

$$\text{Ex } \mathbf{F}(x, y, z) = (xy, y^2 - z^2, yz)$$

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(y^2 - z^2) + \frac{\partial}{\partial z}(yz) =$$

$$= y + 2y + y = 4y$$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & y^2 - z^2 & yz \end{vmatrix} =$$

$$= \left( \frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(y^2 - z^2), \frac{\partial}{\partial z}(xy) - \frac{\partial}{\partial x}(yz), \frac{\partial}{\partial x}(y^2 - z^2) - \frac{\partial}{\partial y}(xy) \right) =$$

$$= (z + 2z, 0, -x) = (3z, 0, -x)$$

### Interpretation of the divergence

Let  $\vec{F}$  be a smooth vector field and  $\vec{N}$  be the unit outward normal vector field of  $S_\epsilon$ , the sphere with radius  $\epsilon$  centered at P. Then

$$\operatorname{div} \vec{F}(P) = \lim_{\epsilon \rightarrow 0^+} \left( \frac{3}{4\pi\epsilon^3} \oint_{S_\epsilon} \vec{F} \cdot \vec{N} dS \right)$$

$\frac{1}{\text{volume of } S_\epsilon}$

"Proof": Let  $\vec{P} = \vec{O}$ . Then  $\vec{N} = \frac{1}{\varepsilon} (x, y, z)$

Taylor expansion of  $(F_1, F_2, F_3)$   $\vec{F}_{x_0}$

$$\vec{F}(x, y, z) = \underbrace{\vec{F}_0(0, 0, 0)}_{\vec{F}_{y_0}} + \underbrace{\left(\frac{\partial F_1}{\partial x}, \frac{\partial F_2}{\partial x}, \frac{\partial F_3}{\partial x}\right)}_{\vec{F}_{x_0}} x +$$

$$+ \underbrace{\left(\frac{\partial F_1}{\partial y}, \frac{\partial F_2}{\partial y}, \frac{\partial F_3}{\partial y}\right)}_{\vec{F}_{y_0}} y + \underbrace{\left(\frac{\partial F_1}{\partial z}, \frac{\partial F_2}{\partial z}, \frac{\partial F_3}{\partial z}\right)}_{\vec{F}_{z_0}} z +$$

+ higher order terms

$$\begin{aligned} \vec{F} \cdot \vec{N} &= \frac{1}{\varepsilon} (\vec{F}_0 \cdot (x, y, z) + \vec{F}_{x_0} x^2 \cdot \vec{e}_1 + \vec{F}_{x_0} xy \cdot \vec{e}_2 \\ &\quad + \vec{F}_{x_0} xz \cdot \vec{e}_3 + \vec{F}_{y_0} yx \cdot \vec{e}_1 + \vec{F}_{y_0} y^2 \cdot \vec{e}_2 + \\ &\quad + \vec{F}_{y_0} yz \cdot \vec{e}_3 + \vec{F}_{z_0} xz \cdot \vec{e}_1 + \vec{F}_{z_0} yz \cdot \vec{e}_2 + \\ &\quad + \vec{F}_{z_0} z^2 \cdot \vec{e}_3 + \dots) \end{aligned}$$

Now integrate termwise

$$\oint_{S_\varepsilon} x dS = \oint_{S_\varepsilon} y dS = \oint_{S_\varepsilon} z dS = 0$$

$$\oint_{S_\epsilon} xy \, dS = \oint_{S_\epsilon} xz \, dS = \oint_{S_\epsilon} yz \, dS = 0$$

$$\begin{aligned} \oint_{S_\epsilon} x^2 \, dS &= \oint_{S_\epsilon} y^2 \, dS = \oint_{S_\epsilon} z^2 \, dS = \\ &= \frac{1}{3} \oint_{S_\epsilon} x^2 + y^2 + z^2 \, dS = \frac{1}{3} \epsilon^2 \cdot 4\pi \epsilon^2 = \frac{4\pi}{3} \epsilon^4 \end{aligned}$$

Higher order terms involve  $\epsilon^k$ ,  $k \geq 5$

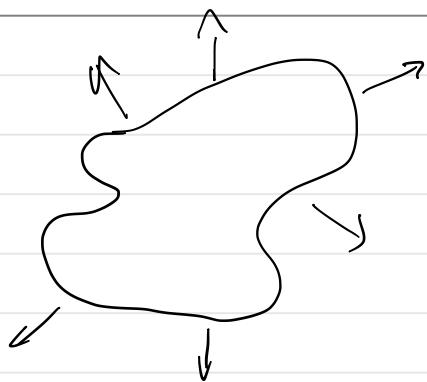
$$S_0 \frac{3}{4\pi \epsilon^3} \oint_{S_\epsilon} \vec{F} \cdot \vec{N} \, dS =$$

$$= \frac{3}{4\pi \epsilon^3} \cdot \frac{1}{\epsilon} \left( \oint_{S_\epsilon} (F_{x0} \cdot \vec{e}_1) x^2 + (F_{y0} \cdot \vec{e}_2) y^2 + (F_{z0} \cdot \vec{e}_3) z^2 \, dS + O(\epsilon^5) \right)$$

$$\text{and } \lim_{\epsilon \rightarrow 0^+} \frac{3}{4\pi \epsilon^3} \oint_{S_\epsilon} \vec{F} \cdot \vec{N} \, dS = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \operatorname{div} \vec{F}$$

⊗

That is  $\operatorname{div} \vec{F}$  measures "how much fluid is created or destroyed in each point".



Flow = "sum" of  $\operatorname{div} \mathbf{F}$   
in the interior.

### Interpretation of $\operatorname{Curl}$

Ex Consider the vector field

$$\vec{v} = (-\omega y, \omega x, 0)$$

Calculate the circulation counter clockwise around the circle  $C_\epsilon$  centered at  $(x_0, y_0)$  with radius  $\epsilon$  in the  $xy$ -plane

$$C_\epsilon(t) = (x_0 + \epsilon \cos t, y_0 + \epsilon \sin t, 0) \quad 0 \leq t \leq 2\pi$$

$$\begin{aligned} \oint_{C_\epsilon} \vec{v} \cdot d\vec{r} &= \int_0^{2\pi} -\omega(y_0 + \epsilon \sin t)(-\epsilon \cos t) + \\ &\quad + \omega(x_0 + \epsilon \sin t)(\epsilon \cos t) dt = \\ &= \int_0^{2\pi} \omega \epsilon (y_0 \sin t + x_0 \cos t) + \omega \epsilon^2 dt = 2\pi \omega \epsilon^2 \end{aligned}$$

Also  $\text{Curl } \vec{v} = \nabla \times \vec{v} = \left( \frac{\partial}{\partial x} (\omega_x) - \frac{\partial}{\partial y} (-\omega_y) \right) \vec{e}_3$

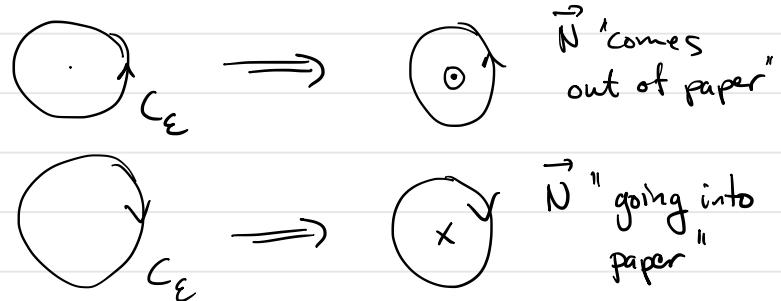
$$= 2\omega \vec{e}_3$$

Note that  $\oint_{C_\epsilon} \vec{v} \cdot d\vec{r} = \text{Area}(C_\epsilon) (\text{Curl } \vec{v} \cdot \vec{e}_3)$

Theorem: If  $\vec{F}$  is a smooth vector field in  $\mathbb{R}^3$  and  $C_\epsilon$  is a circle of radius  $\epsilon$  centered at  $P$  and bounding a disk  $D_\epsilon$  with unit normal  $\vec{N}$  (orientation inherited from  $C_\epsilon$ ) then

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{\pi \epsilon^2} \oint_{C_\epsilon} \vec{F} \cdot d\vec{r} = \vec{N} \cdot \text{Curl } \vec{F}(P)$$

Orientation inherited from  $C_\epsilon$ ?



Identities involving  $\text{div}$ ,  $\text{grad}$  and  $\text{curl}$

$\phi, \psi$  functions,  $\vec{F}, \vec{G}$  vector fields

$$a) \nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$$

$$b) \text{div}(\phi\vec{F}) = \nabla \cdot (\phi\vec{F}) = (\nabla\phi) \cdot \vec{F} + \phi(\nabla \cdot \vec{F})$$

$$c) \nabla \times (\phi\vec{F}) = (\nabla\phi) \times \vec{F} + \phi(\nabla \times \vec{F})$$

$$d) \nabla \cdot (\vec{F} \times \vec{G}) = (\nabla \times \vec{F}) \cdot \vec{G} - \vec{F} \cdot (\nabla \times \vec{G})$$

$$e) \nabla \times (\vec{F} \times \vec{G}) = (\nabla \cdot \vec{G})\vec{F} + (\vec{G} \cdot \nabla)\vec{F} - (\nabla \cdot \vec{F})\vec{G} - (\vec{F} \cdot \nabla)\vec{G}$$

$$f) \nabla(\vec{F} \cdot \vec{G}) = \vec{F} \times (\nabla \times \vec{G}) + \vec{G} \times (\nabla \times \vec{F}) + (\vec{F} \cdot \nabla)\vec{G} + (\vec{G} \cdot \nabla)\vec{F}$$

$$g) \nabla \cdot (\nabla \times \vec{F}) = 0 \quad \text{div curl} = 0$$

$$h) \nabla \times (\nabla \phi) = 0 \quad \text{curl grad} = 0$$

$$i) \nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - (\nabla \cdot \nabla)\vec{F}$$

$\nabla^2 = \Delta = \text{Laplace operator}$

$$\text{curl curl} = \text{grad div} - \text{Laplace}$$

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}$$

A function satisfying  $\Delta f = 0$  is called harmonic.

Proof g) (Do the rest by yourself)

$$\nabla \times \vec{F} = \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$\begin{aligned} \nabla \cdot (\nabla \times \vec{F}) &= \frac{\partial}{\partial x} \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) + \frac{\partial}{\partial y} \left( \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) + \\ &+ \frac{\partial}{\partial z} \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) = 0 \end{aligned}$$



Green's Theorem in the plane

This can be seen as a higher-dimensional version of the Fundamental Theorem of Calculus.

- Classical version  $\int_a^b \frac{d}{dx} f(x) dx = f(b) - f(a).$

- Version for line integrals in conservative fields



$$\int_{\gamma} \nabla \phi \cdot d\vec{r} = \phi(B) - \phi(A).$$

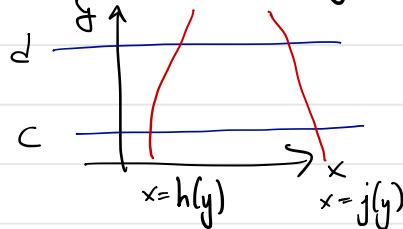
### Green's Theorem

Let  $R$  be a regular, closed region in the plane whose boundary  $\gamma$  consists of one or more piecewise smooth curves. Also assume that  $\gamma$  is simple and positively oriented with respect to  $R$ . If  $\mathbf{F}(x,y) = (F_1, F_2)$  is a smooth vector field on  $R$ , then

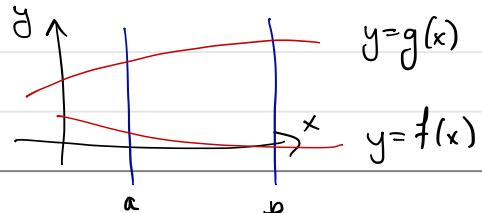
$$\oint_{\gamma} F_1 dx + F_2 dy = \iint_R \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$$

- Regular  $\gamma$ : You can cut  $R$  into pieces that are  $x$ -simple and  $y$ -simple.

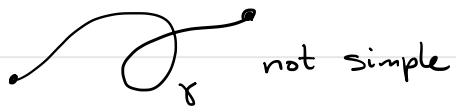
-  $x$ -simple?



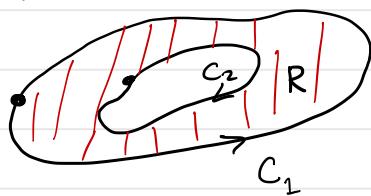
-  $y$ -simple?



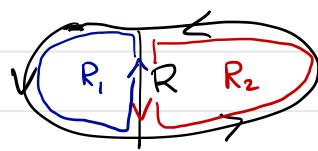
•  $\gamma$  simple?



• Positively oriented?



Proof:



If the theorem holds for  $R_1$  &  $R_2$  it holds for  $R$ .

Since  $R$  is regular we get the theorem if we can show it for regions being both  $x$ -simple and  $y$ -simple.

We assume that

$$R = \{(x, y) \in \mathbb{R}^2; a \leq x \leq b, f(x) \leq y \leq g(x)\}$$

$$= \{(x, y) \in \mathbb{R}^2; c \leq y \leq d, h(y) \leq x \leq j(y)\}$$

$$\begin{aligned} \iint_R -\frac{\partial F_1}{\partial y} dx dy &= - \int_a^b \left( \int_{f(x)}^{g(x)} \frac{\partial F_1}{\partial y} dy \right) dx = \\ &= \int_a^b -F_1(x, g(x)) + F_1(x, f(x)) dx \end{aligned}$$

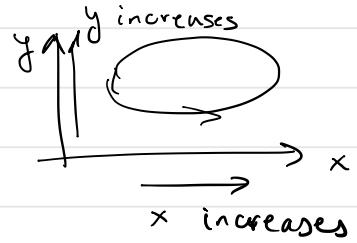
Now,

$$\oint_C F_1(x,y) dx = \int_a^b F_1(x, f(x)) - F_1(x, g(x)) dx$$

$$\text{So } \oint_C F_1(x,y) dx = \iint_R -\frac{\partial F_1}{\partial y} dxdy$$

$$\text{Also } \oint_C F_2(x,y) dy = \iint_R \frac{\partial F_2}{\partial x} dxdy$$

Why different signs?



$$\implies \oint_C F_1 dx + F_2 dy = \iint_R \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA. \quad \otimes$$

Ex Area bounded by a simple closed curve  $\gamma$ .  
Try to find  $(F_1, F_2)$  such that

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1.$$

$$\begin{aligned} \text{Area} &= \iint_R 1 dA = \oint_{\gamma} x dy = \oint_{\gamma} -y dx \\ &= \frac{1}{2} \oint_{\gamma} x dy - y dx \end{aligned}$$

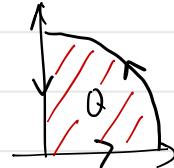
Area of a disk with radius R.

$$\gamma(t) = (R \cos t, R \sin t)$$

$$\begin{aligned} \text{Area} &= \oint_{\gamma} x \, dy = \int_0^{2\pi} R \cos t \cdot R \cos t \, dt = \\ &= R^2 \int_0^{2\pi} \cos^2 t \, dt = R^2 \int_0^{2\pi} \frac{1 + \cos 2t}{2} \, dt = \pi R^2 \end{aligned}$$

Ex Evaluate  $I = \oint_{\gamma} (x-y^3) dx + (y^3+x^3) dy$

where  $\gamma$  is the positively oriented boundary of the quarter disk  $Q : 0 \leq x^2+y^2 \leq a^2, x \geq 0, y \geq 0$ .



$$\vec{F} = (x-y^3, y^3+x^3)$$

$$\begin{aligned} I &= \iint_Q \left( \frac{\partial}{\partial x} (y^3+x^3) - \frac{\partial}{\partial y} (x-y^3) \right) dA = \iint_Q 3x^2+3y^2 \, dA \\ &= \int_0^{\pi/2} \int_0^a 3r^2 \cdot r \, dr \, d\theta = \frac{3\pi}{2} \int_0^a r^3 \, dr = \frac{3\pi a^4}{8}. \end{aligned}$$

Ex: Let  $C$  be a positively oriented simple bounded curve in the plane bounding a regular region  $R$  and not passing through the origin. Show that

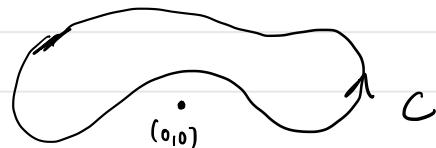
$$\oint_C \frac{-y \, dx + x \, dy}{x^2 + y^2} = \begin{cases} 0 & \text{if } 0 \notin R \\ 2\pi & \text{if } 0 \in R \end{cases}$$

Solution: If  $(x, y) \neq (0, 0)$

$$\frac{\partial}{\partial x} \left( \frac{x}{x^2 + y^2} \right) - \frac{\partial}{\partial y} \left( \frac{-y}{x^2 + y^2} \right) =$$

$$= \frac{1}{x^2 + y^2} - \frac{2x^2}{x^2 + y^2} - \frac{2y^2}{x^2 + y^2} + \frac{1}{x^2 + y^2} = 0$$

Green's Theorem  $\Rightarrow \oint_C \frac{-y \, dx + x \, dy}{x^2 + y^2} = 0$  if  $0 \notin R$



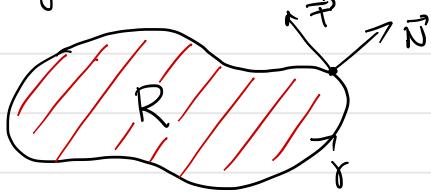
Now assume that the origin is inside  $R$



Put a small circle  $C_\epsilon$  around the origin

$$\oint_C \frac{-y \, dx + x \, dy}{x^2 + y^2} = - \oint_{C_\epsilon} \frac{-y \, dx + x \, dy}{x^2 + y^2} \xrightarrow{\text{Exercise}} = -(-2\pi) = 2\pi$$

## Divergence Theorem in the plane



$\vec{T}$  = tangential unit vector field

$\vec{N}$  = unit normal outward (from R) vector field

$$\text{Note that } \vec{T} = (T_1, T_2) \Rightarrow \vec{N} = (T_2, -T_1)$$

$$\text{Given } \vec{F} = (F_1, F_2) \text{ define } \vec{G} = (-F_2, F_1)$$

$$\text{We have } \vec{G} \cdot \vec{T} = -F_2 \cdot T_1 + F_1 \cdot T_2 = \vec{F} \cdot \vec{N}$$

$$\begin{aligned} \text{Now, } \iint_R \operatorname{div} \vec{F} dA &= \iint_R \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} dA = \\ &= \iint_R \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} dA = \oint_{\gamma} \vec{G} \cdot d\vec{r} = \\ &= \oint_{\gamma} \vec{G} \cdot \vec{T} ds = \underbrace{\oint_{\gamma} \vec{F} \cdot \vec{N} ds}_{\text{Flow out of } R}. \end{aligned}$$

↑  
Green's Thm

Gauss's Theorem  
(Divergence Theorem in 3-space)