1. **Complex dielectric constant**

Note: ε_0 - vacuum permittivity, $\epsilon_0 = \epsilon(0) = \epsilon(\omega = 0)$.

Equation of motion:

$$\ddot{\boldsymbol{u}} + \gamma\dot{\boldsymbol{u}} = -\frac{k}{\mu}\boldsymbol{u} + \frac{q}{\mu}\boldsymbol{E}_{\text{loc}}, \tag{1}$$

where $\frac{k}{\mu} = \omega_0^2$.

By inserting the ansatz $\boldsymbol{u} = \boldsymbol{u}_0 e^{-i\omega t}$ (and implicitly $\boldsymbol{E}_{\text{loc}} = \boldsymbol{E}_0 e^{-i\omega t}$) we get

$$-\omega^2\boldsymbol{u} - i\omega\gamma\boldsymbol{u} = -\omega_0^2\boldsymbol{u} - \frac{q}{\mu}\boldsymbol{E}_{\text{loc}} \tag{2}$$

$$\Rightarrow \boldsymbol{u} = \frac{q}{\mu}\frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}\boldsymbol{E}_{\text{loc}}. \tag{3}$$

The polarization can be expressed microscopically as $\boldsymbol{P} = nq\boldsymbol{u} + n\alpha\boldsymbol{E}_{\text{loc}}$ and macroscopically as $\boldsymbol{P} = (\epsilon - 1)\varepsilon_0\boldsymbol{E}_{\text{mac}}$. Because we are ultimately interested in understanding the behaviour of a single oscillator, we assume a sparse (and isotropic and spherical) sample so that $\boldsymbol{E}_{\text{loc}} = \boldsymbol{E}_{\text{mac}} = \boldsymbol{E}_{\text{ext}} = \boldsymbol{E}$. Then, we get by setting "$\boldsymbol{P} = \boldsymbol{P}$"

$$nq\boldsymbol{u} + n\alpha\boldsymbol{E} = (\epsilon - 1)\varepsilon_0\boldsymbol{E} \tag{4}$$

$$\frac{nq^2}{\mu}\frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}\boldsymbol{E} + n\alpha\boldsymbol{E} = (\epsilon - 1)\varepsilon_0\boldsymbol{E} \tag{5}$$

$$\epsilon = 1 + \frac{nq^2}{\mu\varepsilon_0}\frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} + \frac{n\alpha}{\varepsilon_0} \tag{6}$$

Next, let's rewrite this in terms of ϵ_0 and ϵ_∞. By taking limits $\omega \to 0$ and $\omega \to \infty$ we get

$$\epsilon_0 = 1 + \frac{nq^2}{\mu\varepsilon_0}\frac{1}{\omega_0^2} + \frac{n\alpha}{\varepsilon_0} \tag{7}$$

and

$$\epsilon_\infty = 1 + \frac{n\alpha}{\varepsilon_0}, \tag{8}$$

respectively. Note that $\epsilon_0 - \epsilon_\infty = \frac{nq^2}{\mu\varepsilon_0}\frac{1}{\omega_0^2}$

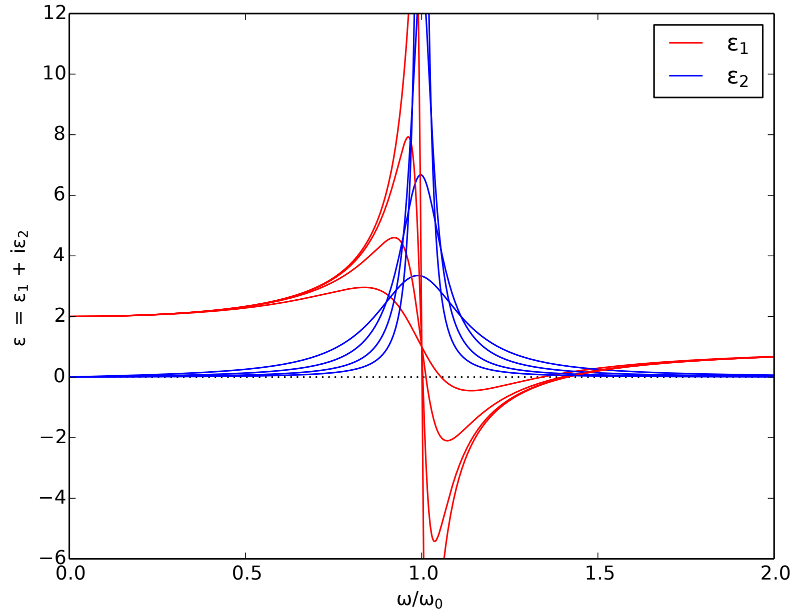
Thus, we get

$$\epsilon(\omega) = \epsilon_\infty + \frac{\omega_0^2(\epsilon_0 - \epsilon_\infty)}{\omega_0^2 - \omega^2 - i\gamma\omega}. \tag{9}$$

Now, let's separate the real and imaginary parts

$$\begin{aligned}\epsilon(\omega) &= \epsilon_\infty + \frac{\omega_0^2(\epsilon_0 - \epsilon_\infty)}{\omega_0^2 - \omega^2 - i\omega\gamma} \cdot \frac{\omega_0^2 - \omega^2 + i\omega\gamma}{\omega_0^2 - \omega^2 + i\omega\gamma} &(10)\\ &= \epsilon_\infty + \frac{(\epsilon_0 - \epsilon_\infty)\omega_0^2(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2} + i\frac{(\epsilon_0 - \epsilon_\infty)\omega_0^2\gamma\omega}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2} &(11)\\ &\equiv \epsilon_1(\omega) + i\epsilon_2(\omega). &(12)\end{aligned}$$

We note that $\epsilon(\omega)$ has divergence when $\gamma \to 0$. Thus, when calculating the peak of ϵ_2 and the zeros of ϵ_1, one needs to first evaluate the condition for peak/zero and only after that take the limit $\gamma \to 0$. This leads to long but straightforward algebra, which we can avoid by remembering to treat the divergence specially. Figure tells more than thousand words, so see below the plot of $\epsilon(\omega)$ with $\epsilon_0 = 2$, $\epsilon_\infty = 1$, and different damping factors γ (peak gets thinner and the divergence develops as $\gamma \to 0$).



Peak of ϵ_2, small damping limit:

When $\gamma \to 0$,

$$\epsilon_2 \to \frac{(\epsilon_0 - \epsilon_\infty)\omega_0^2 \gamma \omega}{(\omega_0^2 - \omega^2)^2}. \tag{13}$$

The peak corresponds to the divergence at $\omega = \omega_0 \equiv \omega_T$.

Zeros of ϵ_1, small damping limit:

When $\gamma \to 0$,

$$\epsilon_1 \to \epsilon_\infty + \frac{(\epsilon_0 - \epsilon_\infty)\omega_0^2}{\omega_0^2 - \omega^2}. \tag{14}$$

This equals to zero when $\omega^2 = \frac{\epsilon_0}{\epsilon_\infty}\omega_0^2 \equiv \omega_L^2$. Additionally, ϵ_1 diverges at ω_0 and changes sign. Thus, there is a zero at $\omega \approx \omega_0$ when γ is non-zero though small (see the figure above).

2. **Plasma frequency in atomic units**

In atomic units, the numeric values of the following physical quantities are set to unity:

$$m_e = e = \hbar = \frac{1}{4\pi\epsilon_0} = 1. \tag{15}$$

The units are can be marked, e.g. in the case of mass, as m_e, a.u., or nothing, with the latter two somewhat ambiguous. It can be inferred, that length is in the units of Bohr radius ($a_0 \approx 0.529$ Å) and energy in Hartree (Ha ≈ 27.211 eV).

The Wigner-Seitz radius r_s is defined by equating the mean volume per particle (here, electron) in a system to a volume of sphere:

$$\frac{4}{3}\pi r_s^3 = \frac{V}{N} = \frac{1}{n}. \tag{16}$$

The plasma frequency is

$$\omega_p = \sqrt{\frac{ne^2}{m_e\varepsilon_0}} = \sqrt{\frac{3e^2}{4\pi r_s^3 m_e \varepsilon_0}} \tag{17}$$

in SI units [rad/s], or, in atomic units

$$\omega_p = \sqrt{\frac{3}{r_s^3}} \tag{18}$$

In silver $r_s = 3.02$ (a_0), and thus $\omega_p = 0.33$ (Ha/$\hbar$) or in the units of energy $\hbar\omega_p = 0.33$ Ha $= 8.98$ eV.

3. **Magnetization**

(a) Let the field be aligned in the z direction, i.e., $\boldsymbol{B} = (0, 0, B_z)$. For the given $J = 1/2$, we have $J_z = \pm\frac{1}{2}$, i.e., the ion has two possible states. The magnetic moment of the ion is $m_z = -g\mu_B J_z = \mp\mu_B$ and the corresponding energy is $E = -\boldsymbol{m}\cdot\boldsymbol{B} = -m_z B_z = \pm\mu_B B_z$.
The probability of a state at a given temperature T is given by the Boltzmann factor $P(E) = e^{-E/k_BT}$. Thus, the thermal average is

$$\langle m_z\rangle = \frac{-\mu_B e^{-\mu_B B_z/k_BT} + \mu_B e^{\mu_B B_z/k_BT}}{e^{-\mu_B B_z/k_BT} + e^{\mu_B B_z/k_BT}} \tag{19}$$

where the denominator normalizes the probabilities. The exponential factors can be rewritten as

$$\langle m_z\rangle = \mu_B \tanh\left(\frac{\mu_B B_z}{k_BT}\right) \tag{20}$$

to yield the magnetization

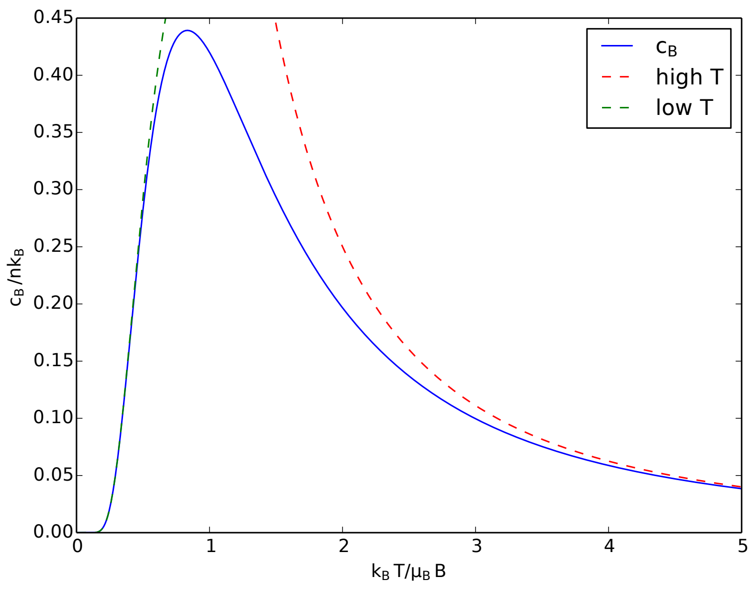
$$M = n\langle m_z\rangle = n\mu_B \tanh\left(\frac{\mu_B B_z}{k_BT}\right), \tag{21}$$

where n is the density of ions.

(b) The internal energy per unit volume is $u = -\boldsymbol{M}\cdot\boldsymbol{B} = -MB_z = -n\mu_B\tanh(\mu_B B_z/k_BT)B_z$. The heat capacity per unit volume in a constant magnetic field B is

$$c_B = \left(\frac{\partial u}{\partial T}\right)_B = nk_B\left(\frac{\mu_B B_z}{k_BT}\right)^2 \frac{1}{\cosh^2\left(\frac{\mu_B B_z}{k_BT}\right)}. \tag{22}$$

See below a plot of c_B and the limiting forms.



By writing $\cosh x$ in terms of exponential functions and using series expansions for them, one gets limiting forms

$$\cosh^2 x = \begin{cases} \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\right)^2 \approx 1, & x \ll 1 \\ \frac{1}{4}(e^x + e^{-x})^2 \approx \frac{1}{4}e^{2x}, & x \gg 1 \end{cases}. \tag{23}$$

Thus

$$c_B \approx \begin{cases} nk_B\left(\frac{\mu_B B_z}{k_BT}\right)^2, & T \gg \mu_B B_z/k_B \text{ (high T)} \\ 4nk_B\left(\frac{\mu_B B_z}{k_BT}\right)^2 e^{-2\mu_B B_z/k_BT}, & T \ll \mu_B B_z/k_B \text{ (low T)} \end{cases}. \tag{24}$$

Let's calculate the temperature at the peak of $c_B(T)$. The derivative at zero gives ($x \equiv \mu_B B_z / k_B T$)

$$\frac{d}{dx}\left(\frac{x^2}{\cosh^2 x}\right) = \frac{2x\cosh x - 2x^2 \sinh x}{\cosh^3 x} = 0 \tag{25}$$

from which

$$x = 0 \quad \text{or} \quad x\sinh x = \cosh x \quad \Rightarrow x \approx 1.2 \text{ (numerically)}. \tag{26}$$

Thus, c_B has the maximum value at $T \approx \mu_B B_z / 1.2 k_B$, so more accurate conditions for high and low temperature are given by $T \gg \mu_B B_z / 1.2 k_B$ and $T \ll \mu_B B_z / 1.2 k_B$, respectively. Thus, with $B_z = 0.5$ T, we get high temperature regime $T \gg 0.28$ K.