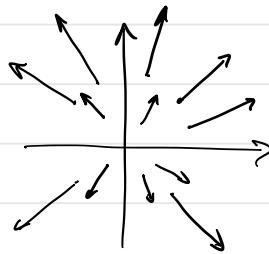


Vector fields

We have studied $f: \mathbb{R}^n \rightarrow \mathbb{R}$. It is also necessary to study $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ (vector-valued functions). These maps are called vector fields if $n=m$.

$$\text{Ex } F(x,y) = x\vec{e}_1 + y\vec{e}_2 = (x,y) = (F_1(x,y), F_2(x,y))$$



Notation and terminology

$$F: \mathbb{R}^n \rightarrow \mathbb{R}^n$$

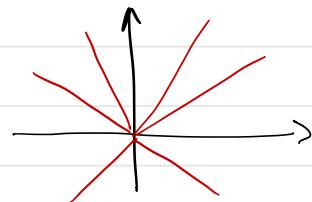
$$F(x_1, \dots, x_n) = (F_1(x_1, \dots, x_n), F_2(x_1, \dots, x_n), \dots, F_n(x_1, \dots, x_n)) \\ = F_1(\vec{x})\vec{e}_1 + \dots + F_n(\vec{x})\vec{e}_n$$

C^k -vector field if $F_i \in C^k$ for $i=1, \dots, n$.
smooth / C^∞ -vector field if F_i are.

Integral curves / Field lines / Trajectories

An integral curve for a vector field is a curve to which the vector field is tangent at all points on the curve.

Ex $F(x,y) = (x,y)$



Integral curves
= half-rays.

What is a curve ?

$$r: \mathbb{R} \rightarrow \mathbb{R}^n$$

Can we easily find tangent vectors for r ?

$$\frac{dr}{dt} = \dot{r}(t) = \left(\frac{dx_1}{dt}, \dots, \frac{dx_n}{dt} \right)$$

For an integral curve we have

$$\dot{r}(t) = J(t) F(r(t))$$

(16)

When $n=3$ (or $n=2$) we find

$$\frac{dx}{dt} = \lambda(t) F_1(x, y, z), \quad \frac{dy}{dt} = \lambda(t) F_2(x, y, z)$$

and $\frac{dz}{dt} = \lambda(t) F_3(x, y, z)$

$$\Rightarrow \lambda(t) dt = \frac{dx}{F_1(x, y, z)} = \frac{dy}{F_2(x, y, z)} = \frac{dz}{F_3(x, y, z)}$$

If we can multiply these equations by a function so we get

$$P(x) dx = Q(y) dy = R(z) dz$$

then we can integrate to find the integral curves.

Ex $F(x, y) = (x, y)$. Integral curves?

$$\frac{dx}{x} = \frac{dy}{y}$$

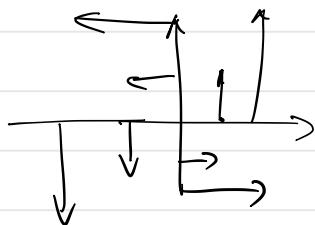
$$\ln|x| = \ln|y| + C$$

$$|y| = A|x| \Rightarrow y = Ax, x > 0$$

or $y = Ax, x < 0$

We can also check that $x=0$ and $y=0$ works

$$\text{Ex } F(x,y) = (-y, x)$$



$$\frac{dx}{-y} = \frac{dy}{x} \Rightarrow$$

$$\Rightarrow x dx = -y dy$$

$$\Rightarrow \frac{x^2}{2} = -\frac{y^2}{2} + \frac{C}{2}$$

$$\Rightarrow x^2 + y^2 = C$$

The integral curves are circles.

Conservative fields

Given a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ it's gradient

$\nabla f = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$ is a vector field. When is a given vector field the gradient of a function?

When the vector field is the gradient of a function it is called conservative. The function(s) are called the potential of the vector field. It is easy to find a necessary condition for a planar vector field to be conservative.

Assume that $\nabla \phi = F$

$$\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right) = (F_1, F_2)$$

$$\begin{aligned} \frac{\partial F_1}{\partial y} &= \frac{\partial^2 \phi}{\partial y \partial x} & \frac{\partial F_2}{\partial x} &= \frac{\partial^2 \phi}{\partial x \partial y} \\ \implies \frac{\partial F_1}{\partial y} &= \frac{\partial F_2}{\partial x} \end{aligned}$$

So if this doesn't hold then F is not conservative.

Ex $F(x,y) = (x,y)$

Is the vector field conservative?

$$\frac{\partial F_1}{\partial y} = 0 \quad \frac{\partial F_2}{\partial x} = 0 \quad \text{So the field can be conservative.}$$

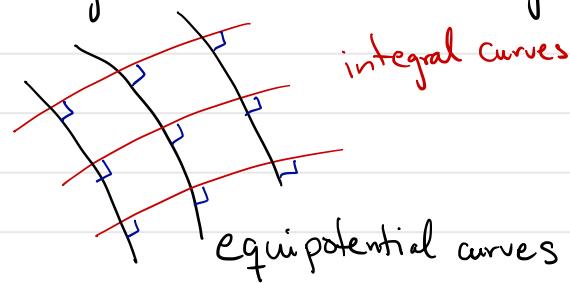
We try to construct the potential ϕ

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= x \implies \phi(x,y) = \frac{x^2}{2} + C(y) \\ \implies \frac{\partial \phi}{\partial y} &= C'(y) = y \implies C(y) = \frac{y^2}{2} + D \\ \phi(x,y) &= \frac{x^2}{2} + \frac{y^2}{2} + D \end{aligned}$$

$F(x,y)$ is conservative since $\nabla \phi = F$.

The sets $\phi(\vec{x}) = C$ are called equipotential curves (in \mathbb{R}^2) / surfaces (in \mathbb{R}^3) / hypersurfaces (in \mathbb{R}^n)
 $n \geq 4$

Fact Equipotential curves are orthogonal trajectories for the integral curves



Ex $F(x,y) = (y/x) = y\vec{e}_1 + x\vec{e}_2$

First integral curves

$$\frac{dx}{y} = \frac{dy}{x}$$

$$\Rightarrow \int x dx = \int y dy \Rightarrow \frac{x^2}{2} = \frac{y^2}{2} + C$$

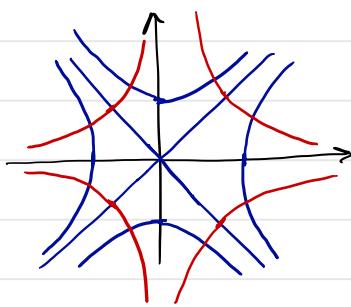
$$\Rightarrow \text{Integral curves } x^2 - y^2 = A \quad \text{hyperbolae}$$

Now equipotential curves

$$\frac{\partial \phi}{\partial x} = y \Rightarrow \phi(x,y) = xy + \alpha(y)$$

$$\frac{\partial \phi}{\partial y} = x + \alpha'(y) \Rightarrow x'(y) = 0$$

$$\Rightarrow \phi(x,y) = xy + C$$



Integral curves

Equipotential curves
(also coordinate axes)

$$\text{Ex} \quad F(x,y) = (x,y)$$

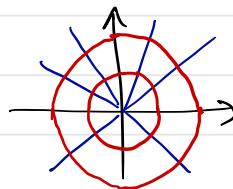
We already know that the integral curves for this vector field are half-rays starting at the origin.
Let's find the equipotential curves.

$$\frac{\partial \phi}{\partial x} = x \implies \phi(x,y) = \frac{x^2}{2} + \alpha(y)$$

$$\frac{\partial \phi}{\partial y} = \alpha'(y) = y \implies \alpha(y) = \frac{y^2}{2} + A$$

$$\phi(x,y) = \frac{x^2}{2} + \frac{y^2}{2} + A$$

$$\implies x^2 + y^2 = C \quad \text{Circles around the origin.}$$

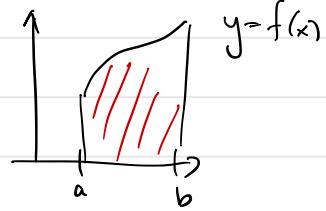


Integral curves

Equipotential curves

Line integrals

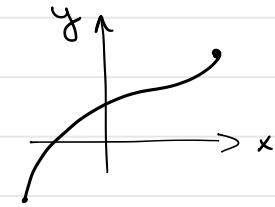
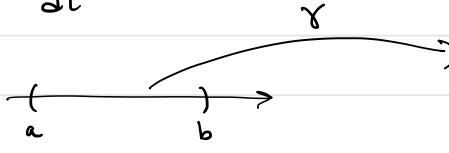
When we first meet $\int_a^b f(x) dx$ we think of this as an area calculation



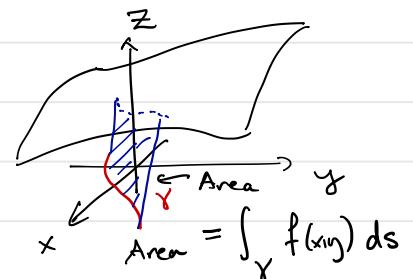
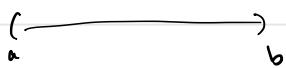
However the applications of integrals goes beyond area calculations. The same is true for line integrals but we begin by think about these integrals as area calculations. We talk a little about curves first.

$\gamma: (a, b) \rightarrow \mathbb{R}^n$ is a C^1 -curve if

$\frac{d\gamma}{dt}$ is continuous and $\frac{d\gamma}{dt} \neq \vec{0}$ for all $t \in (a, b)$



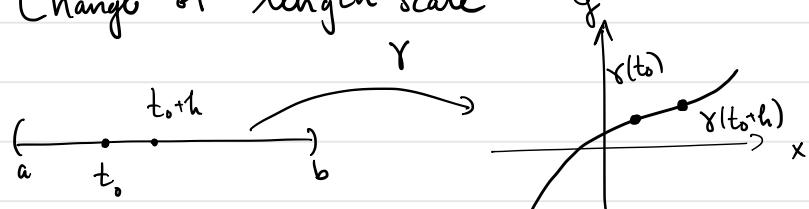
Assume $z = f(x, y)$



$$\text{Area} = \int_Y f(x, y) ds$$

We use the parametrization to reduce this calculation to an ordinary integral.

Change of length scale



$$|\gamma(t_0+h) - \gamma(t_0)| \approx \left| \frac{d\gamma}{dt}(t_0) \right| h$$

$$\int_{\gamma} f(x,y) ds = \int_a^b f(\gamma(t)) \left| \frac{d\gamma}{dt} \right| dt$$

Ex Calculate $\int (x^2+y^2) ds$ where γ is the straight line from $(0,0)$ to $(1,3)$.

Solution: First we parametrize γ .

$$\gamma(t) = (t, 3t) \quad 0 \leq t \leq 1$$

$$\frac{d\gamma}{dt} = (1, 3)$$

$$\left| \frac{d\gamma}{dt} \right| = \sqrt{1^2 + 3^2} = \sqrt{10}$$

$$\begin{aligned} \int_{\gamma} x^2 + y^2 ds &= \int_0^1 (t^2 + (3t)^2) \sqrt{10} dt = 10\sqrt{10} \int_0^1 t^2 dt \\ &= \frac{10\sqrt{10}}{3} \end{aligned}$$

Ex Calculate the length of a circle with radius $r (>0)$.

$$\gamma(t) = (r \cos t, r \sin t) \quad 0 \leq t \leq 2\pi$$

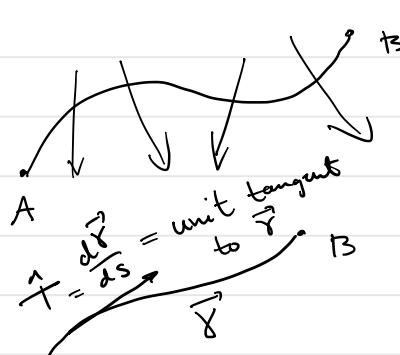
$$\begin{aligned} \text{length of } \gamma &= \int_{\gamma} 1 \, ds = \int_0^{2\pi} 1 \cdot \left| \frac{d\gamma}{dt} \right| dt = \\ &= \Gamma \left| \frac{d\gamma}{dt} \right| = r \int_0^{2\pi} r \, dt = 2\pi r \end{aligned}$$

One thing to notice. The chain rule gives that different parametrizations of γ gives the same value for the integrals. Therefore $\int_{\gamma} f(x,y) \, ds$ is independent of the parametrization. Another thing to note: going from a to b or from b to a gives same value ($\gamma(t)$ or $\gamma(-t)$ have the same $\left| \frac{d\gamma}{dt} \right|$)

Line integrals of vector fields

In physics work done is "force times distance"





\vec{F} varies in space

How to calculate work?

$$W = \int_{\gamma} \vec{F} \cdot \hat{T} \, ds = \int_{\gamma} \vec{F} \cdot d\vec{r} = \int_{\gamma} F_1 dx + F_2 dy$$

Notice $\int_{\gamma} \vec{F} \cdot d\vec{r} = - \int_{-\gamma} \vec{F} \cdot d\vec{r}$

Ex Let $\vec{F}(x,y) = (y^2, 2xy)$. Calculate

$$\int_{\gamma} \vec{F} \cdot d\vec{r} \text{ when } \gamma \text{ is}$$

- a straight line from $(0,0)$ to $(1,1)$.
- the curve $y=x^2$ from $(0,0)$ to $(1,1)$.

Solutions: a) $\gamma(t) = (t, t) \quad 0 \leq t \leq 1$

$$\frac{d\gamma}{dt} = (1,1) \quad \hat{T} = \frac{1}{\sqrt{2}} (1,1)$$

$$\vec{F} \cdot \hat{T} = (y^2, 2xy) \cdot \frac{1}{\sqrt{2}} (1,1) = \frac{1}{\sqrt{2}} (y^2 + 2xy)$$

$$\int_0^1 \frac{1}{\sqrt{2}} \cdot 3t^2 \cdot \sqrt{2} dt = \left[t^3 \right]_0^1 = 1$$

Notice that

$$\text{"} \hat{T} ds = \hat{T} \left| \frac{d\vec{\gamma}}{dt} \right| dt = \frac{d\vec{\gamma}}{dt} dt \text{"}$$

b) $\gamma(t) = (t, t^2)$ $\frac{d\vec{\gamma}}{dt} = (1, 2t)$ $F(t) = (t^4, 2t^3)$

$$\begin{aligned} \int_{\gamma} F \cdot d\vec{r} &= \int_0^1 (t^4, 2t^3) \cdot (1, 2t) dt = \\ &= \int_0^1 t^4 + 4t^4 dt = \int_0^1 5t^4 dt = [t^5]_0^1 = 1. \end{aligned}$$

Alternative: $\int_{\gamma} y^2 dx + 2xy dy = \int_0^1 (t^2)^2 \frac{dx}{dt} + 2t \cdot t^2 \frac{dy}{dt} dt$
 $= \int_0^1 t^4 \cdot 1 + 2t^3 \cdot 2t dt = \int_0^1 5t^4 dt = 1$

We will now investigate when $\int_{\gamma} F \cdot d\vec{r}$ is dependent only on the end-points of γ .

First let γ be a closed curve

$\oint_{\gamma} F \cdot d\vec{r}$ = circulation of F around γ .

\oint_{γ} indicates that the curve is closed.

Theorem If D is an open connected domain and F is a smooth vector field then the following are equivalent

1) F is conservative in D .

2) $\oint_{\gamma} F \cdot d\vec{r} = 0$ for every piecewise smooth closed curve γ in D .

3) $\int_{\gamma} F \cdot d\vec{r}$ depends only on the end-points of γ .

Proof: 1) \Rightarrow 2) (in \mathbb{R}^2)

$$\nabla \phi = F$$

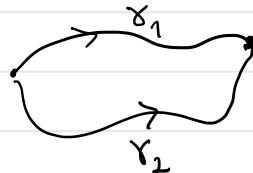
$$F \cdot d\vec{r} = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt} \right) dt = \frac{d\phi(\gamma(t))}{dt} dt$$

$$\int_{\gamma} F \cdot d\vec{r} = \int_a^b \frac{d\phi(\gamma(t))}{dt} dt = \phi(\gamma(b)) - \phi(\gamma(a))$$

$$\gamma \text{ closed} \Rightarrow \gamma(b) = \gamma(a)$$

$$\Rightarrow \oint_{\gamma} F \cdot d\vec{r} = 0$$

2) \Rightarrow 3)



$$\int_{\gamma_1} F \cdot d\vec{r} \neq \int_{\gamma_2} F \cdot d\vec{r}$$

$$\Rightarrow \int_{\gamma_1 - \gamma_2} F \cdot d\vec{r} \neq 0 \quad \left(\neg 3) \Rightarrow \neg 2) \right)$$

3) \Rightarrow 1) Fix $(x_0, y_0) \in D$. For $(x, y) \in D$ choose γ from (x_0, y_0) to (x, y) .

$$\text{Define } \phi(x, y) = \int_{\gamma} \mathbf{F} \cdot d\vec{r}$$

$\phi(x, y)$ is well-defined (independent of γ)

$$\phi(x+h, y) - \phi(x, y) = \int_x^{x+h} F_1(\xi, y) d\xi$$

$$\frac{\partial \phi}{\partial x} = \lim_{h \rightarrow 0} \frac{\phi(x+h, y) - \phi(x, y)}{h} = F_1(x, y)$$

In the same way we get $\frac{\partial \phi}{\partial y} = F_2(x, y)$ \otimes

This shows why conservative vector fields are so pleasant to work with.