# HOMEWORK 5 - Solutions

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(a) I guess a simple and good example is the same dyadic that Sajjad found for 4(b): $\alpha \bar{u}\bar{n} + \beta \bar{v}\bar{w}$

For example: $\bar{\bar{A}} = \bar{u}_x\bar{u}_x + \bar{u}_y\bar{u}_z$

it is strictly planar: $\det \bar{\bar{A}} = 0$ but $\bar{\bar{A}} {}^\times_\times \bar{\bar{A}} = -2\bar{u}_z\bar{u}_y \neq 0$

However, it is not two-dimensional, because the r. eigenvector corresponding to zero eigenvalue is not the same as the l. e. vector:

$$\bar{\bar{A}} \cdot \bar{u}_y = 0 \quad \& \quad \bar{u}_z \cdot \bar{\bar{A}} = 0$$

(for a two-dimensional dyadic we require $\bar{n}\cdot\bar{\bar{A}} = 0$ & $\bar{\bar{A}}\cdot\bar{n} = 0$)

(b) $\bar{\bar{D}} = \alpha \bar{\bar{I}} + \bar{a}\bar{b}$ $\Rightarrow$

$$\operatorname{tr} \bar{\bar{D}} = 3\alpha + \bar{a}\cdot\bar{b}$$
$$\operatorname{spm} \bar{\bar{D}} = \alpha(3\alpha + 2\bar{a}\cdot\bar{b})$$
$$\det \bar{\bar{D}} = \alpha^2(\alpha + \bar{a}\cdot\bar{b})$$

Eigenvalues from (2.107)

$$\gamma^3 - \gamma^2 \operatorname{tr}\bar{\bar{D}} + \gamma \operatorname{spm}\bar{\bar{D}} - \det\bar{\bar{D}} = 0$$

$\Rightarrow$ (some algebra) $(\gamma-\alpha)^2(\gamma-\alpha-\bar{a}\cdot\bar{b}) = 0$

$\Rightarrow$ eigenvalues $\gamma_1 = \alpha$ $\gamma_2 = \alpha$ $\gamma_3 = \alpha + \bar{a}\cdot\bar{b}$

$\bar{\bar{D}}\cdot\bar{c} = \alpha\bar{c}$

$= \alpha\bar{c} + \bar{a}\,\bar{b}\cdot\bar{c} = \alpha\bar{c}$

$\Rightarrow \bar{b}\cdot\bar{c} = 0$

↑ eigenvector

$\bar{c}_1 = \bar{a}\times\bar{b}$ and $\bar{c}_2 = (\bar{a}\times\bar{b})\times\bar{b}$

$\bar{\bar{D}}\cdot\bar{c}_3 = (\alpha + \bar{a}\cdot\bar{b})\bar{c}_3$

$= (\alpha\bar{\bar{I}} + \bar{a}\bar{b})\cdot\bar{c}_3$

$= \alpha\bar{c}_3 + \bar{a}\,\bar{b}\cdot\bar{c}_3$

$\Rightarrow (\bar{a}\cdot\bar{b})\bar{c}_3 = \bar{a}(\bar{b}\cdot\bar{c})$

$\Rightarrow \bar{c}_3 = \dfrac{\bar{b}\cdot\bar{c}}{\bar{a}\cdot\bar{b}}\,\bar{a}$

eigenvector

(c) An electromagnetic material is

- **(bi-)isotropic** if $\overline{\overline{\epsilon}} = \epsilon\overline{\overline{\mathrm{I}}}$, $\overline{\overline{\mu}} = \mu\overline{\overline{\mathrm{I}}}$, $\overline{\overline{\xi}} = \xi\overline{\overline{\mathrm{I}}}$, $\overline{\overline{\zeta}} = \zeta\overline{\overline{\mathrm{I}}}$
  (isotropic if $\xi = \zeta = 0$ and bi-isotropic otherwise),
- **lossless** if $\overline{\overline{\epsilon}} = \overline{\overline{\epsilon}}^H$, $\overline{\overline{\mu}} = \overline{\overline{\mu}}^H$, $\overline{\overline{\xi}} = \overline{\overline{\zeta}}^H$, and
- **reciprocal** if $\overline{\overline{\epsilon}} = \overline{\overline{\epsilon}}^T$, $\overline{\overline{\mu}} = \overline{\overline{\mu}}^T$, $\overline{\overline{\xi}} = -\overline{\overline{\zeta}}^T$.

where $\overline{\overline{\mathrm{A}}}^T$ denotes the transpose and $\overline{\overline{\mathrm{A}}}^H$ the complex conjugate of the transpose $\overline{\overline{\mathrm{A}}}^{H*}$, of the dyadic $\overline{\overline{\mathrm{A}}}$.

i. $\overline{\overline{\epsilon}} = a\overline{\overline{\mathrm{I}}} + b(\mathbf{u}_x\mathbf{u}_y + \mathbf{u}_y\mathbf{u}_x)$, $\overline{\overline{\mu}} = \mu_0\overline{\overline{\mathrm{I}}}$, $\overline{\overline{\xi}} = \overline{\overline{\zeta}} = 0$
– anisotropic, lossless, reciprocal

ii. $\overline{\overline{\epsilon}} = a\overline{\overline{\mathrm{I}}}$, $\overline{\overline{\mu}} = b\overline{\overline{\mathrm{I}}} + c\mathbf{u}_x\mathbf{u}_y$, $\overline{\overline{\xi}} = \overline{\overline{\zeta}} = 0$
– anisotropic, lossy, non-reciprocal

iii. $\overline{\overline{\epsilon}} = a\overline{\overline{\mathrm{I}}}$, $\overline{\overline{\mu}} = b\overline{\overline{\mathrm{I}}}$, $\overline{\overline{\xi}} = c\overline{\overline{\mathrm{I}}}$, $\overline{\overline{\zeta}} = -c\overline{\overline{\mathrm{I}}}$
– bi-isotropic, lossy, reciprocal

iv. $\overline{\overline{\epsilon}} = a\overline{\overline{\mathrm{I}}}$, $\overline{\overline{\mu}} = b\overline{\overline{\mathrm{I}}}$, $\overline{\overline{\xi}} = \mathrm{j}\mathbf{d} \times \overline{\overline{\mathrm{I}}}$, $\overline{\overline{\zeta}} = \mathrm{j}\mathbf{d} \times \overline{\overline{\mathrm{I}}}$
– bi-anisotropic, lossless, reciprocal [Note that $(\mathbf{d} \times \overline{\overline{\mathrm{I}}})^T = -\mathbf{d} \times \overline{\overline{\mathrm{I}}}$.]

v. Hall effect:

$$\mathbf{J} = \sigma(\mathbf{E} + R\sigma\mathbf{E} \times \mathbf{B}_0)$$

The permittivity dyadic is:

$$\begin{aligned}
\nabla \times \mathbf{H} &= \mathbf{J} + \mathrm{j}\omega\epsilon\mathbf{E} = \sigma(\mathbf{E} + R\sigma\mathbf{E} \times \mathbf{B}_0) + \mathrm{j}\omega\epsilon\mathbf{E} \\
&= \left(\sigma\overline{\overline{\mathrm{I}}} - R\sigma^2\mathbf{B}_0 \times \overline{\overline{\mathrm{I}}}\right) \cdot \mathbf{E} + \mathrm{j}\omega\epsilon\mathbf{E} \\
&= \mathrm{j}\omega\left[\left(\epsilon - \mathrm{j}\frac{\sigma}{\omega}\right)\overline{\overline{\mathrm{I}}} + \mathrm{j}\frac{R\sigma^2}{\omega}\mathbf{B}_0 \times \overline{\overline{\mathrm{I}}}\right] \cdot \mathbf{E} = \mathrm{j}\omega\overline{\overline{\epsilon}} \cdot \mathbf{E} \\
\Rightarrow \quad \overline{\overline{\epsilon}} &= \left(\epsilon - \mathrm{j}\frac{\sigma}{\omega}\right)\overline{\overline{\mathrm{I}}} + \mathrm{j}\frac{R\sigma^2}{\omega}\mathbf{B}_0 \times \overline{\overline{\mathrm{I}}}
\end{aligned}$$

– anisotropic, lossy, non-reciprocal