ARCH models

ARCH= Autoregressive conditionally heteroscedastic

Allows $Var(u_t)$ depend on t: $Var(u_t) = \sigma_t^2$

$E(u_t)=0$

Ex. 1. AR(1) $y_t = \mu + \phi_1 y_{t-1} + u_t$

↗ level model

History Ω_{t-1} up to time $t-1$

Conditionally variance $\boxed{\sigma_t^2} = Var(y_t | \overset{\Omega_{t-1}}{y_{t-1}})$

$= Var_{t-1}(y_t)$ ← history

$= Var_{t-1}(u_t) = Var_{t-1}(y_t - \mu - \phi_1 y_{t-1})$

$= \boxed{E_{t-1}(u_t^2)}$

Model $u_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + v_t$ ↙ WN

$$\boxed{\sigma_t^2} = E_{t-1}(u_t^2) = \boxed{\alpha_0 + \alpha_1 u_{t-1}^2}$$

ARCH (1)

similar idea as in AR(1)

Note: If $\alpha_1 = 0$, then $\sigma_t^2 = \alpha_0 = \sigma^2$ constant

Note: Sometimes simplify notation: $h_t = \sigma_t^2$

Non-conditional (long-term) variance (equilibrium)

$$E\sigma_t^2 = E(\alpha_0 + \alpha_1 u_{t-1}^2)$$

$$\sigma^2 = \alpha_0 + \alpha_1 \underbrace{E(u_{t-1}^2)}_{\sigma^2}$$

Solving $\sigma^2 = \alpha_0 + \alpha_1 \sigma^2$

$$(1-\alpha_1)\sigma^2 = \alpha_0$$

$$\boxed{\sigma^2 = \frac{\alpha_0}{1-\alpha_1}}$$

Must be:

$$\begin{cases} \alpha_0 > 0 \\ 0 < \alpha_1 < 1 \end{cases}$$

Note: Two models

1) level model for y_t
2) Conditional variance model for σ_t^2

GARCH models

Generalised

WN

$$u_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta \sigma_{t-1}^2 + v_t$$

$$\boxed{\sigma_t^2} = \mathrm{Var}_{t-1}(u_t) = E_{t-1}(u_t^2) = \boxed{\alpha_0 + \alpha_1 u_{t-1}^2 + \beta \sigma_{t-1}^2}$$

GARCH(1,1)

Repeatingly substituting $\sigma_{t-1}^2, \sigma_{t-2}^2, \dots$

$$\sigma_t^2 = \alpha_0(1+\beta+\beta^2+\dots) + \alpha_1 u_{t-1}^2 + \alpha_1 \beta u_{t-2}^2 + \dots$$
$$+ \alpha_1 \beta^2 u_{t-3}^2 + \dots)$$

$$= \frac{\alpha_0}{1-\beta} + \alpha_1 \sum_{k=0}^{\infty} \beta^k u_{t-k}^2 \quad \text{ARCH}(\infty)$$

Assuming $|\beta| < 1$

Non-conditional variance (equilibrium)

$$\boxed{\sigma^2 = \frac{\alpha_0}{1-(\alpha_1+\beta)}}$$

Must have: $\alpha_1 + \beta < 1$, $\alpha_0 > 0$