MEC-E8003 Beam, Plate and Shell models, exam 09.04.2019

1. Derive the expressions of linear strain components ε_{rr}, $\varepsilon_{r\phi}$, $\varepsilon_{\phi r}$ and $\varepsilon_{\phi\phi}$ of the polar coordinate system. Use the displacement representation $\vec{u} = u_r\vec{e}_r + u_\phi\vec{e}_\phi$ where the components depend on the polar coordinates r and ϕ. Use definitions

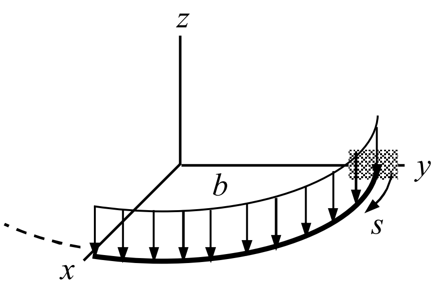
$$\vec{\varepsilon} = \frac{1}{2}[\nabla\vec{u} + (\nabla\vec{u})_{\mathrm{c}}],\ \nabla = \vec{e}_r\frac{\partial}{\partial r} + \vec{e}_\phi\frac{\partial}{r\partial\phi},\ \frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix} = \begin{Bmatrix}\vec{e}_\phi\\ -\vec{e}_r\end{Bmatrix},\ \frac{\partial}{\partial r}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix} = 0.$$

2. Virtual work expression of a Bernoulli beam, clamped at the left end $x=0$ and loaded by force F and moment R at the right end $x=L$ of solution domain $\Omega = (0,L)$, is given by

$$\delta W = \int_0^L (M\frac{d^2\delta w}{dx^2} + b\delta w)dx + (F\delta w - R\frac{d\delta w}{dx})_{x=L}.$$

Use the principle of virtual work $\delta W = 0\ \forall\delta w \in U$ to derive the beam equilibrium equation in Ω, natural boundary conditions on $x=L$, and essential boundary conditions on $x=0$. Functions of set U have continuous derivatives up to the fourth order in Ω. In addition, a function of U vanishes at $x=0$ as does also its first derivative.

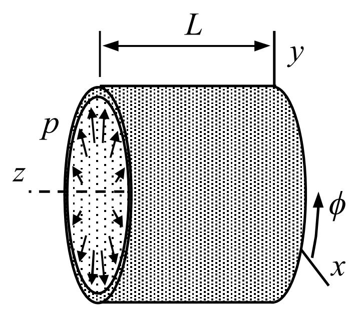
3. Consider the curved beam of the figure forming a 90-degree circular segment of radius R in the horizontal plane. Find the stress resultants $N(s)$, $Q_n(s)$, $Q_b(s)$, $T(s)$, $M_n(s)$, and $M_b(s)$. Use the equilibrium equations of the beam model in the $(s,n,b)-$coordinate system. The distributed constant load of magnitude b is acting to the negative direction of the $z-$axis.



4. Show that the vertical displacement $w(x,y)$ of the Kirchhoff plate model satisfies the biharmonic equation $D\nabla_0^2\nabla_0^2 w = b_n$. Start with the Reissner-Mindlin plate model equations for the bending mode:

$$\begin{Bmatrix}\frac{\partial}{\partial x}Q_x + \frac{\partial}{\partial y}Q_y + b_n\\ \frac{\partial}{\partial x}M_{xx} + \frac{\partial}{\partial y}M_{xy} - Q_x\\ \frac{\partial}{\partial y}M_{yy} + \frac{\partial}{\partial x}M_{xy} - Q_y\end{Bmatrix} = 0,\ \begin{Bmatrix}M_{xx}\\ M_{yy}\\ M_{xy}\end{Bmatrix} = D\begin{Bmatrix}\frac{\partial\theta}{\partial x} - \nu\frac{\partial\phi}{\partial y}\\ -\frac{\partial\phi}{\partial y} + \nu\frac{\partial\theta}{\partial x}\\ \frac{1-\nu}{2}(\frac{\partial\theta}{\partial y} - \frac{\partial\phi}{\partial x})\end{Bmatrix} \text{and} \begin{Bmatrix}Q_x\\ Q_y\end{Bmatrix} = Gt\begin{Bmatrix}\frac{\partial w}{\partial x} + \theta\\ \frac{\partial w}{\partial y} - \phi\end{Bmatrix}.$$

5. A steel ring of length L, radius R, and thickness t is loaded by radial surface force p acting on the inner surface. No forces are acting on the ends. Model the ring as a cylindrical membrane, write down the equilibrium and constitutive equations, and solve for the radial displacement. Assume rotation symmetry. Young's modulus E and Poisson's ratio ν of the material are constants.



Derive the expressions of linear strain components ε_{rr}, $\varepsilon_{r\phi}$, $\varepsilon_{\phi r}$ and $\varepsilon_{\phi\phi}$ of the polar coordinate system. Use the displacement representation $\vec{u} = u_r \vec{e}_r + u_\phi \vec{e}_\phi$ where the components depend on the polar coordinates r and ϕ. Use definitions

$$\vec{\varepsilon} = \frac{1}{2}[\nabla\vec{u} + (\nabla\vec{u})_{\mathrm{c}}], \; \nabla = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{\partial}{r\partial\phi}, \; \frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_r \\ \vec{e}_\phi\end{Bmatrix} = \begin{Bmatrix}\vec{e}_\phi \\ -\vec{e}_r\end{Bmatrix}, \; \frac{\partial}{\partial r}\begin{Bmatrix}\vec{e}_r \\ \vec{e}_\phi\end{Bmatrix} = 0 .$$

Solution

In manipulation of vector expression containing vectors and tensors, it is important to remember that tensor ($\otimes$), cross ($\times$), inner ($\cdot$) products are non-commutative (order may matter). The basis vectors of a curvilinear coordinate system are not constants which should be taken into account if gradient operator is a part of expression. Otherwise, simplifying an expression or finding a specific form in a given coordinate system is a straightforward (sometimes tedious) exercise. For simplicity of presentation, outer (tensor) products like $\vec{a} \otimes \vec{b}$ are denoted by $\vec{a}\vec{b}$. Otherwise, the usual rules of algebra apply: Gradient operator ∇ acts on everything on its right-hand side, the operator is treated like a vector etc.

2p Let us start with the gradient of displacement (an outer product). Substitute first the representations in the polar coordinate system

$$\nabla\vec{u} = (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{\partial}{r\partial\phi})(u_r \vec{e}_r + u_\phi \vec{e}_\phi) .$$

Then expand to have a term-by-term representation. Keep the order of the basis vectors and the position of derivatives

$$\nabla\vec{u} = \vec{e}_r \frac{\partial}{\partial r}(u_r \vec{e}_r) + \vec{e}_r \frac{\partial}{\partial r}(u_\phi \vec{e}_\phi) + \vec{e}_\phi \frac{\partial}{r\partial\phi}(u_r \vec{e}_r) + \vec{e}_\phi \frac{\partial}{r\partial\phi}(u_\phi \vec{e}_\phi)$$

2p Use the derivative rule of products. Notice that the basis vectors are not constants and may have non-zero derivatives

$$\nabla\vec{u} = \vec{e}_r(\frac{\partial u_r}{\partial r}\vec{e}_r + u_r \frac{\partial \vec{e}_r}{\partial r}) + \vec{e}_r(\frac{\partial u_\phi}{\partial r}\vec{e}_\phi + u_\phi \frac{\partial \vec{e}_\phi}{\partial r}) + \vec{e}_\phi(\frac{\partial u_r}{r\partial\phi}\vec{e}_r + u_r \frac{\partial \vec{e}_r}{r\partial\phi}) + \vec{e}_\phi(\frac{\partial u_\phi}{r\partial\phi}\vec{e}_\phi + u_\phi \frac{\partial \vec{e}_\phi}{r\partial\phi}) .$$

Substitute the derivatives of the basis vectors

$$\nabla\vec{u} = \vec{e}_r(\frac{\partial u_r}{\partial r}\vec{e}_r) + \vec{e}_r(\frac{\partial u_\phi}{\partial r}\vec{e}_\phi) + \vec{e}_\phi(\frac{\partial u_r}{r\partial\phi}\vec{e}_r + \frac{u_r}{r}\vec{e}_\phi) + \vec{e}_\phi(\frac{\partial u_\phi}{r\partial\phi}\vec{e}_\phi - \frac{u_\phi}{r}\vec{e}_r) .$$

Combine the terms having the same pair of basis vectors (order matters so terms containing $\vec{e}_\phi\vec{e}_r$ and $\vec{e}_r\vec{e}_\phi$ cannot be combined)

$$\nabla\vec{u} = \frac{\partial u_r}{\partial r}\vec{e}_r\vec{e}_r + \frac{\partial u_\phi}{\partial r}\vec{e}_r\vec{e}_\phi + (\frac{\partial u_r}{r\partial\phi} - \frac{u_\phi}{r})\vec{e}_\phi\vec{e}_r + (\frac{u_r}{r} + \frac{\partial u_\phi}{r\partial\phi})\vec{e}_\phi\vec{e}_\phi .$$

2p Conjugate of a second order tensor can be obtained by swapping the basis vectors in all the pairs. Conjugate is a kind of transpose and can also be obtained by transposing the matrix of the component representation.

$$(\nabla\vec{u})_{\mathrm{c}} = \frac{\partial u_r}{\partial r}\vec{e}_r\vec{e}_r + (\frac{\partial u_r}{r\partial\phi} - \frac{u_\phi}{r})\vec{e}_r\vec{e}_\phi + \frac{\partial u_\phi}{\partial r}\vec{e}_\phi\vec{e}_r + (\frac{u_r}{r} + \frac{\partial u_\phi}{r\partial\phi})\vec{e}_\phi\vec{e}_\phi$$

Finally using the definition $\vec{\varepsilon} = \frac{1}{2}\left[\nabla\vec{u} + (\nabla\vec{u})_{\mathrm{c}}\right]$

$$\vec{\varepsilon} = \frac{\partial u_r}{\partial r}\vec{e}_r\vec{e}_r + \frac{1}{2}(\frac{\partial u_r}{r\partial\phi} - \frac{u_\phi}{r} + \frac{\partial u_\phi}{\partial r})(\vec{e}_r\vec{e}_\phi + \vec{e}_\phi\vec{e}_r) + (\frac{\partial u_\phi}{r\partial\phi} + \frac{u_r}{r})\vec{e}_\phi\vec{e}_\phi .$$

In the components of strain ε_{rr}, $\varepsilon_{r\phi}$, $\varepsilon_{\phi r}$ and $\varepsilon_{\phi\phi}$, indices are in the same order as the indices in the basis vector pairs. Hence

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\phi\phi} = \frac{\partial u_\phi}{r\partial\phi} + \frac{u_r}{r}, \quad \varepsilon_{r\phi} = \varepsilon_{\phi r} = \frac{1}{2}(\frac{\partial u_r}{r\partial\phi} - \frac{u_\phi}{r} + \frac{\partial u_\phi}{\partial r}) . \leftarrow$$

Virtual work expression of a Bernoulli beam, clamped at the left end $x=0$ and loaded by force F and moment R at the right end $x=L$ of solution domain $\Omega=(0,L)$, is given by

$$\delta W=\int_0^L (\frac{d^2\delta w}{dx^2}M+\delta w b)dx+(F\delta w-R\frac{d\delta w}{dx})_{x=L}.$$

Use the principle of virtual work $\delta W=0\ \forall\delta w\in U$ to derive the beam equilibrium equation in Ω, natural boundary conditions on $x=L$, and essential boundary conditions on $x=0$. Functions of set U have continuous derivatives up to the fourth order in Ω. In addition, a function of U vanishes at $x=0$ as does also its first derivative.

Solution

In MEC-E8003, principle of virtual work is used to derive the equilibrium equation(s) in terms of the stress resultants (like shear forces and bending moments). The constitutive equation, giving the relationship between the stress resultants and kinetic quantities (like displacements and rotations), is a separate story. The mathematical tools needed in the derivation are (one-dimensional case $\Omega\subset\mathbb{R}$) $a,b\in C^0(\Omega)$

$$\int_\Omega \frac{d}{dx}(ab)dx=\sum_{\partial\Omega} nab,$$ where $n=\pm1$ is the unit outward normal to Ω (on $\partial\Omega$)

$$\int_\Omega abdx=0 \quad \forall b \quad \Leftrightarrow \quad a=0 \ \text{ in } \Omega.$$

2p Integration by parts once in the first term gives an equivalent form (notice that $\delta w\in U$ and therefore $\delta w=d\delta w/dx=0$ at $x=0$

$$\delta W=\int_0^L (M\frac{d^2\delta w}{dx^2}+b\delta w)dx+(F\delta w-R\frac{d\delta w}{dx})_{x=L} \quad \Leftrightarrow$$

$$\delta W=\int_0^L (-\frac{dM}{dx}\frac{d\delta w}{dx}+b\delta w)dx+[(M-R)\frac{d\delta w}{dx}]_{x=L}+(F\delta w)_{x=L}.$$

Integration by parts second time in the first term gives also an equivalent form

$$\delta W=\int_0^L (\frac{d^2M}{dx^2}+b)\delta w dx+[(-\frac{dM}{dx}+F)\delta w]_{x=L}+[(M-R)\frac{d\delta w}{dx}]_{x=L}.$$

2p According to the principle of virtual work $\delta W=0\ \ \forall\delta w\in U$. Let us first consider a subset $U_0\subset U$ for which $\delta w=d\delta w/dx=0$ at $x=L$ so that the boundary terms vanish. The equilibrium equation follows from the fundamental lemma of variation calculus:

$$\delta W=\int_0^L (\frac{d^2M}{dx^2}+b)\delta w dx=0 \quad \forall\delta w\in U_0 \qquad \Leftrightarrow \qquad \frac{d^2M}{dx^2}+b=0 \quad \text{in} \quad (0,L).$$

Let us next consider a subset $U_0\subset U$ for which only $d\delta w/dx=0$ at $x=0$ so that the last boundary term of the virtual work expression vanishes. Also, the first term can be omitted due to the equilibrium equation. The natural boundary condition follows from the fundamental lemma of variation calculus:

$$\delta W = [(-\frac{dM}{dx} + F)\delta w]_{x=L} = 0 \quad \forall \delta w \in U_0 \quad \Leftrightarrow \quad -\frac{dM}{dx} + F = 0 \quad \text{at} \quad x = L .$$

Finally, let us consider a subset $U_0 \subset U$ for which only $\delta w = 0$ at $x = L$ and use the equations already obtained to simplify the virtual work expression. The natural boundary condition follows from the fundamental lemma of variation calculus:

$$\delta W = [(M - R)\frac{d\delta w}{dx}]_{x=L} = 0 \quad \forall \delta w \in U_0 \quad \Leftrightarrow \quad M - R = 0 \quad \text{at} \quad x = L .$$

2p As the last step, the essential boundary conditions follow from the problem definition (clamped). They can also partly be deduced from the definition of U. Vanishing of variation $d\delta w / dx$ and δw at $x = 0$ imply that dw / dx and w are given at $x = 0$.

A beam boundary value problem is composed of the equations implied by the principle of virtual work

$$\frac{d^2 M}{dx^2} + b = 0 \quad \text{in} \quad (0, L). \quad \leftarrow$$

$$-\frac{dM}{dx} + F = 0 \quad \text{and} \quad M - R = 0 \quad \text{at} \quad x = L . \quad \leftarrow$$

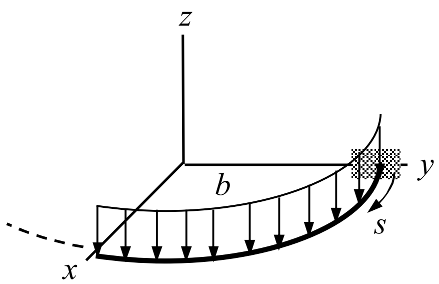
$$w = 0 \quad \text{and} \quad \frac{dw}{dx} = 0 \quad \text{at} \quad x = 0 . \quad \leftarrow$$

Definition of stress resultant, stress-strain relationship, and elasticity tensor for the beam problem gives the constitutive equation

$$M = -EI \frac{d^2 w}{dx^2}$$

which is needed for a closed system.

Consider the curved beam of the figure forming a 90-degree circular segment of radius R in the horizontal plane. Find the stress resultants $N(s)$, $Q_n(s)$, $Q_b(s)$, $T(s)$, $M_n(s)$, and $M_b(s)$. Use the equilibrium equations of the beam model in the $(s,n,b)-$coordinate system. The distributed constant load of magnitude b is acting to the negative direction of the $z-$axis.



Solution

In a statically determined case, stress resultants follow from the equilibrium equations and boundary conditions at the free end of the beam (or directly from a free body diagram). In $(s,n,b)-$coordinate system, equilibrium equations are

$$\begin{Bmatrix} N' - Q_n\kappa + b_s \\ Q_n' + N\kappa - Q_b\tau + b_n \\ Q_b' + Q_n\tau + b_b \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} T' - M_n\kappa + c_s \\ M_n' + T\kappa - M_b\tau - Q_b + c_n \\ M_b' + M_n\tau + Q_n + c_b \end{Bmatrix} = 0.$$

2p For a circular beam, curvature and torsion are $\kappa = 1/R$ (constant) and $\tau = 0$. As external distributed forces and moments $b_s = b_n = c_s = c_n = c_b = 0$ and $b_b = b$, equilibrium equations and the boundary conditions at the free end simplify to (here $L = \pi R/2$)

$$\begin{Bmatrix} N' - Q_n/R \\ Q_n' + N/R \\ Q_b' + b \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} T' - M_n/R \\ M_n' + T/R - Q_b \\ M_b' + Q_n \end{Bmatrix} = 0 \quad \text{in} \quad (0, L)$$

$$\begin{Bmatrix} N \\ Q_n \\ Q_b \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} T \\ M_n \\ M_b \end{Bmatrix} = 0 \quad \text{at} \quad s = L.$$

2p Equations constitute a boundary value problem which can be solved one equation at a time by following certain order

$$Q_b' = -b \quad \text{in} \quad (0,L) \quad \text{and} \quad Q_b(L) = 0 \quad \Rightarrow \quad Q_b(s) = b(L-s). \quad \leftarrow$$

Eliminating Q_n and N from the remaining two connected force equilibrium equations and using the original equations to find the missing boundary condition gives

$$N'' + \frac{1}{R^2}N = 0 \quad \text{in} \quad (0,L) \quad \text{and} \quad N'(L) = N(L) = 0 \quad \Rightarrow \quad N(s) = 0. \quad \leftarrow$$

Knowing the result above, the first equilibrium equation gives

$$Q_n(s) = 0. \quad \leftarrow$$

2p After that, continuing with the moment equilibrium equations with the already known solutions to the force equilibrium equations

$$M_b' = -Q_n = 0 \quad \text{in} \quad (0,L) \quad \text{and} \quad M_b(L) = 0 \quad \Rightarrow \quad M_b(s) = 0 \,. \quad \Leftarrow$$

Eliminating M_n and T from the remaining two connected moment equilibrium equations and using the original equations to find the missing boundary condition gives $(L = \pi R / 2)$

$$T'' + \frac{1}{R^2} T = \frac{1}{R} Q_b = \frac{b}{R}(L-s) \quad \text{in} \quad (0,L) \quad \text{and} \quad T'(L) = T(L) = 0 \quad \Rightarrow$$

$$T(s) = -bR^2 \cos(\frac{s}{R}) + bR(L-s) \,. \quad \Leftarrow$$

Knowing this, the first moment equilibrium equation gives

$$M_n(s) = RT' = bR^2 \sin(\frac{s}{R}) - bR^2 \,. \quad \Leftarrow$$

Show that the vertical displacement $w(x,y)$ of the Kirchhoff plate model satisfies the biharmonic equation $D\nabla_0^2\nabla_0^2 w = b_n$. Start with the Reissner-Mindlin plate model equations for the bending mode:

$$\begin{Bmatrix} \frac{\partial}{\partial x}Q_x + \frac{\partial}{\partial y}Q_y + b_n \\ \frac{\partial}{\partial x}M_{xx} + \frac{\partial}{\partial y}M_{xy} - Q_x \\ \frac{\partial}{\partial y}M_{yy} + \frac{\partial}{\partial x}M_{xy} - Q_y \end{Bmatrix} = 0, \quad \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = D \begin{Bmatrix} \frac{\partial\theta}{\partial x} - \nu\frac{\partial\phi}{\partial y} \\ -\frac{\partial\phi}{\partial y} + \nu\frac{\partial\theta}{\partial x} \\ \frac{1-\nu}{2}(\frac{\partial\theta}{\partial y} - \frac{\partial\phi}{\partial x}) \end{Bmatrix} \text{ and } \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = Gt \begin{Bmatrix} \frac{\partial w}{\partial x} + \theta \\ \frac{\partial w}{\partial y} - \phi \end{Bmatrix}.$$

Solution

2p Kirchhoff constraints are first used to write the constitutive equations in terms of the transverse displacement

$$\theta = -\frac{\partial w}{\partial x} \text{ and } \phi = \frac{\partial w}{\partial y} \quad \Rightarrow \quad \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = D \begin{Bmatrix} \frac{\partial\theta}{\partial x} - \nu\frac{\partial\phi}{\partial y} \\ -\frac{\partial\phi}{\partial y} + \nu\frac{\partial\theta}{\partial x} \\ \frac{1-\nu}{2}(\frac{\partial\theta}{\partial y} - \frac{\partial\phi}{\partial x}) \end{Bmatrix} = -D \begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2} \\ \frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2} \\ (1-\nu)\frac{\partial^2 w}{\partial x\partial y} \end{Bmatrix}.$$

2p In the Kirchhoff model, shear forces Q_x and Q_y are in the role of constraint forces to be solved from the moment equations. Eliminating the shear forces from the equilibrium equation in the transverse direction by using the moment equations gives

$$\begin{Bmatrix} \frac{\partial}{\partial x}Q_x + \frac{\partial}{\partial y}Q_y + b_z \\ \frac{\partial}{\partial x}M_{xx} + \frac{\partial}{\partial y}M_{yx} - Q_x \\ \frac{\partial}{\partial y}M_{yy} + \frac{\partial}{\partial x}M_{xy} - Q_y \end{Bmatrix} = 0 \quad \Rightarrow \quad \frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + b_n = 0.$$

2p The biharmonic equation for the transverse displacement follows from the equilibrium equation above, when the constitutive equations for the moments are substituted there

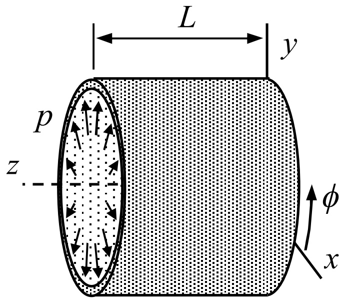
$$\frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + b_n = 0 \quad \Leftrightarrow$$

$$\frac{\partial^2}{\partial x^2}(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}) + 2(1-\nu)\frac{\partial^2}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y} + \frac{\partial^2}{\partial y^2}(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}) - \frac{b_n}{D} = 0 \quad \Leftrightarrow$$

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\partial y^2} + \frac{\partial^4 w}{\partial y^4} - \frac{b_n}{D} = 0 \quad \text{or} \quad \nabla_0^2\nabla_0^2 w = \frac{b_n}{D}. \leftarrow$$

The last invariant form holds also, e.g., in the polar coordinate system.

A steel ring of length L, radius R, and thickness t is loaded by radial surface force p acting on the inner surface. No forces are acting on the ends. Model the ring as a cylindrical membrane, write down the equilibrium and constitutive equations, and solve for the radial displacement. Assume rotation symmetry and $u_\phi = 0$. Young's modulus E and Poisson's ratio ν of the material are constants.



Solution

According to the formulae collection, equilibrium and constitutive equations of a cylindrical membrane in (z,ϕ,n) coordinates are (notice that $\vec{e}_n$ is directed inwards)

$$\begin{Bmatrix} \frac{1}{R}\frac{\partial}{\partial\phi}N_{z\phi} + \frac{\partial}{\partial z}N_{zz} + b_z \\ \frac{\partial}{\partial z}N_{z\phi} + \frac{1}{R}\frac{\partial}{\partial\phi}N_{\phi\phi} + b_\phi \\ \frac{1}{R}N_{\phi\phi} + b_n \end{Bmatrix} = 0, \quad \begin{Bmatrix} N_{zz} \\ N_{\phi\phi} \\ N_{z\phi} \end{Bmatrix} = \frac{tE}{1-\nu^2}\begin{Bmatrix} \frac{\partial}{\partial z}u_z + \nu\frac{1}{R}(\frac{\partial}{\partial\phi}u_\phi - u_n) \\ \frac{1}{R}(\frac{\partial}{\partial\phi}u_\phi - u_n) + \nu\frac{\partial}{\partial z}u_z \\ \frac{1-\nu}{2}(\frac{1}{R}\frac{\partial}{\partial\phi}u_z + \frac{\partial}{\partial z}u_\phi) \end{Bmatrix}.$$

2p Due to the rotation symmetry, the derivatives with respect to the angular coordinate vanish and $u_\phi = 0$. External distributed force $b_n = -p$ is due to the traction acting on the inner boundary. Therefore, the equilibrium equations and constitutive equations simplify to a set of ordinary differential equations

$$\frac{dN_{zz}}{dz} = 0, \quad \frac{dN_{z\phi}}{dz} = 0, \quad \frac{1}{R}N_{\phi\phi} - p = 0 \quad \text{in} \quad (0,L),$$

$$N_{zz} = \frac{tE}{1-\nu^2}\frac{1}{R}(R\frac{du_z}{dz} - \nu u_n), \quad N_{\phi\phi} = \frac{tE}{1-\nu^2}\frac{1}{R}(R\nu\frac{du_z}{dz} - u_n), \quad N_{z\phi} = 0 \quad \text{in} \quad (0,L).$$

2p As the edges are stress-free i.e.

$$N_{zz} = 0 \text{ and } N_{z\phi} = 0 \quad \text{on} \quad \{0,L\}.$$

Solution to the stress resultants, as obtained from the equilibrium equations, are

$$N_{zz} = 0, \; N_{z\phi} = 0, \text{ and } N_{\phi\phi} = Rp.$$

2p Constitutive equations give

$$N_{zz} = \frac{tE}{1-\nu^2}\frac{1}{R}(R\frac{du_z}{dz} - \nu u_n) = 0 \quad \Rightarrow \quad \frac{du_z}{dz} = \frac{\nu}{R}u_n \quad \text{and}$$

$$Rp = N_{\phi\phi} = \frac{tE}{1-\nu^2}\frac{1}{R}(R\nu\frac{du_z}{dz} - u_n) = \frac{tE}{1-\nu^2}\frac{1}{R}(\nu^2 - 1)u_n = -\frac{tE}{R}u_n \quad \Leftrightarrow \quad u_n = -\frac{pR^2}{tE}. \quad \leftarrow$$