

MS-C1650 Numeerinen analyysi, Exercise 2, Guidelines

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- These are not model solutions, but “getting started guidelines”.
- You are allowed, and supposed, to use MATLAB (or Octave/Python/whatever), unless otherwise stated. It is probably a good idea to bring a laptop with you to the exercise session.
- Ask questions at exercise sessions, by email, or at office M329 (afternoon is more likely)
- Errors and stupidities (in these notes): contact lauri.perkkio@aalto.fi

Exercise 1

The exercise wants us to interpolate the function f with a piecewise polynomial p . This means that we have two different polynomials for each interval, p_1 and p_2 . Third degree Hermite interpolation constructs a polynomial, which has the same value and first derivative at the end points of the interval as the original function f . There is a clear formula to construct such a polynomial, for example in the lecture notes section 4.4.:

$$p(x) = \frac{f'(a)}{h} \int_a^x (t-b)dt + \frac{f'(b)}{h} \int_a^x (t-b)dt + \alpha \int_a^x (t-a)(t-b)dt + C,$$

where the interval is $[a, b]$, $h = b - a$ (the length of the interval), $C = f(a)$ and

$$\alpha = \frac{3}{h^2} (f'(a) + f'(b)) + \frac{6}{h^3} (f(a) - f(b))$$

Exercise 2

In the first part of the exercise we must simply apply the functions $p(x)$ and $q(x)$ to all the x -values of the data points and then verify that the resulting values are the same as for $f(x)$ in the table.

For the second part read the uniqueness theorem of polynomial interpolation and its proof carefully. In the lecture notes it can be found in section 4.1. proposition 4.1.4. How are the number of interpolation points and the degree of the interpolation polynomial related to each others?

Exercise 3

Divided differences (fin. jaetut erotukset) is closely related to Newton interpolation. The following should be regarded only as quick guide for the exercise. The topic is covered in the lecture notes in section 4.2. with examples and more detailed explanations.

If we are interpolating data points $((x_0, y_0), (x_1, y_1), (x_2, y_2), \dots, (x_{n-1}, y_{n-1}))$ with Newton interpolation, we first construct a new basis functions for the polynomials, namely $1, x - x_0, (x - x_0)(x - x_1), \dots, \prod_{j=0}^{n-1} (x - x_j)$. Each basis function is the previous basis function multiplied by a first degree polynomial $(x - x_j)$, where x_j is the x-coordinate of previously unused data point (x_j, y_j) . Now every polynomial of degree $n - 1$ can be represented in this basis:

$$p_n(x) = a_0 + a_1(x - x_0) + \dots + a_{n-1} \prod_{j=0}^{n-1} (x - x_j),$$

where $a_i \in \mathbb{R}$ for all indices $i \in \{0, 1, 2, \dots, n - 1\}$.

Only problem left is to figure out all coefficients a_0, a_1, \dots, a_{n-1} that the interpolation polynomial should have. If the coefficient except a_0, a_1, a_2 and a_3 are all zeros, the resulting interpolation polynomial is clearly cubic at most and we are done with the exercise.

These coefficients can be calculated with divided differences. The divided difference of degree k is defined as

$$f[x_0, x_1, x_2, \dots, x_k] = a_k$$

and they can be calculated recursively with the formulas

$$f[x_0, x_1, x_2, \dots, x_k] = \frac{f[x_1, x_2, \dots, x_k] - f[x_0, x_1, x_2, \dots, x_{k-1}]}{x_k - x_0}$$
$$f[x_j] = f(x_j) = y_j \quad \forall j \in \{0, 1, 2, \dots, n - 1\}.$$

Thus every coefficient can be calculated recursively using these formulae.

Exercise 4

If we have used $n + 1$ points (x_0, x_1, \dots, x_n) on an interval $[a, b]$ to interpolate a function $f(x)$, the error of the interpolation polynomial p_n can be expressed as

$$f(\hat{x}) - p_n(\hat{x}) = \frac{1}{(n + 1)!} f^{(n+1)}(\zeta) \prod_{i=0}^n (\hat{x} - x_i), \quad (1)$$

where $\zeta \in [a, b]$.

Now we are interpolating the cosine function $f(x) = \cos(x)$ on the interval $[0, \pi/4]$ with second order polynomials $p_2(x)$. This means that $n = 2$ and we need $n + 1 = 3$ nearest points to \hat{x} to construct the interpolating polynomial.

To make sure that the absolute error $|f(x) - p_n(x)|$ stays below 10^{-5} , consider what is the maximum absolute value of the function $f^{(3)}(\zeta)$ in the interval and what is the maximum absolute value of the product $\prod_{i=0}^2 (x - x_i)$. Note that here the point x_i ($i = 0, 1, 2$) refer to the three closest point to the point \hat{x} .

MATLAB

a) Using a high order polynomial interpolant can oscillate (often an unwanted thing), if used carelessly.

b) Using Chebyshev interpolation points reduces the oscillation.

c) Barycentric weights. See Lecture notes 4.1.5, or Greenbaum & Chartier, or skim through this (especially formula 4.2)

<https://people.maths.ox.ac.uk/trefethen/barycentric.pdf>

CHALLENGE

A very ugly letter L, made in 5 minutes using MATLAB's spline (consists of 6 splines).

