Solution Suggestions – Homework 4

(a)

From (2.47)

$$(\bar{\bar{A}} \overset{\times}{\times} \bar{\bar{B}}) \overset{\times}{\times} (\bar{\bar{C}} \overset{\times}{\times} \bar{\bar{D}}) = (\bar{\bar{A}} \overset{\times}{\times} \bar{\bar{B}} : \bar{\bar{C}})\bar{\bar{D}} + (\bar{\bar{A}} \overset{\times}{\times} \bar{\bar{B}} : \bar{\bar{D}})\bar{\bar{C}} - \bar{\bar{C}}\cdot(\bar{\bar{A}} \overset{\times}{\times} \bar{\bar{B}})^T\cdot\bar{\bar{D}} - \bar{\bar{D}}\cdot(\bar{\bar{A}} \overset{\times}{\times} \bar{\bar{B}})^T\cdot\bar{\bar{C}}$$

$$= (\bar{\bar{C}} \overset{\times}{\times}\bar{\bar{D}}:\bar{\bar{A}})\bar{\bar{B}} + (\bar{\bar{C}}\overset{\times}{\times}\bar{\bar{D}}:\bar{\bar{B}})\bar{\bar{A}} - \bar{\bar{A}}\cdot(\bar{\bar{C}}\overset{\times}{\times}\bar{\bar{D}})^T\cdot\bar{\bar{B}} - \bar{\bar{B}}\cdot(\bar{\bar{C}}\times\bar{\bar{D}})^T\cdot\bar{\bar{A}}$$

$$\bar{\bar{C}}=\bar{\bar{D}}=\bar{\bar{I}} \Rightarrow 2(\underbrace{\bar{\bar{A}}\overset{\times}{\times}\bar{\bar{B}}:\bar{\bar{I}}}_{(2.48)\,\to\,(\mathrm{tr}\bar{\bar{A}})(\mathrm{tr}\bar{\bar{B}})-\bar{\bar{A}}:\bar{\bar{B}}^T})\bar{\bar{I}} - 2(\bar{\bar{A}}\overset{\times}{\times}\bar{\bar{B}})^T = 2(\mathrm{tr}\bar{\bar{A}})\bar{\bar{B}} + 2(\mathrm{tr}\bar{\bar{B}})\bar{\bar{A}} - 2(\bar{\bar{A}}\cdot\bar{\bar{B}}+\bar{\bar{B}}\cdot\bar{\bar{A}})$$

$\Rightarrow$ (2.51)

(b) Let's try a symmetric dyadic. $\bar{\bar{A}} = a\bar{u}\bar{u}+b\bar{v}\bar{v}+c\bar{w}\bar{w}$

$(\bar{u},\bar{v},\bar{w})$ orthonormal. $\mathrm{spm}\,\bar{\bar{A}}=0 \Rightarrow c = -\frac{ab}{a+b}$

For example: $\bar{\bar{A}} = 2\bar{u}\bar{u} + 2\bar{v}\bar{v} - \bar{w}\bar{w} \Rightarrow \bar{\bar{A}}\overset{\times}{\times}\bar{\bar{A}} = -4\bar{u}\bar{u} - 4\bar{v}\bar{v} + 8\bar{w}\bar{w} \neq 0$

(This has $\det\bar{\bar{A}} = -4 \neq 0$)

But an even simpler dyadic is $\bar{\bar{B}} = \alpha\bar{u}\bar{u} + \beta\bar{v}\bar{v}$

for which $\bar{\bar{B}}\overset{\times}{\times}\bar{\bar{B}} = -2\alpha\beta\bar{w}\bar{w}$

and $\mathrm{spm}\,\bar{\bar{B}} = \frac{1}{2}\bar{\bar{B}}\overset{\times}{\times}\bar{\bar{B}}:\bar{\bar{I}} = 0$ & $\det\bar{\bar{B}}=0$

(c) Because $\bar{u}\cdot\bar{u} = \bar{v}\cdot\bar{v} = \bar{w}\cdot\bar{w} = 1$ & $\bar{u}\cdot\bar{v} = \bar{v}\cdot\bar{w} = \bar{w}\cdot\bar{u} = 0$, we have the eigenvector expansion of the permittivity dyadic $\bar{\bar{\varepsilon}}$.

Hence the inverse is straightforward:

$$\left(\bar{\bar{\varepsilon}}/\varepsilon_0 + 2\bar{\bar{I}}\right)^{-1} = \frac{1}{\varepsilon_u+2}\,\bar{u}\bar{u} + \frac{1}{\varepsilon_v+2}\,\bar{v}\bar{v} + \frac{1}{\varepsilon_w+2}\,\bar{w}\bar{w}$$

as well as the dyadic product:

$$\bar{\bar{\alpha}} = 3\varepsilon_0 V\left(\frac{\varepsilon_u-1}{\varepsilon_u+2}\,\bar{u}\bar{u} + \frac{\varepsilon_v-1}{\varepsilon_v+2}\,\bar{v}\bar{v} + \frac{\varepsilon_w-1}{\varepsilon_w+2}\,\bar{w}\bar{w}\right)$$

(in other words, the polarizability dyadic has the same eigenvectors as the permittivity dyadic!)

$$\bar{\bar{\alpha}} = \alpha_u\,\bar{u}\bar{u} + \alpha_v\,\bar{v}\bar{v} + \alpha_w\,\bar{w}\bar{w}$$