PHYS-E0421 Solid-State Physics (5 cr)
Exercise set 6
Model solutions

Dielectric properties of solids
Spring 2019

1. **Depolarization field** ($\sim$ Elliot 7.1 b)

   The depolarization field is of form

$$\boldsymbol{E}_1 = -\frac{N_i}{\varepsilon_0}\boldsymbol{P}, \tag{1}$$

   when the external field $\boldsymbol{E}_{\text{ext}}$ is in the direction $i \in \{x, y, z\}$. The depolarization factors N_i fulfill sum rule $N_x + N_y + N_z = 1$.

   (i) The sphere is symmetric in x, y, and z directions. The depolarization field must follow the same symmetry. Thus $N_x = N_y = N_z = 1/3$ in order to the sum rule to be fulfilled.

   (ii) Assume that the long cylinder ("infinitely long and thin cylinder“) is aligned along z axis. Then, the depolarization field in z direction must be vanishing due to the neglible areas of the cylinder ends (and their long distance). Thus $N_z = 0$, whence $N_x + N_y = 1$. In x and y directions the cylinder is symmetric, so $N_x = N_y = 1/2$ in order to the sum rule to be fulfilled.

   (iii) Assume that the thin disc ("infinitely large and thin disc“) is on a (x, y) plane. Due to the similar reasoning as above, it must be that $N_x = N_y = 0$. Thus $N_z = 1$.

2. **Orientational polarizability** ($\sim$ Elliott 7.3)

   (a) Interaction energy of the dipole moment with the local electric field is

$$U = -\boldsymbol{p} \cdot \boldsymbol{E}_{\text{loc}} = -p\cos\theta E_{\text{loc}}, \tag{2}$$

   where θ is the angle between the dipole and the electric field.
   Probability of finding the dipole at angle θ is given by the Boltzmann factor:

$$P(\theta) = Ne^{-U/k_BT} = Ne^{x\cos\theta}, \tag{3}$$

   where N is the normalization constant and we have used a short-hand $x = pE_{\text{loc}}/k_BT$. The dipole's parallel component to the electric field is $p\cos\theta$, and its thermal average is given by

$$p_{\text{par}} = \langle p\cos\theta\rangle = \frac{\int p\cos\theta P(\theta)dA}{\int P(\theta)dA}. \tag{4}$$

   The integral is over the whole configuration space of the dipole. Now, the dipole is able to adopt any orientation (but its length is fixed), so that the configuration space is the surface of a sphere. Thus, in the present case the integral $\int dA$ can be presented in spherical coordinates as

$$\int dA[\cdot] = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta[\cdot] = \int_0^{\pi} 2\pi\sin\theta d\theta[\cdot], \tag{5}$$

   where the last equality results from the fact that the integrand does not depend on the angle φ in the present case. Thus, we get

$$p_{\text{par}} = \frac{\int_0^{\pi} p\cos\theta e^{x\cos\theta} 2\pi\sin\theta d\theta}{\int_0^{\pi} e^{x\cos\theta} 2\pi\sin\theta d\theta}. \tag{6}$$

Let's make a change of variable $t \equiv x\cos\theta$

$$p_{\text{par}} = \frac{p}{x}\frac{\int_{-x}^{x} te^t dt}{\int_{-x}^{x} e^t dt}. \tag{7}$$

Integration by parts gives

$$\begin{aligned} p_{\text{par}} &= \frac{p}{x}\frac{\left[te^t\right]_{t=-x}^{t=x} - \int_{-x}^{x} e^t dt}{\int_{-x}^{x} e^t dt} & (8)\\ &= \frac{p}{x}\left(x\frac{e^x+e^{-x}}{e^x-e^{-x}} - 1\right) & (9)\\ &= p\left(\coth(x) - \frac{1}{x}\right) = pL(x), & (10)\end{aligned}$$

by the definition of the Langevin function $L(x)$.

(b) With the given values, $x \ll 1$, and then

$$\coth(x) \approx \frac{1}{x} + \frac{x}{3}, \tag{11}$$

so that

$$p_{\text{par}} = \frac{px}{3} = \frac{p^2 E_{\text{loc}}}{3k_B T}. \tag{12}$$

(c) Now the dipole is able to adopt only two orientations: parallel ($\theta = 0$) and antiparallel ($\theta = \pi$). Thus, in this case the integral in the thermal average Eq. (4) reduces to a sum over all the possible configurations, and we get

$$p_{\text{par}} = \langle p\cos\theta\rangle = p\frac{\cos 0\, P(0) + \cos\pi P(\pi)}{P(0) + P(\pi)} = p\frac{e^x - e^{-x}}{e^x + e^{-x}} = p\tanh(x) \approx px = \frac{p^2 E_{\text{loc}}}{k_B T}, \tag{13}$$

where "$\approx$" is evaluated in the limit $x \ll 1$.

3. **Reflectivity** ($\sim$ Elliott 4.13)

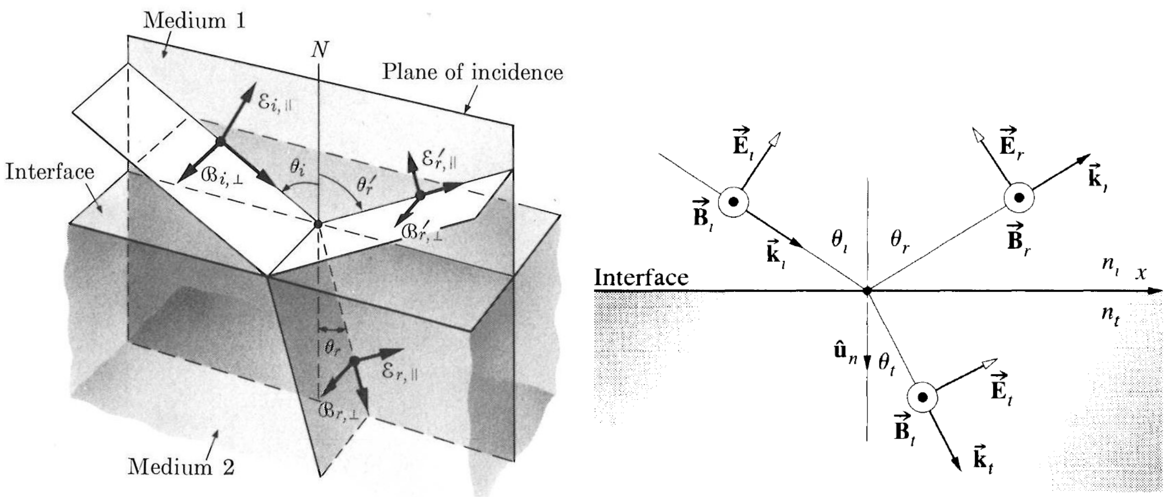

Figure 1: A p-polarized light reflecting and transmitting through an interface.

(a) Consider a p-polarized light wave passing from a medium with complex refractive index n_1 into a medium with index n_2, see Fig. 1. The values of E and $H = nE/\mu_0 c$ (we assume that for the materials $\mu = \mu_0$) of the electromagnetic wave parallel to the interface are continuous across the boundary. This gives equations

$$\begin{cases} E_i\cos\theta_i - E_r\cos\theta_i = E_t\cos\theta_t \\ H_i + H_r = H_t \end{cases} \Rightarrow \begin{cases} E_i\cos\theta_i - E_r\cos\theta_i = E_t\cos\theta_t \\ n_1 E_i + n_1 E_r = n_2 E_t \end{cases}. \tag{14}$$

The reflectivity and transmission are defined as $r = E_r/E_i$ and $t = E_t/E_i$, respectively. Thus, we get

$$\begin{cases} \cos\theta_i(1-r) = t\cos\theta_t \\ n_1(1+r) = n_2 t \end{cases} . \tag{15}$$

From this pair of equations we can solve r and t to obtain

$$r = \frac{n_2\cos\theta_i - n_1\cos\theta_t}{n_1\cos\theta_t + n_2\cos\theta_i} \tag{16}$$

and

$$t = \frac{2n_1\cos\theta_i}{n_1\cos\theta_t + n_2\cos\theta_i}. \tag{17}$$

These are Fresnel equations for p-polarized light. (Note that the signs depend on the choice of the directions of the vectors, see Fig. 1. For reflectivity and transmission factors for *intensities* there is no such dependence.)

(b) The asked factors t_1, t_2, and r are illustrated in Fig. 2.

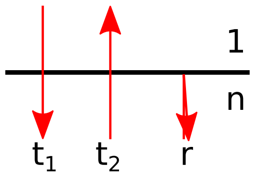

Figure 2: Interface between materials with refractive indices 1 and $n = n_r + i\kappa_i$, and transmission and reflection factors for normal incidence.

By using the results of (a) with $\theta_i = \theta_t = 0$ and $n_{1/2} = 1$ and $n_{2/1} = n$ dependening on the direction of the light, we get $t_1 = 2/(n+1)$, $t_2 = 2n/(n+1)$, and $r = -(n-1)/(n+1)$. For the intensity $I = EE^* = |E|^2$, we have

$$R = \frac{I_r}{I_i} = \frac{|E_r|^2}{|E_i|^2} = |r|^2 = \frac{|n-1|}{|n+1|} = \frac{(n_r-1)^2+\kappa_i^2}{(n_r+1)^2+\kappa_i^2}. \tag{18}$$

(c) The propagation of light is decribed by the plane wave

$$E_i = E_0 e^{i\omega\left(\frac{nx}{c}-t\right)}, \tag{19}$$

so that the amplitude of the light is absorbed by factor

$$e^{i\frac{n\omega d}{c}} \tag{20}$$

during the propagation through the medium of thickness d. The light can travel through a medium after a series of reflections within the medium as illustrated in Fig. 3.

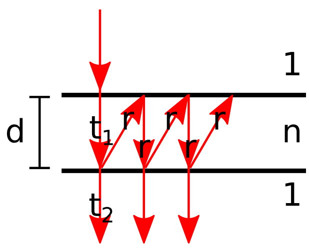

Figure 3: Slab of thickness d and light transmitting through. Note that the reflected waves are tilted for visual convenience.

Thus, the total transmitted light is a sum of all these components:

$$E_t = \left[E_0 t_1 e^{in\omega d/c} t_2\right] + \left[E_0 t_1 e^{in\omega d/c} r e^{in\omega d/c} r e^{in\omega d/c} t_2\right] \\ + \left[E_0 t_1 e^{in\omega d/c} r e^{in\omega d/c} r e^{in\omega d/c} r e^{in\omega d/c} r e^{in\omega d/c} t_2\right] + \dots \tag{21}$$

$$= E_0 t_1 t_2 e^{in\omega d/c} \sum_{m=0}^{\infty} \left(r^2 e^{2in\omega d/c}\right)^m \tag{22}$$

$$= E_0 t_1 t_2 \frac{e^{in\omega d/c}}{1 - r^2 e^{2in\omega d/c}}, \tag{23}$$

where we the sum was evaluated as a geometric sum ($|re^{in\omega d/c}|^2 = |r|^2 \, |e^{-\kappa_i \omega d/c}|^2 < 1$, $\kappa_i > 0$).

For an optically thin sample ($n \sim 1+i0$), we have $t_1 t_2 \sim 1$ and $r^2 \sim 0$ (we approximate the polynomial terms and keep only the dominant exponential term). Then $E_t \sim E_0 e^{in\omega d/c}$, and $I_t = |E_t|^2 \sim I_0 e^{-2\kappa_i \omega d/c} = I_0 e^{-K(\omega) d}$, where $K(\omega) = 2\omega\kappa_i/c$.