CIV-E4080, Material Modelling in Civil Engineering L

Written examination, total obligatory points = 3×5, today: 27/5/2019

Read me:

- EXTRA exercises are not obligatory and will not affect your points if they are not answered. They just may give you the missing one point.
- Exercises # 2 & # 3: only one of them is obligatory to solve

1. Exercise - Elasticity [5 points]

Isotropy and plane stress: Assume an isotropic linear elastic material and the generalized 3D Hooke's law $\sigma = \mathbf{D}\epsilon$ using the elasticity constants E and ν. Shear modulus $G = E/[2(1+\nu)]$. Inverting the Hooke's law for strain components one obtains (Voigt's notation) the flexibility relation $\epsilon = \mathbf{D}^{-1}\sigma$ in component form as

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{bmatrix} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{bmatrix} \tag{1}$$

where the axes $x \to 1$, $y \to 2$ and $z \to 3$.

Question: Consider *plane stress* case in plane $1-2$ (or $x-y$) in which the non-zero stress components act ($\Longrightarrow$ 3 is the direction of zero stress).

1. what stress components are null?
2. account for that in the flexibility Equation (1) and derive the reduced form (3×3 matrix) of the flexibility relation (1).
3. which strain components are null?
4. determine ϵ_{33} in terms of ϵ_{11} and ϵ_{22}

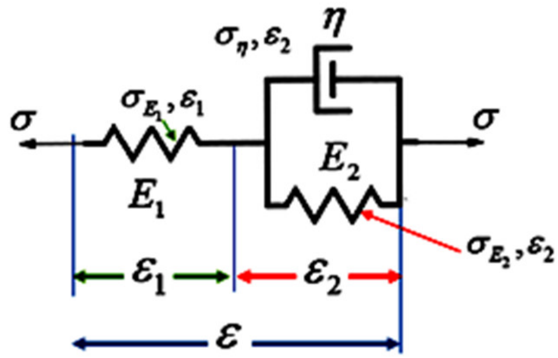
EXTRA [1 point]: Explain in few lines and with an illustration (drawing) what is meant by *orthotropy* in elasticity? You can use equations, too.

2. Exercise - Viscoelasticity [5 points]

Consider the *standard linear model* (SLM) (Figure above). Derive the constitutive equation below

$$\dot{\epsilon} + \frac{E_2}{\eta}\epsilon = \frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta}[1 + \frac{E_2}{E_1}] \tag{2}$$

EXTRA [1 point]: Determine the expression of the relaxation modulus $G(t)$ for the above (SLM) model (Equation 2)



3. Exercise - Viscoelasticity [5 points]

Stress relaxation: In a stress relaxation test, a straight steel cable formed by steel wire strands, is pre-stressed by applying an initial stress of 100 MPa after what the ends of the wire were blocked from any motion. After **three weeks**, a **loss** of 2 MPa is observed in the cable.

Question: Use a simple *Maxwell model* and derive the relaxation function (or modulus). What should be the new initial pre-stress level in order to keep the stress in the cable > 200 MPa over one year? (Assume that the temperature remains constant.)

Hint: The relaxation time is defined as $\tau_R = \eta/E$, where η being the viscosity coefficient and E the elastic modulus, respectively.

EXTRA [1 point]: Explain concisely the two concepts of *creep* and *stress relaxation* tests – use graphical illustrations and equations.

4. Exercise - Engineering Plasticity [5 points]

Yielding criteria: Consider the following stress state in a steel structure at some location being

$$\sigma = \begin{bmatrix} 200 & 110 & \tau_{xz} \\ 110 & 125 & 0 \\ \tau_{xz} & 0 & 125 \end{bmatrix}, \quad \text{units are in MPa.} \tag{3}$$

The material is isotropic and its yield stress in uniaxial test $\sigma_Y = 280$ MPa.

Question: How much the stress component τ_{xz} can be before yield occurs, according to von-Mises criteria?

EXTRA [1 point]: How much the stress component τ_{xz}, in exercise #4, can be before yield occurs, according Tresca yield criteria?

EXTRA [1 point]: What are the three main ingredients of the theory of engineering plasticity? Explain each of them concisely – what they are and what they describe? (Not more than 1/3 of a page)

Formulary: *von-Mises*: $\sigma_e - \sigma_Y = 0$, $\sigma_e \equiv \sqrt{3J_2}$

$$\frac{1}{2}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]-\sigma_Y^2=0, \quad \text{in principle stresses, or} \tag{4}$$

$$\frac{1}{2}\left[(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2\right]+3\left[(\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2)\right]-\sigma_Y^2=0. \tag{5}$$

Tresca's criteria: the student should remember it.

Generalized Hooke's Law – Examples of problems

Linear isotropic material: $\boldsymbol{\sigma} = \lambda\,\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}$

$$\boldsymbol{\varepsilon} = \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1}$$

Constitutive law

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$$

$$\boldsymbol{\varepsilon} = \mathbf{D}^{-1}\boldsymbol{\sigma} \equiv \mathbf{S}\boldsymbol{\sigma}$$

The Navier-Lame equations:

$$(\lambda+\mu)\ \mathrm{grad\ div}\ \mathbf{u} + \mu\nabla^2\mathbf{u} + \mathbf{F} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2},$$

and initial & boundary conditions IBC

$$G\left(\Delta u + \frac{1+\nu}{1-\nu}\frac{\partial e}{\partial x}\right) + f_x = 0,$$

$$G\left(\Delta v + \frac{1+\nu}{1-\nu}\frac{\partial e}{\partial y}\right) + f_y = 0,$$

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y},$$

Component form:

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y), \quad \varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y),$$

$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x), \quad \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x),$$

$$\tau_{xy} = G\gamma_{xy}. \quad \gamma_{xy} = \frac{1}{G}\tau_{xy},$$

Plane stress: $\sigma_{33} = \sigma_{23} = \sigma_{13} = 0$

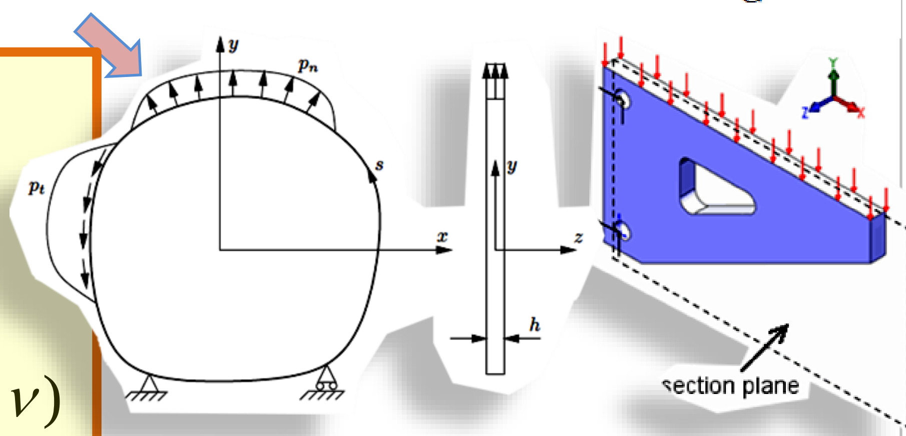
$\boldsymbol{\varepsilon} = \mathbf{D}^{-1}\boldsymbol{\sigma} \equiv \mathbf{S}\boldsymbol{\sigma}$ **reduces to:**

$$\sigma_{ij} = \lambda'\varepsilon_{kk} + 2\mu\varepsilon_{ij}$$

$$\lambda' = \nu E/(1-\nu^2)$$

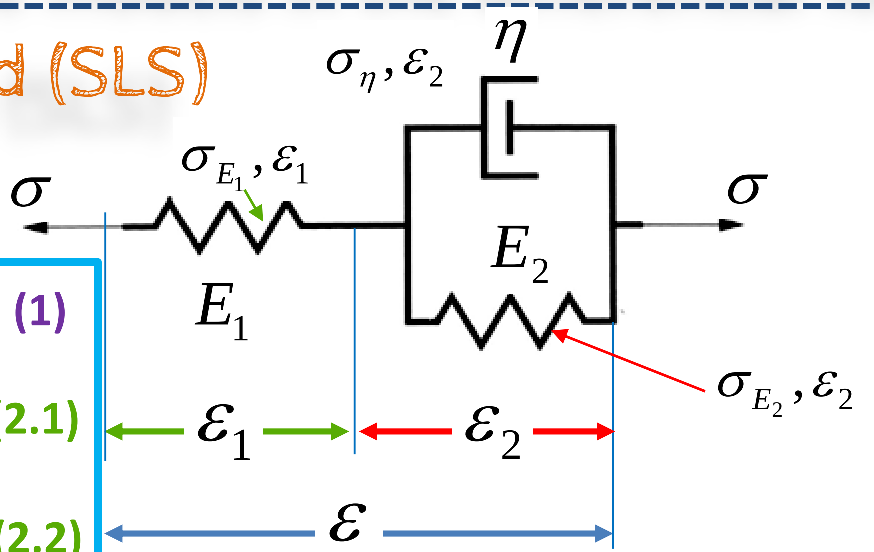
$$\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{12}\end{bmatrix} = \begin{bmatrix}1/E & -\nu/E & 0\\ -\nu/E & 1/E & 0\\ 0 & 0 & (1+\nu)/E\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix}$$

$$\varepsilon_{13} = \varepsilon_{23} = 0 \qquad \varepsilon_{33} = -\nu(\varepsilon_{11}+\varepsilon_{22})/(1-\nu)$$



Deriving the constitutive models in the differential form

Standard Linear Solid (SLS)



Compatibility: $\varepsilon = \varepsilon_1 + \varepsilon_2$ **(1)**

Equilibrium:

$\sigma_{E_1} = \sigma$ **(2.1)**

$\sigma_{E_2} + \sigma_\eta = \sigma$ **(2.2)**

(2)

$$\sigma_{E_1} = E_1\varepsilon_1$$

$$\sigma_{E_2} + \sigma_\eta = E_2\varepsilon_2 + \eta\dot{\varepsilon}_2$$

$$\Longrightarrow$$

(2.1)-(2.2) $\Rightarrow$

$$\sigma - \sigma_{E_2} - \sigma_\eta = 0 \Rightarrow \sigma - E_2\varepsilon_2 - \eta\dot{\varepsilon}_2 = 0$$

$\Rightarrow$

$$\sigma - E_2(\varepsilon \overset{(1)}{-} \varepsilon_1) - \eta(\dot{\varepsilon} - \dot{\varepsilon}_1) = 0$$

$$\varepsilon_1 = \sigma_{E_1} / E_1 \qquad \dot{\varepsilon}_1 = \dot{\sigma}_{E_1} / E_1$$

$\Longrightarrow$

The constitutive equation:

$$\dot{\varepsilon} + \frac{E_2}{\eta}\varepsilon = \frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta}\left(1 + \frac{E_2}{E_1}\right)$$

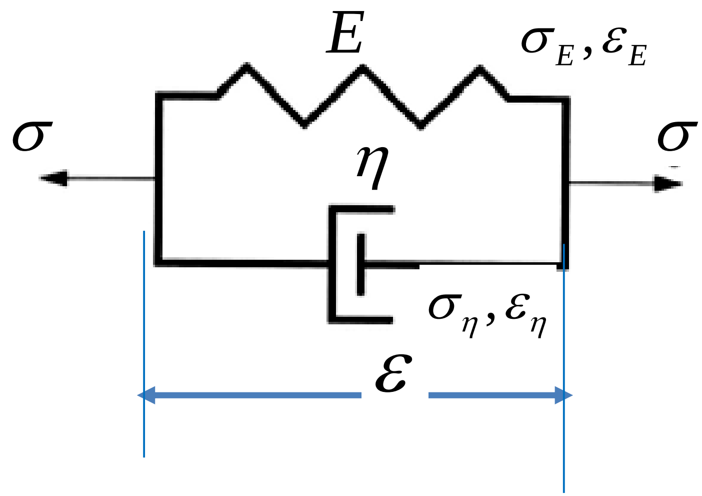
A linear first-order ordinary differential equation (ODE)

Deriving the constitutive models in the differential form

Derive the constitutive models in the differential form for the Kelvin-Voigt model

Kelvin-Voigt element

(viscos & elastic)



Applications: organic polymers, rubber, wood when load is not to high /*ref*: Lemaitre & Chaboche in the course textbook/

$$\sigma = E\varepsilon + \eta\dot{\varepsilon}$$

Use equilibrium and compatibility (of strains) equations:

Compatibility: $\varepsilon_E = \varepsilon_\eta = \varepsilon$

Equilibrium: $\sigma_E + \sigma_\eta = \sigma$

$$\sigma_E = E\varepsilon_E = E\varepsilon \qquad \sigma_\eta = \eta\dot{\varepsilon}_\eta = \eta\dot{\varepsilon}$$

Constitutive equation in the differential form:

$$\sigma = E\varepsilon + \eta\dot{\varepsilon}$$

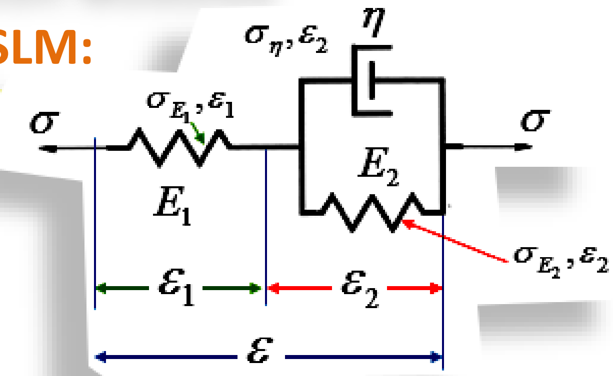
A linear first-order ordinary differential equation (ODE)

What are the
Creep response?
Relaxation response?

What is the Creep response?

Creep Compliance and Relaxation Modulus

SLM:



Creep response: Apply a **constant stress** $\sigma(0) = \sigma_0$ at $t = 0$

… and determine (measure) the time dependent strain $\varepsilon(t)$

Initial condition: $\varepsilon(0) = \sigma_0 / E_1$

Initially, only he Hooke element reacts. The dash element is infinitely stiff at t = 0, since $\dot{\varepsilon}_2(0) \to \infty$

$\Downarrow$

Homework

Solve the ODE:

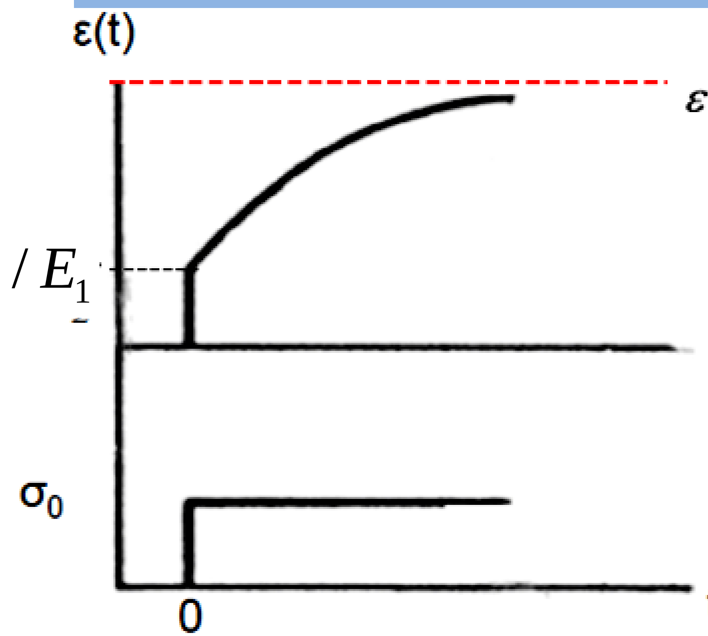
$$\dot{\varepsilon} + \frac{E_2}{\eta}\varepsilon = \frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta}(1 + \frac{E_2}{E_1})$$

and show that the creep function $J(t)$ is:

$$\left[\frac{1}{E_1} + \frac{1}{E_2}[1 - e^{-\frac{E_2}{\eta}t}]\right]$$

Creep response:

$$\varepsilon(t) = \sigma_0\left[\frac{1}{E_1} + \frac{1}{E_2}[1 - e^{-\frac{E_2}{\eta}t}]\right] \equiv \sigma_0 J(t)$$



$\varepsilon(\infty) = \sigma_0\left(\frac{1}{E_1} + \frac{1}{E_2}\right)$

$\varepsilon(0) = \sigma_0 / E_1$

E› Von Mises Yield Criteria:

strands … and *stress relaxation*

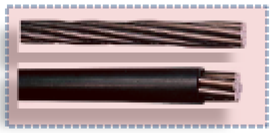
Solution: do not distribute

Relaxation experiment for the material: (observation)

- After 2 weeks, a loss of 2 MPa is observed in the material of the cable while the initial stress was 100 MPa.

Problem:

- **Derive relaxation function** (modulus) – use simple *Maxwell model*
- Determine from relaxation experiment characteristic relaxation time
- What should be the initial pre-stress level in order to keep > 150 MPa over one year?
- Assume a constant operating temperature of ~20 °C



Solution:

$$\sigma(t_w) = \sigma_0 e^{-t_w/\tau_R} \Rightarrow \tau_R = 99 \text{ weeks}$$

$(100-2)$ MPa; $t_w = 7$ weeks; 100 MPa

$t_{Year} = 52$ weeks

Initial pre-stress:

$$\sigma(t_{Year}) = \sigma_0' e^{-t_{Year}/\tau_R} \Rightarrow \sigma_0' > 254 \text{ MPa}$$

Maxwell model relaxation function – the simplest model:

$$\sigma(t) = \varepsilon_0 \cdot E e^{-\frac{E}{\eta}t} \equiv \sigma_0 e^{-t/\tau_R}$$

τ_R *-the characteristic relaxation time of the material*

Maxwell models are suitable for modelling stress relaxation phenomena

Exercises – Example

Consider the following stress state (units in MPa): $\sigma_x = 200 \equiv \sigma_{11}$

$$\sigma_y = \sigma_z = 120$$

$$\tau_{xy} = 110, \quad \tau_{yz} = 0$$

Yield stress in uniaxial test: $\sigma_Y = 240 \text{ MPa}$

Question: how much the stress component τ_{xz} can be increased before yield occurs? *a)* according to von Mises and *b)* Tresca yield condition.

Tresca Yield Criteria:

$$F = \max\left\{\frac{1}{2}|\sigma_1-\sigma_2|, \frac{1}{2}|\sigma_2-\sigma_3|, \frac{1}{2}|\sigma_3-\sigma_1|\right\} - k$$

Von Mises Yield Criteria: $F = \sigma_e^2 - \sigma_Y^2 = 0$

$$F = \underbrace{\frac{1}{2}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]}_{=\sigma_e^2 = 3J_2} - \sigma_Y^2 = 0$$

... or in component form

$$\sigma_e^2 = \frac{1}{2}[(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2]+\dots$$

$$\dots+3[\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2] \quad \Downarrow$$

Let use the notations:

$$\tau_{xz} = \alpha\sigma_0$$
$$\sigma_x = 200 = \sigma_{11} = 2\sigma_0$$
$$\sigma_y = \sigma_z = 1.2\sigma_0$$
$$\tau_{xy} = 1.1\sigma_0, \quad \tau_{yz} = 0$$

$$\boldsymbol{\sigma} = \sigma_0\begin{bmatrix} 2 & 1.1 & \alpha \\ & 1.2 & 0 \\ Symm. & & 1.2 \end{bmatrix}$$

HW b) do the same for Tresca yield

$$\sigma_e^2 = \frac{1}{2}\sigma_0^2 \cdot \underbrace{[(2-1.2)^2+(1.2-1.2)^2+(1.2-2)^2]}_{=a}+\dots$$

$$\dots+3\sigma_0^2[1.1^2+0+\alpha^2] = \sigma_Y^2 \Rightarrow$$

$$3\alpha^2 = (\sigma_Y/\sigma_0)^2 - 4.27 \rightarrow \left[(240/100)^2 - 4.27\right]/3 =$$

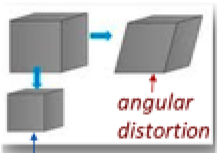
$$\alpha \approx 0.705$$

$$\Rightarrow \tau_{xz} = \alpha\sigma_0 = 0.705 \cdot 100 \approx 70 \text{ MPa}$$

With the slightly different data of the new exam,
here the results:
V-Mises = tau <= 110 MPa
Tresca : 78 MPa

Distortional Energy Criteria:

Yielding will occur when the ***distortion energy*** in a unit volume of the material equals the distortion energy in the same volume which is uniaxially stressed till yielding occurs



Von Mises Yield criteria: $F = J_2 - k^2 = 0 \Rightarrow$

$\tau_Y \equiv k = \sigma_Y/\sqrt{3}$ (yield in uniaxial test)

$$F = \frac{1}{6}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2] - k^2 = 0 \Rightarrow$$

$$\Rightarrow \frac{1}{\sqrt{3}}\underbrace{\sqrt{\frac{1}{2}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]}}_{\equiv\sigma_e} = k = \frac{\sigma_Y}{\sqrt{3}}$$

von Mises equivalent stress

$$\Rightarrow \sigma_e = \sigma_Y \quad \text{yielding criteria}$$

The material yields when *von Mises equivalent stress* σ_e exceeds the uniaxial material yield strength $\quad \sigma_e = \sigma_Y$

$$F = \frac{1}{2}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2] - \sigma_Y^2 = 0$$