MS-E1999 Special Topics in the Finite Element Method

Exercise 2.
Do (at least) 3 of the exercise below and hand them in on Thursday May 16.
Be also prepared to present the solution at the blackboard.

1. (Left from last week) Assume that $\Gamma_N = \emptyset$, and that the regularity estimate
$$\|u\|_2 \lesssim \|f\|_0$$
holds. Using the Nitsche trick and the Lagrange interpolation operator to show that
$$\|u - u_h\|_0 \lesssim \Big(\sum_{K \in \mathcal{C}_h} h_K^4 \|\Delta u_h + f\|_{0,K}^2 + \sum_{E \subset \Omega} h_E^3 \|[\![\frac{\partial u_h}{\partial n_E}]\!]\|_{0,E}^2 \Big)^{1/2}.$$

2. Let us recall the following notation. $A \lesssim B$ means: there exists a positive constant C, independent of the mesh size h (or the local mesh since h_K, such that $A \le CB$. With $A \approx B$ we mean $A \lesssim B$ and $B \lesssim A$.

   Let K be a triangle or tetrahedron, $P_k(K)$ the polynomials of degree k on K, and $b_K \in P_{d+1}(K)$ be the bubble function on K ($d =$ the space dimension.)

   Prove by scaling arguments that
$$\|v\|_{0,K} \approx \|b_K^{1/2} v\|_{0,K} \approx \|b_K v\|_{0,K} \quad \forall v \in P_k(K),$$
   and
$$\|\nabla v\|_{0,K} \lesssim h_K^{-1} \|v\|_{0,K} \quad \forall v \in P_k(K).$$

3. Read the section in the lecture notes where it is shown that the discrete linear system for Stokes is of the form:
$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}. \tag{0.1}$$
   $U \in \mathbb{R}^N$, $P \in \mathbb{R}^M$. Note that A is symmetric and positively definite.

   Show that this can be interpreted as the discrete optimisation problem: find U which minimises the object function
$$\frac{1}{2} V^T A V - F^T V \tag{0.2}$$
   subject to the linear constraint
$$B^T V = G. \tag{0.3}$$
   Show that the problem has a unique solution if, and only if, $N(B) = \{0\}$, or equvalently $R(B^T) = \mathbb{R}^M$, with N and R denoting the nullspace and range, respectively.

4. In the lectures we proved the stability of the lowest order Crouzeix-Raviart element, i.e. the FE pair

$$\begin{aligned} \boldsymbol{V}_h &= \{\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \,|\, \boldsymbol{v}|_K \in [P_2(K)]^2 \; K \in \mathcal{C}_h \,\}, \\ P_h &= \{\, q \in L_0^2(\Omega) \,|\, q|_K \in P_0(K) \; K \in \mathcal{C}_h \,\}. \end{aligned}$$

A common (miss)belief is the the same method works in 3D, i.e, $\Omega \subset I\!R^3$ and

$$\begin{aligned} \boldsymbol{V}_h &= \{\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \,|\, \boldsymbol{v}|_K \in [P_2(K)]^3 \; K \in \mathcal{C}_h \,\}, \\ P_h &= \{\, q \in L_0^2(\Omega) \,|\, q|_K \in P_0(K) \; K \in \mathcal{C}_h \,\}. \end{aligned} \tag{0.4}$$

Question: does the 2D proof of stability (or even uniqueness) carry over to 3D?

5. Let $\mathcal{C}_h$ be a partitioning into quadrilaterals and consider the Stokes pair

$$\begin{aligned} \boldsymbol{V}_h &= \{\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \,|\, \boldsymbol{v}|_K \in [Q_2(K)]^2 \; K \in \mathcal{C}_h \,\}, \\ P_h &= \{\, q \in L_0^2(\Omega) \,|\, q|_K \in P_1(K) \; K \in \mathcal{C}_h \,\}. \end{aligned}$$

Verify the stability. How is it with the method in 3D?

6. The lowest order quadrilateral Taylor-Hood method (continuous pressures) consists of the following spaces

$$\begin{aligned} \boldsymbol{V}_h &= \{\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \,|\, \boldsymbol{v}|_K \in [Q_2(K)]^2 \; K \in \mathcal{C}_h \,\}, \\ P_h &= \{\, q \in L_0^2(\Omega) \cap C(\Omega) \,|\, q|_K \in Q_1(K) \; K \in \mathcal{C}_h \,\}. \end{aligned}$$

with the mesh $\mathcal{C}_h$ consists of quadrilaterals. For the case of rectangles, prove the uniqueness of the solution. Hint: use a patch of two element and the fact that Simpson's rule is exact for cubic polynomials.