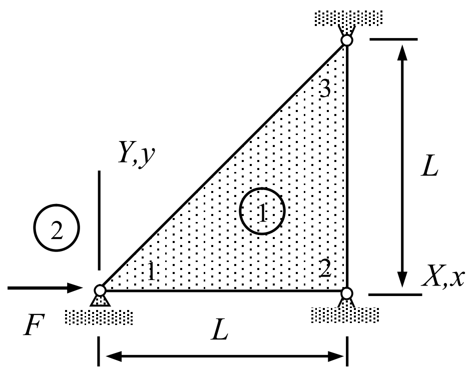
2. A plate is simply supported on two edges and free on the other two edges as shown. Use the approximation $w(x,y,t) = a(t)xy/L^2$ to determine the transverse displacement as function of time $t > 0$. Material properties E, ν, and ρ are constants and thickness of the plate is h. At $t = 0$, initial conditions are $\dot{w}(x,y,0) = 0$ and $w(x,y,0) = Uxy/L^2$. Assume that the plate is thin so that the rotation part of the inertia term is negligible.



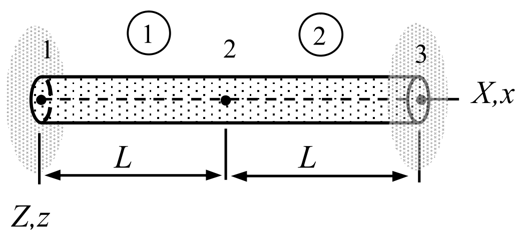
3. Determine the buckling force p_{cr} and the buckled shape of the structure shown by using one beam element. Displacements are confined to the xz – plane. Parameters E, A, and I are constants.



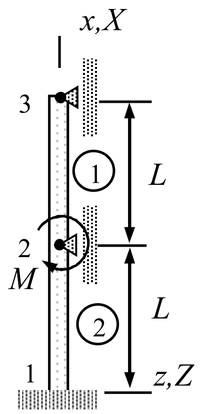
4. A thin triangular slab (assume plane stress conditions) loaded by a horizontal force is allowed to move horizontally at node 1 and nodes 2 and 3 are fixed. Derive the equilibrium equation for the structure according to the large displacement theory. Material parameters C, ν and thickness t at the initial geometry of the slab are constants.



5. Electric current causes heat generation in element 2 of the bar shown. Calculate the temperature ϑ_2 at the center point, if the wall temperature (nodes 1 and 3) is ϑ°. Cross sectional area A and thermal conductivity k are constants. Heat production rate per unit length vanishes in element 1 and it is constant s in element 2.



Beam structure of the figure is loaded by a point moment acting on node 2. Determine the rotations θ_{Y2} and θ_{Y3} by using two beam elements. Displacements are confined to the XZ-plane. The cross-section properties of the beam A, I and Young's modulus of the material E are constants.



Solution

Virtual work expression for the displacement analysis consists of parts coming from internal and external forces δW^{int} and δW^{ext}. For the beam bending mode in xz-plane and cubic interpolation of the nodal values, the element contributions are

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}, \quad \delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{f_z h}{12} \begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix},$$

The element contribution of the point force/moment follows from the definition or work and is given by

$$\delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \\ \delta u_{Z1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix} + \begin{Bmatrix} \delta\theta_{X1} \\ \delta\theta_{Y1} \\ \delta\theta_{Z1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} M_X \\ M_Y \\ M_Z \end{Bmatrix}.$$

4p For beam 1, $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}}$ is given by

$$\delta W^1 = -\begin{Bmatrix} 0 \\ \delta\theta_{Y2} \\ 0 \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ \theta_{Y2} \\ 0 \\ \theta_{Y3} \end{Bmatrix} = -\begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 4\frac{EI}{L} & 2\frac{EI}{L} \\ 2\frac{EI}{L} & 4\frac{EI}{L} \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix},$$

For beam 2, $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}}$ is given by (when written in a form which is compatible with the first element contribution)

$$\delta W^2 = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_{Y2} \end{Bmatrix} = -\begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 4\frac{EI}{L} & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix}.$$

Virtual work expression of the point moment (written in a form which is compatible with the other element contributions)

$$\delta W^3 = \begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} -M \\ 0 \end{Bmatrix}.$$

2p Virtual work expression of structure is sum of the element contributions

$$\delta W = \delta W^1 + \delta W^2 + \delta W^3 = -\begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} 8\dfrac{EI}{L} & 2\dfrac{EI}{L} \\ 2\dfrac{EI}{L} & 4\dfrac{EI}{L} \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix} - \begin{Bmatrix} -M \\ 0 \end{Bmatrix} \right).$$

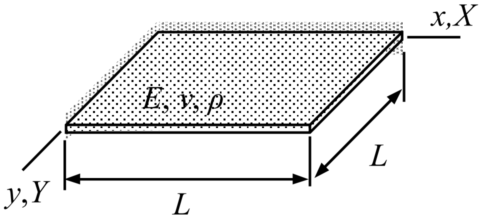
Principle of virtual work and the fundamental lemma of variation calculus imply that

$$\begin{bmatrix} 8\dfrac{EI}{L} & 2\dfrac{EI}{L} \\ 2\dfrac{EI}{L} & 4\dfrac{EI}{L} \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix} - \begin{Bmatrix} -M \\ 0 \end{Bmatrix} = 0.$$

Solution to the linear equations system is given by

$$\theta_{Y2} = -\frac{1}{7}\frac{ML}{EI} \quad \text{and} \quad \theta_{Y3} = \frac{1}{14}\frac{ML}{EI}. \quad \leftarrow$$

A plate is simply supported on two edges and free on the other two edges as shown. Use the approximation $w(x,y,t)=a(t)xy/L^2$ to determine the transverse displacement as function of time $t>0$. Material properties E, ν, and ρ are constants and thickness of the plate is h. At $t=0$, initial conditions are $\dot{w}(x,y,0)=0$ and $w(x,y,0)=Uxy/L^2$. Assume that the plate is thin so that the rotation part of the inertia term is negligible.



Solution

4p Only the bending mode of the plate matters. When the approximation $w=a(t)xy/L^2$ is substituted there, virtual work densities of internal and inertia forces (without the rotation part) of the plate simplify to (shear modulus $G=E/(2+2\nu)$)

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \\ 2\partial^2\delta w/\partial x\partial y \end{Bmatrix}^{\mathrm{T}} \frac{h^3}{12}\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\partial y \end{Bmatrix} = -\delta a\frac{1}{L^4}\frac{h^3}{3}Ga\,,$$

$$\delta w_\Omega^{\text{ine}} = -\begin{Bmatrix} \partial\delta w/\partial x \\ \partial\delta w/\partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}\rho\begin{Bmatrix} \partial\ddot{w}/\partial x \\ \partial\ddot{w}/\partial y \end{Bmatrix} - \delta w t\rho\ddot{w} = -\delta a(\frac{x}{L})^2(\frac{y}{L})^2 h\rho\ddot{a}$$

in which h is thickness of the plate. Integration over the domain occupied by the element gives the virtual work expressions

$$\delta W^{\text{int}} = \int_0^L\int_0^L \delta w_\Omega^{\text{int}} dydx = \int_0^L\int_0^L -\delta a\frac{1}{L^4}\frac{h^3}{3}Gadydx = -\delta a\frac{1}{L^2}\frac{h^3}{3}Ga\,,$$

$$\delta W^{\text{ine}} = \int_0^L\int_0^L \delta w_\Omega^{\text{ine}} dydx = \int_0^L\int_0^L -\delta a(\frac{x}{L})^2(\frac{y}{L})^2 h\rho\ddot{a}dxdy = -\delta a\frac{L^2}{9}h\rho\ddot{a}\,.$$

Virtual work expression of the structure consists of the internal and inertia parts

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\delta a(\frac{1}{L^2}\frac{h^3}{3}Ga + \frac{L^2}{9}h\rho\ddot{a})\,.$$

Principle of virtual work $\delta W = 0 \quad \forall\delta\mathbf{a}$ and the fundamental lemma of variation calculus $\delta\mathbf{a}^{\mathrm{T}}\mathbf{F}=0 \ \forall\delta\mathbf{a} \quad \Leftrightarrow \quad \mathbf{F}=0$ imply

$$\frac{1}{L^2}\frac{h^3}{3}Ga + \frac{L^2}{9}h\rho\ddot{a} = 0\,.$$

2p What remains, is solving for the displacement from the initial value problem

$$\ddot{a}+3\frac{Gh^2}{\rho L^4}a=0 \quad t>0, \quad a(0)=U, \quad \dot{a}(0)=0.$$

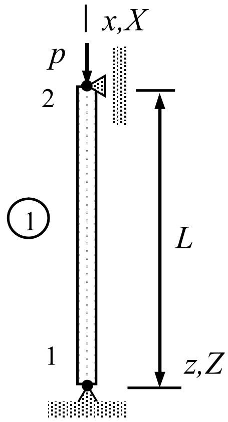
Solution to equations is (this can be shown e.g. by substituting the solution in the equations above)

$$a(t)=U\cos(\sqrt{3\frac{G}{\rho}}\frac{h}{L^2}t) \quad t>0.$$

Finally, substituting the solution to parameter $a(t)$ into the approximation gives

$$w(x,y,t)=U\cos(\sqrt{3\frac{G}{\rho}}\frac{h}{L^2}t)\frac{xy}{L^2}.$$

Determine the buckling force $p_{\rm cr}$ and the buckled shape of the structure shown by using one beam element. Displacements are confined to the $xz-$plane. Parameters E, A, and I are constants.



Solution

4p The non-zero displacement/rotation components of the structure are and $\theta_{y1}=\theta_{Y1}$, $\theta_{y2}=\theta_{Y2}$, and $u_{x2}=u_{X2}$. The normal force in the beam $N=-p$ can be deduced without calculations on the axial displacement. Therefore, it is enough to consider only the bending and coupling terms of the virtual work expression. As buckling is confined to the $xz-$plane

$$\delta W=-\begin{Bmatrix}0\\ \delta\theta_{Y1}\\ 0\\ \delta\theta_{Y2}\end{Bmatrix}^{\rm T}(\frac{EI}{L^3}\begin{bmatrix}12 & -6L & -12 & -6L\\ -6L & 4L^2 & 6L & 2L^2\\ -12 & 6L & 12 & 6L\\ -6L & 2L^2 & 6L & 4L^2\end{bmatrix}-\frac{p}{30L}\begin{bmatrix}36 & -3L & -36 & -3L\\ -3L & 4L^2 & 3L & -L^2\\ -36 & 3L & 36 & 3L\\ -3L & -L^2 & 3L & 4L^2\end{bmatrix})\begin{Bmatrix}0\\ \theta_{Y1}\\ 0\\ \theta_{Y2}\end{Bmatrix}\Leftrightarrow$$

$$\delta W=-\begin{Bmatrix}\delta\theta_{Y1}\\ \delta\theta_{Y2}\end{Bmatrix}^{\rm T}(\frac{EI}{L}\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix}-\frac{pL}{30}\begin{bmatrix}4 & -1\\ -1 & 4\end{bmatrix})\begin{Bmatrix}\theta_{Y1}\\ \theta_{Y2}\end{Bmatrix}.$$

According to the principle of virtual work

$$(\frac{EI}{L}\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix}-\frac{pL}{30}\begin{bmatrix}4 & -1\\ -1 & 4\end{bmatrix})\begin{Bmatrix}\theta_{Y1}\\ \theta_{Y2}\end{Bmatrix}=0.$$

2p A homogeneous equation system has a non-trivial solution only if the matrix is singular

$$\det(\begin{bmatrix}4\frac{EI}{L}-4\frac{pL}{30} & 2\frac{EI}{L}+\frac{pL}{30}\\ 2\frac{EI}{L}+\frac{pL}{30} & 4\frac{EI}{L}-4\frac{pL}{30}\end{bmatrix})=(4\frac{EI}{L}-4\frac{pL}{30})^2-(2\frac{EI}{L}+\frac{pL}{30})^2=0\quad\Rightarrow\quad\frac{pL^2}{EI}\in\{12,60\}.$$

The smallest eigenvalue gives the critical loading

$$p_{\rm cr}=12\frac{EI}{L^2}.\quad\leftarrow$$

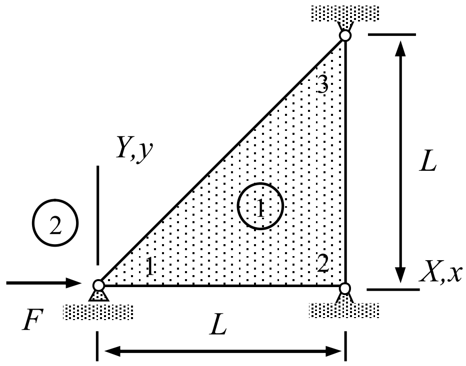
The corresponding eigenvector (mode) is given by

$$\begin{bmatrix}4\frac{EI}{L}-4\frac{p_{\rm cr}L}{30} & 2\frac{EI}{L}+\frac{p_{\rm cr}L}{30}\\ 2\frac{EI}{L}+\frac{p_{\rm cr}L}{30} & 4\frac{EI}{L}-4\frac{p_{\rm cr}L}{30}\end{bmatrix}\begin{Bmatrix}\theta_{Y1}\\ \theta_{Y2}\end{Bmatrix}=\frac{72}{30}\frac{EI}{L}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{Bmatrix}\theta_{Y1}\\ \theta_{Y2}\end{Bmatrix}=0\quad\Rightarrow\quad\begin{Bmatrix}\theta_{Y1}\\ \theta_{Y2}\end{Bmatrix}=\begin{Bmatrix}1\\ -1\end{Bmatrix}\text{ (say).}$$

Shape of the buckled beam follows from approximation when the mode is substituted there (see the formulae collection)

$$w(x)=\begin{Bmatrix}(1-\xi)^2(1+2\xi)\\ L(1-\xi)^2\xi\\ (3-2\xi)\xi^2\\ L\xi^2(\xi-1)\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}0\\ -1\\ 0\\ 1\end{Bmatrix}=-L(\xi-1)\xi \quad \text{where} \quad \xi=\frac{x}{L}.$$

A thin triangular slab (assume plane stress conditions) loaded by a horizontal force is allowed to move horizontally at node 1 and nodes 2 and 3 are fixed. Derive the equilibrium equation for the structure according to the large displacement theory. Material parameters C, ν and thickness t at the initial geometry of the slab are constants.



Solution

Virtual work density of internal force, when modified for large displacement analysis with the same constitutive equation as in the linear case of plane stress, is given by

$$\delta w_{\Omega^\circ}^{\text{int}} = -\begin{Bmatrix} \delta E_{xx} \\ \delta E_{yy} \\ 2\delta E_{xy} \end{Bmatrix}^{\mathrm{T}} \frac{tC}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} E_{xx} \\ E_{yy} \\ 2E_{xy} \end{Bmatrix}, \quad \begin{Bmatrix} E_{xx} \\ E_{yy} \\ 2E_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} + \dfrac{1}{2}(\dfrac{\partial u}{\partial x})^2 + \dfrac{1}{2}(\dfrac{\partial v}{\partial x})^2 \\ \dfrac{\partial v}{\partial y} + \dfrac{1}{2}(\dfrac{\partial u}{\partial y})^2 + \dfrac{1}{2}(\dfrac{\partial v}{\partial y})^2 \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\dfrac{\partial v}{\partial y} \end{Bmatrix}.$$

2p Let us start with the approximations and the corresponding components of the Green-Lagrange strain. Linear shape functions can be deduced from the figure. Only the shape function $N_1 = (1 - x/L)$ of node 1 is needed. Displacement components $v = w = 0$ and

$$u = (1 - \frac{x}{L})u_{X1} \;\Rightarrow\; \frac{\partial u}{\partial x} = -\frac{u_{X1}}{L}, \quad \frac{\partial u}{\partial y} = 0, \quad E_{yy} = E_{xy} = 0 \;\text{ and }\; E_{xx} = -\frac{u_{X1}}{L} + \frac{1}{2}(-\frac{u_{X1}}{L})^2.$$

2p When the strain component expression are substituted there, virtual work density simplifies to

$$\delta w_{\Omega^\circ}^{\text{int}} = -\delta E_{xx}\frac{tC}{1-\nu^2}E_{xx} = -\frac{\delta u_{X1}}{L}(-1 + \frac{u_{X1}}{L})\frac{tC}{1-\nu^2}\frac{u_{X1}}{L}(-1 + \frac{1}{2}\frac{u_{X1}}{L}).$$

Integration over the (initial) domain gives the virtual work expression. As the integrand is constant

$$\delta W^1 = -\frac{L^2}{2}\frac{\delta u_{X1}}{L}(-1 + \frac{u_{X1}}{L})\frac{tC}{1-\nu^2}\frac{u_{X1}}{L}(-1 + \frac{1}{2}\frac{u_{X1}}{L})$$

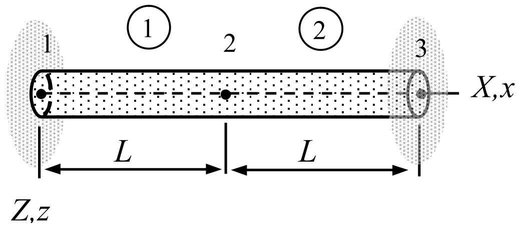
Virtual work expression of the point force follows from the definition of work

$$\delta W^2 = \delta u_{X1}F = \frac{\delta u_{X1}}{L}LF.$$

2p Virtual work expression of the structure is obtained as sum over the element contributions. In terms of the dimensionless displacement $\mathrm{a} = u_{X1}/L$

$$\delta W = -\frac{L^2}{2}\delta\mathrm{a}(-1+\mathrm{a})\frac{tC}{1-\nu^2}\mathrm{a}(-1+\frac{1}{2}\mathrm{a}) + \delta\mathrm{a}LF \;\Rightarrow\; \frac{L}{2}(-1+\mathrm{a})\frac{tC}{1-\nu^2}(-\mathrm{a}+\frac{1}{2}\mathrm{a}^2) - F = 0.$$

Electric current causes heat generation in element 2 of the bar shown. Calculate the temperature ϑ_2 at the center point, if the wall temperature (nodes 1 and 3) is $\vartheta°$. Cross sectional area A and thermal conductivity k are constants. Heat production rate per unit length vanishes in element 1 and it is constant s in element 2



Solution

In a pure heat conduction problem, density expressions of the bar model are given by

$$\delta p_{\Omega}^{\text{int}} = -\frac{d\delta\vartheta}{dx} kA \frac{d\vartheta}{dx} \text{ and } \delta p_{\Omega}^{\text{ext}} = \delta\vartheta s$$

in which ϑ is the temperature, k the thermal conductivity, and s the rate of heat production (per unit length).

2p For bar 1, the nodal temperatures are $\vartheta_1 = \vartheta°$ and ϑ_2 of which the latter is unknown. With a linear interpolation to temperature (notice that variation of $\vartheta°$ vanishes)

$$\vartheta = \begin{Bmatrix} 1 - x/L \\ x/L \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vartheta° \\ \vartheta_2 \end{Bmatrix} = (1 - \frac{x}{L})\vartheta° + \frac{x}{L}\vartheta_2 \quad \Rightarrow \quad \frac{d\vartheta}{dx} = \frac{\vartheta_2 - \vartheta°}{L},$$

$$\delta\vartheta = \frac{x}{L}\delta\vartheta_2 \quad \Rightarrow \quad \frac{d\delta\vartheta}{dx} = \frac{\delta\vartheta_2}{L}.$$

4p When the approximation is substituted there, density expression $\delta p_{\Omega} = \delta p_{\Omega}^{\text{int}} + \delta p_{\Omega}^{\text{ext}}$ simplifies to

$$\delta p_{\Omega} = -\frac{\delta\vartheta_2}{L} kA \frac{\vartheta_2 - \vartheta°}{L},$$

Virtual work expression is the integral of the density over the element domain

$$\delta P^1 = \int_0^L \delta p_{\Omega} dx = -\delta\vartheta_2 kA \frac{\vartheta_2 - \vartheta°}{L}.$$

The nodal temperatures of bar 2 are ϑ_2 and $\vartheta_3 = \vartheta°$. Linear interpolation gives (variations of the given quantities like $\vartheta°$ vanish)

$$\vartheta = \begin{Bmatrix} 1 - x/L \\ x/L \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vartheta_2 \\ \vartheta° \end{Bmatrix} = (1 - \frac{x}{L})\vartheta_2 + \frac{x}{L}\vartheta° \quad \Rightarrow \quad \frac{d\vartheta}{dx} = \frac{\vartheta° - \vartheta_2}{L},$$

$$\delta\vartheta = (1 - \frac{x}{L})\delta\vartheta_2 \quad \Rightarrow \quad \frac{d\delta\vartheta}{dx} = -\frac{\delta\vartheta_2}{L}.$$

When the approximation is substituted there, density expression $\delta p_{\Omega} = \delta p_{\Omega}^{\text{int}} + \delta p_{\Omega}^{\text{ext}}$ simplifies to

$$\delta p_{\Omega} = -(-\frac{\delta\vartheta_2}{L})kA\frac{\vartheta° - \vartheta_2}{L} + (1 - \frac{x}{L})\delta\vartheta_2 s .$$

Element contribution to the variational expressions is the integral of density over the element domain

$$\delta P^2 = \int_0^L \delta p_{\Omega} dx = -\delta\vartheta_2 kA\frac{\vartheta_2 - \vartheta°}{L} + \delta\vartheta_2 \frac{L}{2} s .$$

Variational expression is sum of the element contributions

$$\delta P = \delta P^1 + \delta P^2 = -\delta\vartheta_2[2\frac{kA}{L}(\vartheta_2 - \vartheta°) - \frac{1}{2}Ls] .$$

Variation principle $\delta P = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

$$2\frac{kA}{L}(\vartheta_2 - \vartheta°) - \frac{1}{2}Ls = 0 \quad \Leftrightarrow \quad \vartheta_2 = \vartheta° + \frac{1}{4}\frac{L^2 s}{kA} .$$