MS-E1999 Special Topics in the Finite Element Method
Lecture notes 2019

Rolf Stenberg and Juha Videman

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Contents

1 Notation

The notes have been written during a couple of years, and hence the notation might change from one section to another. However, I will list the main notation, and I will also try to homogenise the notation.

The domain $\Omega \subset \mathbb{R}^d$ is most often assumed to be a polygonal ($d = 2$) or polyhedral domain ($d = 3$). The finite partitioning $\mathcal{C}_h$ consists of triangles/tetrahedron or quadrilaterals/hexahedrons. An element is denoted by K and an edge/face by E. With $P_k(K)$ we denote the polynomials of degree k on K. For K a quadrilateral/hexahedron we use the polynomial space Q_k defined as follows. Let $\hat{K}$ be the reference element, i.e. $\hat{K} = [0,1]^d$ and let $\boldsymbol{F}_K$ be the bi/tri-linear mapping from $\hat{K}$ onto K. On $\hat{K}$ we define

$$Q_k(\hat{K}) = \{\, v \,|\, v(\hat{x}_1, \hat{x}_2) = \sum_{i=0}^{k}\sum_{j=0}^{k} a_{ij}\hat{x}_1^i \hat{x}_2^j,\ a_{ij} \in I\!R \,\}. \tag{1.1}$$

for $d = 2$, and

$$Q_k(\hat{K}) = \{\, v \,|\, v(\hat{x}_1, \hat{x}_2, \hat{x}_3) = \sum_{i=0}^{k}\sum_{j=0}^{k}\sum_{l=0}^{k} a_{ijl}\hat{x}_1^i \hat{x}_2^j \hat{x}_3^l,\ a_{ijl} \in I\!R \,\}. \tag{1.2}$$

for $d = 3$. Using these, we define

$$Q_k(\hat{K}) = \{\, v \,|\, v(\boldsymbol{x}) = \hat{v}(\boldsymbol{F}_K^{-1}(\boldsymbol{x}))\ \hat{v} \in Q_k(\hat{K}) \,\}. \tag{1.3}$$

In finite element analysis we encounter statements as: there exist a constant $C > 0$, indepent of the mesh size h and the solution u, such that $A \leq CB$. Statement like this we will often write as $A \lesssim B$. Likewise, a statement, there exists C_1, $C_2 > 0$ such that ..., is written as $A \lesssim B \lesssim D$.

For Sobolev space we use the following notation for the norms: $\|\cdot\|_{0,D}$ for the $L^2(D)$ - norm, and $\|\cdot\|_{k,D}$, $|\cdot|_{k,D}$ for the $H^1(D)$ norm and seminorm. The subscript D is dropped when $D = \Omega$.

2 Interpolation of non smooth functions. A posteriori error analysis

2.1 The Clément interpolation operator

We denote

$$H_D^1(\Omega) = \{\, v \in H^1(\Omega) \,|\, v|_{\Gamma_D} = 0 \,\} \tag{2.1}$$

and

$$V_h^1 = \{\, v \in H_D^1(\Omega) \,|\, v|_K \in P_1(K)\ \forall K \in \mathcal{C}_h \,\}. \tag{2.2}$$

Denote the vertices of the elements (i.e. the nodes used in V_h^1) by x_j, $j = 1, 2, \ldots, N$, and let

$$h_j = \max_{x_j \in K \in \mathcal{C}_h} h_K. \tag{2.3}$$

By ω_j we denote the set (draw figure to see what I mean)

$$\omega_j = \{\ x \in \overline{\Omega}\ |\ \|x - x_j\| \leq h_j\ \}. \tag{2.4}$$

By $\tilde{\omega}_K$ we denote

$$\tilde{\omega}_K = \cup\{\, \omega_j \,|\, x_j \in K \,\}. \tag{2.5}$$

If $x_j \notin \Gamma_D$ we by $\mathcal{P}_j v$ denote the value of the L^2-projection of $v \in L^2(\omega_j)$ onto $P_0(\omega_j)$ (=the constant functions on ω_j), i.e.

$$\mathcal{P}_j v = \frac{1}{\text{meas}(\omega_j)} \int_{\omega_j} v(x)\, dx. \tag{2.6}$$

When $x_j \in \Gamma_D$ we let $\mathcal{P}_j v = 0$.

Denote by $\varphi_j, j = 1, 2, \ldots, N$, the basis functions of V_h^1 (the Courant hat functions). A simple Clément interpolation operator $I_h : H_D^1(\Omega) \to V_h^1$ can now be defined as

$$I_h v = \sum_{i=1}^{N} \mathcal{P}_j v\, \varphi_j. \tag{2.7}$$

It holds

Theorem 2.1. *It holds*

$$\Big(\sum_{K \in \mathcal{C}_h} \{h_K^{-2} ||v - I_h v||_{0,K}^2 + h_K^{-1} ||v - I_h v||_{0,\partial K}^2\}\Big)^{1/2} \lesssim |v|_1. \tag{2.8}$$

The steps of the *Proof*: First the Bramble-Hilbert technique gives

$$||v - \mathcal{P}_j v||_{0,K} \lesssim h_K |v|_{1,\omega_j} \tag{2.9}$$

and

$$||v - \mathcal{P}_j v||_{0,\partial K} \lesssim h_K^{1/2} |v|_{1,\omega_j}, \tag{2.10}$$

for $K \subset \omega_j$. Let x_1, x_2, x_3 be the nodes of K. It holds $\sum_{j=1}^{3} \varphi_j = 1$. Hence, we can write

$$(v - I_h v)_{|K} = \sum_{j=1}^{3} v\varphi_j - \sum_{j=1}^{3} (\mathcal{P}_j v)\varphi_j = \sum_{j=1}^{3} (v - \mathcal{P}_j v)\varphi_j \tag{2.11}$$

Using the triangle inequality we have

$$\|v - I_h v\|_{0,K} \leq \sum_{j=1}^{3} \|(v - \mathcal{P}_j v)\varphi_j\|_{0,K}. \tag{2.12}$$

Next, since $|\varphi_j| \leq 1$, we have

$$\|(v - \mathcal{P}_j v)\varphi_j\|_{0,K} \leq \|v - \mathcal{P}_j v\|_{0,K}.$$

We thus have

$$\|v - I_h v\|_{0,K} \leq \sum_{j=1}^{3} \|v - \mathcal{P}_j v\|_{0,K}. \tag{2.13}$$

Hence, from (2.9) it follows it follows that

$$||v - I_h v||_{0,K} \lesssim h_K |v|_{1,\tilde{\omega}_K}. \tag{2.14}$$

Using this and the scaled trace inequality (equation (2.17) below) we have

$$||v - I_h v||_{0,\partial K} \lesssim h_K^{1/2} |v|_{1,\tilde{\omega}_K}. \tag{2.15}$$

The asserted estimate then follows by adding the above estimates and noting that there exists C_1, C_2 such that

$$\sum_{K\in\mathcal{C}_h} \int_K w^2\,dx \lesssim \sum_{K\in\mathcal{C}_h} \int_{\tilde{\omega}_K} w^2\,dx \lesssim \sum_{K\in\mathcal{C}_h} \int_K w^2\,dx. \quad \Box \tag{2.16}$$

Lemma 2.1. *For $v \in H^1(K)$ it holds*

$$\|v\|^2_{0,\partial K} \lesssim (h_K^{-1}\|v\|^2_{0,K} + h_K\|\nabla v\|^2_{0,K}). \tag{2.17}$$

Exercise 2.1. *Prove this result in one dimension.*

Exercise 2.2. *Show by scaling that the result follows from the corresponding inequality on the reference element.*

2.2 A posteriori error analysis

We will treat the Poisson problem; find $u \in H^1(\Omega)$ such that

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \Gamma_D, \\ \frac{\partial u}{\partial n} &= g \text{ on } \Gamma_N,\ \Gamma_D \cup \Gamma_N = \partial\Omega. \end{aligned} \tag{2.18}$$

The variational form and the FE method are

$$(\nabla u, \nabla v) = (f, v) + \langle g, v\rangle_{\Gamma_N} \quad \forall v \in H^1_D(\Omega), \tag{2.19}$$

and

$$(\nabla u_h, \nabla v) = (f, v) + \langle g, v\rangle_{\Gamma_N} \quad \forall v \in V_h, \tag{2.20}$$

respectively.

Next, we define the error indicators. By E we denote an edge/face of an element in the mesh. The "jump" in the normal derivative (flux) on an edge in the interior of the domain we denote by

$$[\![\frac{\partial v}{\partial n_E}]\!] = \frac{\partial v}{\partial n_K} + \frac{\partial v}{\partial n_{K'}}, \tag{2.21}$$

with $E = \partial K \cap \partial K'$, K, $K' \in \mathcal{C}_h$. The local indicators are

$$\eta_K = h_K\|\Delta u_h + f\|_{0,K} \tag{2.22}$$

for $K \in \mathcal{C}_h$,

$$\eta_{E,\Omega} = h_E^{1/2}\|[\![\frac{\partial v}{\partial n_E}]\!]\|_{0,E} \tag{2.23}$$

for an edge/face in Ω,[1] and

$$\eta_{E,N} = h_E^{1/2} \| \frac{\partial v}{\partial n} - g \|_{0,E} \tag{2.24}$$

for $E \subset \Gamma_N$. The error indicator is then defined as

$$\eta^2 = \sum_{K \in \mathcal{C}_h} \eta_K^2 + \sum_{E \subset \Omega} (\eta_{E,\Omega})^2 + \sum_{E \subset \Gamma_N} (\eta_{E,N})^2. \tag{2.25}$$

The a posteriori estimate is the following.

Theorem 2.2. *It holds*

$$\|\nabla u - \nabla u_h\|_0 \lesssim \eta. \tag{2.26}$$

Proof. Let us use the shorthand notation $e = u - u_h$. Let $I_h e$ be the Clément interpolant to e. By the Galerkin orthogonality it holds $(\nabla e, \nabla I_h e) = 0$. Hence we obtain

$$\begin{aligned}
\|\nabla e\|_0^2 &= (\nabla e, \nabla e) = (\nabla e, \nabla (e - I_h e)) \\
&= (\nabla u, \nabla (e - I_h e)) - (\nabla u_h, \nabla (e - I_h e)) \\
&= (f, e - I_h e) + \langle g, e - I_h e \rangle_{\Gamma_N} - (\nabla u_h, \nabla (e - I_h e)) \\
&= \sum_{K \in \mathcal{C}_h} (f, e - I_h e)_K - \sum_{K \in \mathcal{C}_h} (\nabla u_h, \nabla (e - I_h e))_K + \langle g, e - I_h e \rangle_{\Gamma_N} \\
&= \sum_{K \in \mathcal{C}_h} (f, e - I_h e)_K + \sum_{K \in \mathcal{C}_h} (\Delta u_h, e - I_h e)_K \\
&\quad + \langle g, e - I_h e \rangle_{\Gamma_N} - \sum_{K \in \mathcal{C}_h} \langle \nabla u_h \cdot \boldsymbol{n}_K, e - I_h e \rangle_{\partial K}.
\end{aligned} \tag{2.27}$$

For the two first terms we use Schwarz inequality for inner products, Cauchy's inequality for sums, and the Clément interpolation estimate

$$\begin{aligned}
&\sum_{K \in \mathcal{C}_h} (f, e - I_h e)_K + \sum_{K \in \mathcal{C}_h} (\Delta u_h, e - I_h e)_K \\
&= \sum_{K \in \mathcal{C}_h} (\Delta u_h + f, e - I_h e)_K \leq \sum_{K \in \mathcal{C}_h} \|\Delta u_h + f\|_{0,K} \|e - I_h e\|_{0,K} \\
&= \sum_{K \in \mathcal{C}_h} h_K \|\Delta u_h + f\|_{0,K} \cdot h_K^{-1} \|e - I_h e\|_{0,K} \\
&\leq \sum_{K \in \mathcal{C}_h} h_K \|\Delta u_h + f\|_{0,K} \cdot h_K^{-1} \|e - I_h e\|_{0,K} \\
&\leq \Big(\sum_{K \in \mathcal{C}_h} h_K^2 \|\Delta u_h + f\|_{0,K}^2 \Big)^{1/2} \Big(\sum_{K \in \mathcal{C}_h} h_K^{-2} \|e - I_h e\|_{0,K}^2 \Big)^{1/2} \\
&\lesssim \eta \|\nabla e\|_0.
\end{aligned} \tag{2.28}$$

[1] As usual Ω is assumed to be open and that means that E is an interior edge/face.

In the same way, rearranging terms and choosing the directions for the normals, we get

$$
\begin{aligned}
&\langle g, e - I_h e\rangle_{\Gamma_N} - \sum_{K\in\mathcal{C}_h} \langle \nabla u_h \cdot \boldsymbol{n}_K, e - I_h e\rangle_{\partial K} \\
&= \sum_{E\subset\Omega} \langle [\![\frac{\partial u_h}{\partial n_E}]\!], e - I_h e\rangle_E + \sum_{E\subset\Gamma_N} \langle g - \nabla u_h \cdot \boldsymbol{n}_E, e - I_h e\rangle_E \\
&\le \sum_{E\subset\Omega} \| [\![\frac{\partial u_h}{\partial n_E}]\!] \|_{0,E}\, \|e - I_h e\|_{0,E} + \sum_{E\subset\Gamma_N} \|g - \nabla u_h \cdot \boldsymbol{n}_E\|_{0,E} \|e - I_h e\|_{0,E} \\
&\le \Big(\sum_{E\subset\Omega} h_E \| [\![\frac{\partial u_h}{\partial n_E}]\!] \|_{0,E}^2 \Big)^{1/2} \Big(\sum_{E\subset\Omega} h_E^{-1} \|e - I_h e\|_{0,E}^2 \Big)^{1/2} \\
&+ \Big(\sum_{E\subset\Gamma_N} h_E \|g - \nabla u_h \cdot \boldsymbol{n}_E\|_{0,E}^2 \Big)^{1/2} \Big(\sum_{E\subset\Gamma_N} h_E^{-1} \|e - I_h e\|_{0,E}^2 \Big)^{1/2} \\
&\lesssim \eta \|\nabla e\|_0.
\end{aligned}
\tag{2.29}
$$

Combining the above estimates proves the claim. □

Exercise 2.3. *Consider the problem with a diffusion coefficient: find $u \in H^1(\Omega)$ such that*

$$
\begin{aligned}
-\mathrm{div}\,(k\nabla u) &= f \text{ in } \Omega, \\
u &= 0 \text{ on } \Gamma_D, \\
k\frac{\partial u}{\partial n} &= g \text{ on } \Gamma_N,\ \Gamma_D \cup \Gamma_N = \partial\Omega.
\end{aligned}
\tag{2.30}
$$

How is the error estimator now defined?

We introduce f_h and g_h as piecewise polynomial interpolants to f and g, respectively. The efficiency of the error indicators are given by the following theorem. By b_K we denote the "bubble function" on K, i.e. the function in $P_{d+1}(K) \cap H_0^1(K)$ taking the value one at the midpoint.

Theorem 2.3. *For every $v \in V_h$ it holds*

$$
h_K \|\Delta v + f\|_{0,K} \lesssim \|\nabla u - \nabla v\|_{0,K} + h_K \|f - f_h\|_{0,K} \quad \forall K \in \mathcal{C}_h. \tag{2.31}
$$

Proof. Define w by $w|_K = b_K(\Delta v + f_h)$, $w = 0$ in $\Omega \setminus K$. By scaling we have

$$
\|\Delta v + f_h\|_{0,K} \lesssim \|b_K^{1/2}(\Delta v + f_h)\|_{0,K}.
$$

Then we have

$$
\begin{aligned}
\|\Delta v + f_h\|_{0,K}^2 &\lesssim \|b_K^{1/2}(\Delta v + f_h)\|_{0,K}^2 \\
&= (\Delta v + f_h, w)_K = (\Delta v + f, w)_K + (f_h - f, w)_K \\
&= (\nabla(u - v), \nabla w)_K + (f_h - f, w)_K \\
&\le \|\nabla(u - v)\|_{0,K} \|\nabla w\|_{0,K} + \|f_h - f\|_{0,K} \|w\|_{0,K} \\
&\lesssim (h_K^{-1} \|\nabla(u - v)\|_{0,K} + \|f_h - f\|_{0,K}) \|w\|_{0,K}
\end{aligned}
\tag{2.32}
$$

□

For an edge E, we denote ω_E the union of the elements in $\mathcal{C}_h$ that contain E. We have:

Theorem 2.4. *For every $v \in V_h$ it holds*

$$h_E^{1/2}\|[\![\frac{\partial v}{\partial n_E}]\!]\|_{0,E} \lesssim \|\nabla u - \nabla v\|_{0,\omega_E} + \sum_{K\subset\omega_E} h_K\|f - f_h\|_{0,K} \quad \forall E \subset \Omega. \tag{2.33}$$

Proof. For a function w, with $\operatorname{supp} w = \omega_E$, it holds

$$\begin{aligned}(\nabla(u-v), \nabla w)_{\omega_E} &= \sum_{K\subset\omega_E} (\nabla(u-v), \nabla w)_K \\ &= \sum_{K\subset\omega_E} \Big((f, w)_K - (\nabla v, \nabla w)_K\Big) \\ &= \sum_{K\subset\omega_E} \Big((f, w)_K + (\Delta v, w)_K\Big) - \langle[\![\frac{\partial v}{\partial n_E}]\!], w\rangle_E.\end{aligned} \tag{2.34}$$

To define w, we proceed as follows. First, we extend $[\![\frac{\partial v}{\partial n_E}]\!]$ to the whole of ω_E, so that the extension $E([\![\frac{\partial v}{\partial n_E}]\!])$ is constant in the direction perpendicular to E. Now let b_E be the edge/face bubble function vanishing on $\partial\omega_E$ and attaining the value one at the midpoint of E. We then let

$$w = b_E E([\![\frac{\partial v}{\partial n_E}]\!]). \tag{2.35}$$

Again, by scaling it holds

$$\|[\![\frac{\partial v}{\partial n_E}]\!]\|_{0,E}^2 \lesssim \|b_E^{1/2}[\![\frac{\partial v}{\partial n_E}]\!]\|_{0,E}^2 = \langle[\![\frac{\partial v}{\partial n_E}]\!], w\rangle_E. \tag{2.36}$$

Combining with (2.34) yields

$$\|[\![\frac{\partial v}{\partial n_E}]\!]\|_{0,E}^2 \le \|\nabla(u-v)\|_{0,\omega_E}\|\nabla w\|_{0,\omega_E} + \sum_{K\subset\omega_E} \|f + \Delta v\|_{0,E}\|w\|_{0,K} \tag{2.37}$$

By scaling it holds

$$\|\nabla w\|_{0,K} \lesssim h_K^{-1}\|w\|_{0,K} \lesssim h_K^{-1/2}\|w\|_{0,E} \lesssim h_K^{-1/2}\|[\![\frac{\partial v}{\partial n_E}]\!]\|_{0,E}. \tag{2.38}$$

Combining the estimates above conclude the proof. □

Theorem 2.5. *For $v \in V_h$ it holds*

$$h_E^{1/2}\|\frac{\partial v}{\partial n} - g\|_{0,E} \lesssim \|\nabla u - \nabla v\|_{0,\omega_E} + h_E^{1/2}\|g - g_h\|_{0,E}. \tag{2.39}$$

Exercise 2.4. *Show by the scaling argument that it holds*

$$\int_K |v|^2\,dx \lesssim \int_K b_K |v|^2\,dx \quad \forall v \in P_k(K).$$

Exercise 2.5. *Assume that* $\Gamma_N = \emptyset$*, and that the regularity estimate*

$$\|u\|_2 \lesssim \|f\|_0 \tag{2.40}$$

holds. Using the Nitsche trick and the Lagrange interpolation operator to show that

$$\|u - u_h\|_0 \lesssim \Big(\sum_{K\in\mathcal{C}_h} h_K^4 \|\Delta u_h + f\|_{0,K}^2 + \sum_{E\subset\Omega} h_E^3 \|[\![\frac{\partial u_h}{\partial n_E}]\!]\|_{0,E}^2 \Big)^{1/2}. \tag{2.41}$$

Exercise 2.6. *Prove Theorem 2.5.*

3 Equations of continuum mechanics

In this section we will give a short review of the basic equations of continuum mechanics. In *The Feynman Lectures on Physics, Vol. II* (http://www.feynmanlectures.caltech.edu) there is a crash-course in elasticity and fluid flow (Chapters 38-41).

3.1 The Cauchy-Navier equations of elasticity

Let $\Omega \subset \mathbb{R}^3$ be the original domain occupied by a body loaded by the volume load $\boldsymbol{f}$ and the traction g along the boundary part Γ_N. Along the complementary part $\Gamma_D = \partial\Omega \setminus \Gamma_N$ the body is assumed fixed, i.e. denoting the displacement by $\boldsymbol{u} : \Omega \to \mathbb{R}^3$, we assume that $\boldsymbol{u} = \boldsymbol{0}$ on Γ_D.

The strain $\boldsymbol{\varepsilon}(\boldsymbol{u})$ related to the displacement field is

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T). \tag{3.1}$$

The Cauchy stress tensor is denoted by $\boldsymbol{\sigma} = \{\sigma_{ij}\}$, $i,j = 1,2,3$. The force equilibrium equations are

$$\mathbf{div}\,\boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0} \tag{3.2}$$

and the moment equilibrium is $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$, i.e. the stress tensor is symmetric. Here $\mathbf{div}$ is the vector valued divergence operating on tensors:

$$(\mathbf{div}\,\boldsymbol{\sigma})_i = \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j}, \quad i = 1,2,3. \tag{3.3}$$

The traction boundary condition is

$$\boldsymbol{\sigma}\boldsymbol{n} = g \qquad \text{on } \Gamma_N \tag{3.4}$$

The stress is a linear function of the strain (Hooke's law), viz.

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{C}\boldsymbol{\varepsilon}(\boldsymbol{u}). \tag{3.5}$$

The strain-stress relationship has to be positively definite, i.e.

$$\boldsymbol{\tau} : \boldsymbol{C}\boldsymbol{\tau} \gtrsim |\boldsymbol{\tau}|^2. \tag{3.6}$$

Here the : is the inner product between two tensors, i.e.

$$\boldsymbol{\tau} : \boldsymbol{\theta} = \sum_{i,j=1}^{3} \tau_{ij}\, \theta_{ij}, \tag{3.7}$$

and the norm is defied as

$$|\boldsymbol{\tau}|^2 = \boldsymbol{\tau} : \boldsymbol{\tau}. \tag{3.8}$$

For an isotropic material this stress-strain relationship is

$$\boldsymbol{\sigma} = 2\mu\, \varepsilon(u) + \lambda \operatorname{div} \boldsymbol{u}\, \boldsymbol{I}, \tag{3.9}$$

where μ, λ are the Lamé parameters, $\boldsymbol{I}$ is the identity tensor and div is the scalar valued divergence operator applied to vectors.

Note that $\boldsymbol{I} : \boldsymbol{\varepsilon}(\boldsymbol{v}) = \operatorname{div} \boldsymbol{v} = \operatorname{tr}(\boldsymbol{\varepsilon}(\boldsymbol{v}))$, with "tr" denoting the trace of a tensor.

The Lamé parameters are related to the Young modulus E and Poisson ration ν by

$$\mu = \frac{E}{2(1+\nu)}\ , \qquad \lambda = \frac{E\nu}{2(1+\nu)(1-2\nu)}\ . \tag{3.10}$$

For these parameters it holds $E > 0$ and $0 \leq \nu < 1/2$ and hence μ, $\lambda > 0$. Note also that $\mu = G$, the shear modulus.

In the inversion of the strain–stress relaionship it is better to use E and ν.

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E}\operatorname{tr}(\boldsymbol{\sigma})\, \boldsymbol{I}. \tag{3.11}$$

For stating the variational problem generalising the boundary value problem we first note that for a symmetric tensor $\boldsymbol{\tau}$ and a vector $\boldsymbol{v}$ it holds

$$\int_\Omega \mathbf{div}\, \boldsymbol{\tau} \cdot \boldsymbol{v}\, dx = \int_{\partial\Omega} \boldsymbol{\tau}\boldsymbol{n}\, ds - \int_\Omega \boldsymbol{\tau} : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx. \tag{3.12}$$

Note that $\boldsymbol{I} : \boldsymbol{\varepsilon}(\boldsymbol{v}) = \operatorname{div} \boldsymbol{v} = \operatorname{tr}(\boldsymbol{\varepsilon}(\boldsymbol{v}))$, with "tr" denoting the trace of a tensor.

The variational formulation is then: find $\boldsymbol{u} \in \boldsymbol{H}^1_D(\Omega)$ such that

$$\int_\Omega \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx + \int_{\Gamma_N} g \cdot \boldsymbol{v}\, ds \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1_D(\Omega), \tag{3.13}$$

with

$$\boldsymbol{H}^1_D(\Omega) = \{\, \boldsymbol{v} \in [H^1(\Omega)]^d \mid \boldsymbol{v}|_{\Gamma_D} = \boldsymbol{0}\, \}. \tag{3.14}$$

The problem is coercive in $\boldsymbol{H}^1_D(\Omega)$ due to (3.6) and the Korn inequality

$$\int_\Omega |\boldsymbol{\varepsilon}(\boldsymbol{v})|^2\,dx \gtrsim \int_\Omega |\nabla \boldsymbol{v}|^2\,dx \tag{3.15}$$

(and Poincaré). The finite element is: find $\boldsymbol{u}_h \in \boldsymbol{V}_h \subset \boldsymbol{H}^1_D(\Omega)$ such that

$$\int_\Omega \boldsymbol{\sigma}(\boldsymbol{u}_h) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,dx = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,dx + \int_{\Gamma_N} g\cdot\boldsymbol{v}\,ds \quad \forall \boldsymbol{v}\in\boldsymbol{V}_h. \tag{3.16}$$

Remark 3.1. *The differential equation we obtain from* (3.2) *and* (3.9) *is*

$$-\mathbf{div}\,(2\mu\,\boldsymbol{\varepsilon}(u) + \lambda\,\mathrm{div}\,\boldsymbol{u}\,\boldsymbol{I}) = \boldsymbol{f}, \tag{3.17}$$

which can be written (the Navier equations)

$$-(\mu\Delta\boldsymbol{u} + (\mu+\lambda)\nabla(\mathrm{div}\,\boldsymbol{u})) = \boldsymbol{f}. \tag{3.18}$$

There is, however, a danger in this. If we try to derive the variational form from this, we easily end up with wrong (unphyscal) boundary conditions. (This error is quite common among mathematicians, cf. [10].)

Theorem 3.1. *The strain vanish if an and only if the displacement is a infinitesimal rigid body motion,*

$$\boldsymbol{\varepsilon}(\boldsymbol{v}) = \mathbf{0} \;\Leftrightarrow\; \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{a} + \boldsymbol{b}\times\boldsymbol{x} \quad \text{for some } \boldsymbol{a},\,\boldsymbol{b}. \tag{3.19}$$

Exercise 3.1. *Prove this result.*

Theorem 3.2. (*Korn's inequality.*) *Assume that meas*$(\Gamma_D) > 0$*. Then it holds*

$$\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_0 \gtrsim \|\nabla\boldsymbol{v}\|_0 \quad \forall \boldsymbol{v}\in\boldsymbol{H}^1_D(\Omega) \tag{3.20}$$

Exercise 3.2. *Prove the Korn inequality in case that* $\Gamma_D = \partial\Omega$*. (Assume a smooth function and integrate by parts a couple of time.)*

3.2 The Navier-Stokes equations in fluid mechanics

Now $\boldsymbol{u}$ denotes the fluid velocity in the domain $\Omega\subset\mathbb{R}^3$. The pressure is denoted by p and then the the stress-velocity relationship is

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) - p\boldsymbol{I}, \tag{3.21}$$

where

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T) \tag{3.22}$$

is the rate of strain. Newton's law gives the equation of motion

$$\rho(\frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}) = \mathbf{div}\,\boldsymbol{\sigma} + \boldsymbol{f}. \tag{3.23}$$

In addition, we have the condition that the fluid is incompressible

$$\operatorname{div} \boldsymbol{u} = 0. \tag{3.24}$$

Eliminating the stress leads to the equation

$$\rho(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) = 2\mu \boldsymbol{A}\boldsymbol{u} - \nabla p + \boldsymbol{f}, \tag{3.25}$$

with

$$\boldsymbol{A}\boldsymbol{u} = \mathbf{div}\, \varepsilon(\boldsymbol{u}). \tag{3.26}$$

Using the incompressibility condition, we get the Navier-Stokes equations in the form they usually are presented:

$$\begin{aligned} \rho(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) &= \mu \Delta \boldsymbol{u} - \nabla p + \boldsymbol{f}, \\ \operatorname{div} \boldsymbol{u} &= 0. \end{aligned} \tag{3.27}$$

Remark 3.2. *Again, if one tries to derive the physically correct variational form from the above equations, one easily makes a big mistake, cf. [9].*

Exercise 3.3. *Carry out the manipulations leading to* (3.18) *and* (3.27)*.*

4 The Stokes problem

4.1 The uniqueness of the continuous and discrete problems

In analysing and discretising the Navier-Stokes problem there are two problems. First, to treat the nonlinear term $\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$. Second, how should the incompressibility condition $\operatorname{div} \boldsymbol{u} = 0$ be treated. In this course we will consider only the second, and we will treat the following scaled Stokes equations

$$\begin{aligned} -\boldsymbol{A}\boldsymbol{u} + \nabla p &= \boldsymbol{f}, \\ \operatorname{div} \boldsymbol{u} &= 0, \quad \text{in } \Omega. \end{aligned} \tag{4.1}$$

For the stability, the worst case are Dirichlet conditions along the whole boundary, and we will assume that, i.e.

$$\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{0}.$$

Let us prove:

Theorem 4.1. *The Stokes problem has an, up to an additive constant pressure, unique solution.*

Proof. By linearity one has show that if $\boldsymbol{f} = \mathbf{0}$, then $\boldsymbol{u} = \mathbf{0}$ and $p = Const$. The equation

$$-\boldsymbol{A}\boldsymbol{u} + \nabla p = \mathbf{0} \tag{4.2}$$

we multiply with $\boldsymbol{u}$, integrate over Ω, and integrate by parts

$$0 = \int_\Omega (-\boldsymbol{A}\boldsymbol{u} + \nabla p) \cdot \boldsymbol{u}\, dx = \int_\Omega |\boldsymbol{\varepsilon}(\boldsymbol{u})|^2\, dx - \int_\Omega p \operatorname{div} \boldsymbol{u}\, dx. \tag{4.3}$$

Due to the incompressibility condition $\operatorname{div} \boldsymbol{u} = 0$, we obtain

$$\int_\Omega |\boldsymbol{\varepsilon}(\boldsymbol{u})|^2\, dx = 0. \tag{4.4}$$

Hence, Theorem 3.1 and the boundary condition imply that the velocity vanish. From this it follows that

$$\nabla p = \mathbf{0} \tag{4.5}$$

and the pressure is a constant. □

The variational form is obtained in the same way. The solution $\boldsymbol{u}$ is in $\boldsymbol{H}_0^1(\Omega)$ and the pressure in $L^2(\Omega)$. The mathematically correct way to have a unique pressure is to assume that the pressure has the average value zero

$$\int_\Omega p\, dx = 0. \tag{4.6}$$

Hence it is in

$$L_0^2(\Omega) = \{\, q \in L^2(\Omega) \,|\, \int_\Omega q\, dx = 0 \,\}.$$

The variational form is: find $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$ and $p \in L_0^2(\Omega)$ such that

$$\begin{aligned} (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v})) - (p, \operatorname{div} \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \\ (\operatorname{div} \boldsymbol{u}, q) = 0 \quad \forall q \in L_0^2(\Omega). \end{aligned} \tag{4.7}$$

To show the uniqueness we assume $\boldsymbol{f} = \mathbf{0}$, choose $\boldsymbol{v} = \boldsymbol{u}$, $q = p$, and add the equations giving

$$(\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{u})) = 0,$$

which again leads to $\boldsymbol{u} = \mathbf{0}$. What remain is the condition

$$(p, \operatorname{div} \boldsymbol{v}) = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega). \tag{4.8}$$

Assuming a smooth solution and integrating by parts gives

$$(\nabla p, \boldsymbol{v}) = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega). \tag{4.9}$$

Now we can,e.g., choose $\boldsymbol{v} = b_\Omega \nabla p$, with a weight function with $b_\Omega(\boldsymbol{x}) > 0,\ \boldsymbol{x} \in \Omega$, and $b_\Omega|_{\partial\Omega} = 0$. This gives

$$\int_\Omega b_\Omega |\nabla p|^2\, dx = 0.$$

This leads to

$$\nabla p = \mathbf{0},$$

which together with the condition (4.6) leads to $p = 0$.

The finite element method is easily formulated: find $\boldsymbol{u}_h \in \boldsymbol{V}_h \subset \boldsymbol{H}_0^1(\Omega)$ and $p_h \in P_h \subset L_0^2(\Omega)$ such that

$$\begin{aligned} (\boldsymbol{\varepsilon}(\boldsymbol{u}_h), \boldsymbol{\varepsilon}(\boldsymbol{v})) - (p_h, \operatorname{div} \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h, \\ (\operatorname{div} \boldsymbol{u}, q) = 0 \quad \forall q \in P_h. \end{aligned} \tag{4.10}$$

Now repeating the same arguments we see that the velocity is always unique, and that the the uniqueness of the pressure is given by the following theorem.

Theorem 4.2. *The solution of the discrete Stokes problem* (4.10) *is unique if and only if the FE spaces satisfy the following condition*

$$(q, \operatorname{div} \boldsymbol{v}) = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h \quad \Rightarrow \quad q = Const. \tag{4.11}$$

That this is not always the case is seen from some simple examples. Note that the basis functions for the pressure can be either discontinuous or continuous.

Example 4.1. Probably the first choice for a FE method, would be to use continuous piecewise linear functions for all velocity components and for the pressure, viz.

$$\begin{aligned} \boldsymbol{V}_h &= \{\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \,|\, \boldsymbol{v}|_K \in [P_1(K)]^d \, K \in \mathcal{C}_h \,\}, \\ P_h &= \{\, q \in L_0^2(\Omega) \cap C(\Omega) \,|\, q|_K \in P_1(K) \, K \in \mathcal{C}_h \,\}, \end{aligned} \tag{4.12}$$

where $\mathcal{C}_h$ is the partitioning into triangles/tetrahedrons. This choice is easily seen to give a non-unique solution. Let Ω be a square which is partitioned into triangles as in the figure below. Let p_c be the pressure taking the nodal values 0 and 1 as in the figure. For this non-zero pressure it holds

$$(p_c, \operatorname{div} \boldsymbol{v}) = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h. \tag{4.13}$$

A second example is with discontinuous pressure.

Example 4.2. We let $\mathcal{C}_h$ a partitioning of a two-dimensional domain into quadrilaterals. The FE spaces are

$$\begin{aligned} \boldsymbol{V}_h &= \{\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \,|\, \boldsymbol{v}|_K \in [Q_1(K)]^2 \, K \in \mathcal{C}_h \,\}, \\ P_h &= \{\, q \in L_0^2(\Omega) \,|\, q|_K \in P_0(K) \, K \in \mathcal{C}_h \,\}. \end{aligned} \tag{4.14}$$

For a square partitioned into equal squares the checkerboard function ± 1 satisfies the equation (4.13).

Example 4.3. The next method is the "MINI" [1] element special designed to yield a unique solution. The pressure space is continuous piecewise linears

$$P_h = \{\, q \in L_0^2(\Omega) \cap C(\Omega) \,|\, q|_K \in P_1(K) \, K \in \mathcal{C}_h \,\}. \tag{4.15}$$

In the velocity space the piecewise linears are augmented with the "bubble functions" b_K in each element and for all components. Let $B(K) = P_{d+1}(K) \cap H_0^1(K)$ and define

$$\boldsymbol{V}_h = \{\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \,|\, \boldsymbol{v}|_K \in [P_1(K) \oplus B(K)]^d \; K \in \mathcal{C}_h \,\}. \tag{4.16}$$

Let us now check the crucial condition.

$$(q, \operatorname{div} \boldsymbol{v}) = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h \quad \Rightarrow \quad q = Const.? \tag{4.17}$$

Let $K \in \mathcal{C}_h$ be arbitrary, and choose now $\boldsymbol{v}$ so that $\boldsymbol{v} = \boldsymbol{0}$ in $\Omega \setminus K$ and $\boldsymbol{v}|_K = b_K \nabla q|_K$. We get

$$0 = (q, \operatorname{div} \boldsymbol{v}) = (q, \operatorname{div} \boldsymbol{v})_K = -(\nabla q, \boldsymbol{v})_K = \int_K b_K |\nabla q|^2 \, dx, \tag{4.18}$$

from which we obtain

$$q|_K = constant \ in \ K. \tag{4.19}$$

Since q is continuous, it is the same constant in the whole domain. The condition for a unique solution is valid.

Example 4.4. The next example is with a discontinuous pressure. We use a triangulation of the two-dimensional domain. The FE spaces are

$$\begin{aligned} \boldsymbol{V}_h &= \{\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \,|\, \boldsymbol{v}|_K \in [P_2(K)]^2 \; K \in \mathcal{C}_h \,\}, \\ P_h &= \{\, q \in L_0^2(\Omega) \,|\, q|_K \in P_0(K) \; K \in \mathcal{C}_h \,\}. \end{aligned} \tag{4.20}$$

Let K and K' be two adjacent elements with E as the common edge. We choose $\boldsymbol{v}$ so that it vanish in $\Omega \setminus (K \cup K')$. Let q_K and $q_{K'}$ be the constant values of q in K and K'. Then the criteria to check is

$$0 = (q, \operatorname{div} \boldsymbol{v}) = (q, \operatorname{div} \boldsymbol{v})_{K \cup K'} = (q_K - q_{K'}) \int_E \boldsymbol{v} \cdot \boldsymbol{n}_K \, ds. \tag{4.21}$$

We can now choose $\boldsymbol{v}$ so that $\int_E \boldsymbol{v} \cdot \boldsymbol{n}_K \, ds \neq 0$, and the conclusion is that

$$q_K = q_{K'}. \tag{4.22}$$

We then repeat the same argument for each edge in the mesh, and conclude that the pressure is a global constant.

Example 4.5. (Crouzeix & Raviart [6]) This example is given by the choice

$$\begin{aligned} \boldsymbol{V}_h &= \{\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \,|\, \boldsymbol{v}|_K \in [P_2(K) \oplus B(K)]^2 \; K \in \mathcal{C}_h \,\}. \\ P_h &= \{\, q \in L_0^2(\Omega) \,|\, q|_K \in P_1(K) \; K \in \mathcal{C}_h \,\}. \end{aligned} \tag{4.23}$$

The uniqueness for this follows from the previous examples.

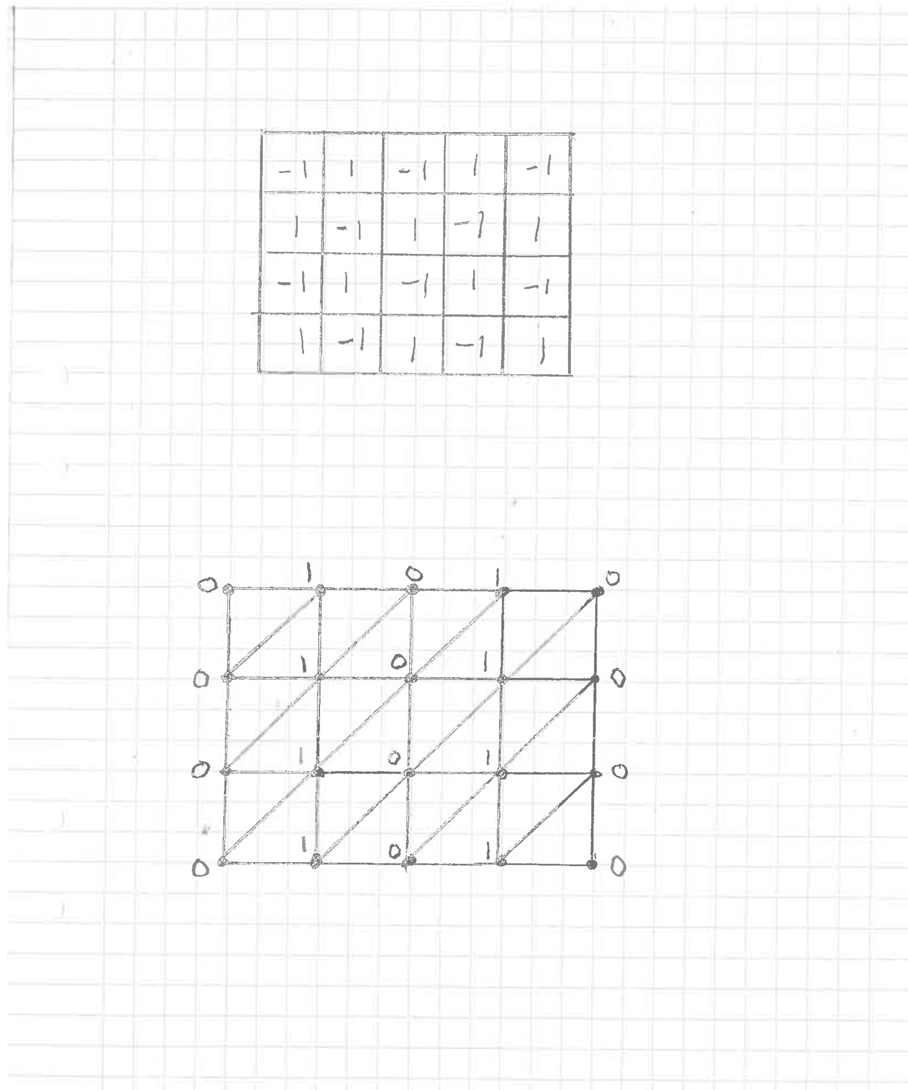

Figure 1: The spurious pressure modes. Top: the $Q_1 - P_0$ element. Bottom: Continuous piecewise linears

The final example is the so-called Taylor–Hood method for which the proof of uniqueness is more tricky.

Example 4.6.

$$\begin{aligned} \boldsymbol{V}_h &= \{\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \,|\, \boldsymbol{v}|_K \in [P_2(K)]^2 \; K \in \mathcal{C}_h \,\}, \\ P_h &= \{\, q \in L_0^2(\Omega) \cap C(\Omega) \,|\, q|_K \in P_1(K) \; K \in \mathcal{C}_h \,\}. \end{aligned} \tag{4.24}$$

To prove the local uniqueness we have to a patch, "macroelement", of three elements, see the figure below. In the proof we use the quadrature formula taking the average of the function at the three midpoints of the triangle times the area. It is easily checked that this is exact for all polynomials of degree two. Now, let us consider $M = K_1 \cup K_2 \cup K_3$. We choose $\boldsymbol{v}$ so that is it vanish outside of $K_1 \cup K_2$. Let $\boldsymbol{t}_{12}$ be the unit tangent on the common edge. The two remaining degrees of freedom, the velocity at the midpoint $\boldsymbol{x}_{02}$, we choose so that the component in the tangential direction $\boldsymbol{t}(\boldsymbol{x}_{02}$ equals one, and the orthogonal component vanish. For $q \in P_h$ $\nabla q \cdot (\boldsymbol{x}_0 \boldsymbol{x}_2)$ is continuous along the common edge $\partial K_1 \cap \partial K_2$. Using the quadrature rule the condition for uniques gives

$$0 = (\operatorname{div} \boldsymbol{v}, q) = -(\boldsymbol{v}, \nabla q)_{K_1 \cup K_2} = \frac{1}{3}(\operatorname{area}(K_1) + \operatorname{area}(K_2)) \nabla q(\boldsymbol{x}_{02}) \cdot \boldsymbol{t}_{12}, \tag{4.25}$$

and hence it holds

$$\nabla q(\boldsymbol{x}_{02}) \cdot \boldsymbol{t}_{12} = 0. \tag{4.26}$$

That means that

$$q(\boldsymbol{x}_0) = q(\boldsymbol{x}_2). \tag{4.27}$$

Repeting the same argument for the elements K_2 and K_3 shows that

$$q(\boldsymbol{x}_0) = q(\boldsymbol{x}_3). \tag{4.28}$$

Hence, q vanish in K_2. Next, we test with functions whose only non-vanishing degrees of freedom are the normal components at $\boldsymbol{x}_{12}$ and $\boldsymbol{x}_{23}$, respectively. This leads to the conclusion that q is a constant in $K_1 \cup K_2 \cup K_3$.

4.2 Stokes as a constrained optimisation problem

Most often an elliptic problem can be posed as a minimisation problem and the Euler-Lagrange equations are the weak formulation.

The Stokes problem is a prototype of a constraint minimisation problem: the solution $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$ is obtained as

$$\min_{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)} \frac{1}{2} \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_0^2 - (\boldsymbol{f}, \boldsymbol{v}) \tag{4.29}$$

subject to

$$\operatorname{div} \boldsymbol{v} = 0. \tag{4.30}$$

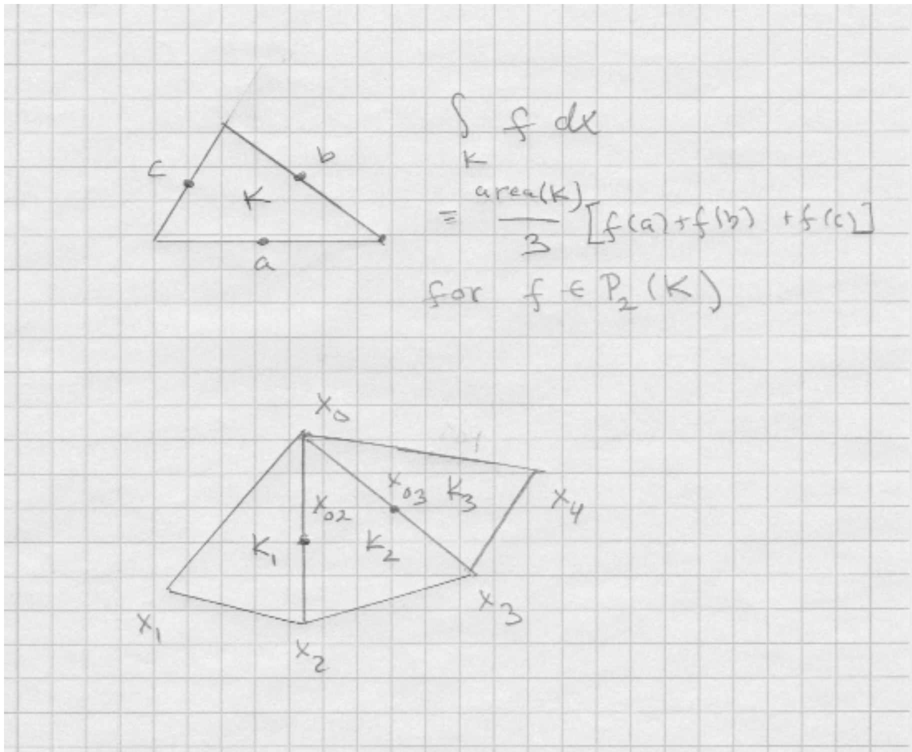

Figure 2: The quadrature rule exact for $P_2(K)$. Bottom: The macroelement.

To solve this we define the Lagrangian $\mathcal{L} : \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega) \to I\!R$ by

$$\mathcal{L}(\boldsymbol{v}, q) = \frac{1}{2}\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_0^2 - (q, \operatorname{div} \boldsymbol{v}) - (\boldsymbol{f}, \boldsymbol{v}). \tag{4.31}$$

The variation with respect to the vector variable gives the first equation

$$(\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v})) - (p, \operatorname{div} \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \tag{4.32}$$

and the variation of the scaler gives the weak form of the incompressibility condition

$$(\operatorname{div} \boldsymbol{u}, q) = 0 \quad \forall q \in L_0^2(\Omega). \tag{4.33}$$

We see that the physical meaning of the pressure is the Lagrange multiplier enforcing the incompressibility. What we have done requires of course some applied functional analysis. However, for the discretisation it is elementary.

Hence, let us turn to that. Let $\{\boldsymbol{w}_i\}_{i=1}^N$, and $\{\phi_i\}_{i=1}^M$, be the basis functions for $\boldsymbol{V}_h$ and P_h, respectively. With

$$\boldsymbol{u}_h = \sum_{j=1}^N u_j \boldsymbol{w}_j, \quad p_h = \sum_{k=1}^M p_k \phi_k, \tag{4.34}$$

and testing with the basis functions gives

$$\begin{aligned} \sum_{j=1}^{N} (\boldsymbol{\varepsilon}(\boldsymbol{w}_i), \boldsymbol{\varepsilon}(\boldsymbol{w}_j))u_j - \sum_{k=1}^{M}(\phi_k, \operatorname{div} \boldsymbol{w}_i)p_k = (\boldsymbol{f}, \boldsymbol{w}_i), \quad i = 1, \dots, N, \\ \sum_{j=1}^{N}(\operatorname{div} \boldsymbol{w}_j, \phi_l)u_j = 0, \quad l = 1, \dots, M. \end{aligned} \tag{4.35}$$

Let $U = (u_1, \dots, u_N)^T$ and $P = (p_1, \dots, p_M)$ and define the matrices A, and B, and the load $F = (F_1, \dots F_N)$, by

$$A_{ij} = (\boldsymbol{\varepsilon}(\boldsymbol{w}_i), \boldsymbol{\varepsilon}(\boldsymbol{w}_j)), \quad B_{ik} = -(\phi_k, \operatorname{div} \boldsymbol{w}_i), \quad F_i = (\boldsymbol{f}, \boldsymbol{w}_i). \tag{4.36}$$

Then the discrete system is then

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}. \tag{4.37}$$

Note that A is symmetric and positively definite.

We see that this is an example of a quadratic optimisation problem: find U which minimises the object function

$$\frac{1}{2} V^T A V - F^T V \tag{4.38}$$

subject to the linear constraint

$$B^T V = G. \tag{4.39}$$

The Lagrangian is (Q is the multiplier)

$$L(V, Q) = \frac{1}{2} V^T A V - F^T V - Q^T (B^T V - G). \tag{4.40}$$

The optimality conditions are then

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}. \tag{4.41}$$

Exercise 4.1. *Show that this system has a unique solution if, and only if, $N(B) = \{0\}$, or equvalently $R(B^T) = \mathbb{R}^M$, with N and R denoting the nullspace and range.*

4.3 Theory of saddle point problems

4.3.1 The Lax-Milgram-Nirenberg Lemma

Theorem 4.3. *Let H be a Hilbert space with inner product $(\cdot, \cdot)_H$ and norm $\|\cdot\|_H$. Suppose that the bilinear form $\mathcal{B} : H \times H \to \mathbb{R}$ satisfies the following conditions.*

- *Continuity: there is a positive constant C such that*

$$|\mathcal{B}(W,V)| \leq C\|W\|_H\|V\|_H \quad \forall W,V \in H. \tag{4.42}$$

- *The "inf-sup" condition: there is a positive constant α such that*

$$\sup_{V\in H,\, V\neq 0} \frac{\mathcal{B}(W,V)}{\|V\|_H} \geq \alpha\|W\|_H \quad \forall W \in H. \tag{4.43}$$

- *The condition*

$$\sup_{W\in H} \mathcal{B}(W,V) \neq 0 \quad \forall V \in H. \tag{4.44}$$

Then the variational problem: find $U \in H$ such that

$$\mathcal{B}(U,V) = F(V) \quad \forall V \in H, \tag{4.45}$$

has a unique solution depending continuously on the data:

$$\|U\|_H \leq \alpha^{-1}\|F\|'_H, \tag{4.46}$$

where $\|\cdot\|'_H$ is the dual norm

$$\|F\|'_H = \sup_{V\in H} \frac{F(V)}{\|V\|_H}. \tag{4.47}$$

Proof. ([2], [7]) Step 1). For every $W \in H$

$$\Phi_W(V) = \mathcal{B}(W,V) \tag{4.48}$$

defines a continuous linear functional on H. By the Riesz representation theorem there is $Z \in H$ such that

$$\Phi_W(V) = (Z,V)_H. \tag{4.49}$$

Hence, we have a linear mapping $A : H \to H$, $Z = A(W)$, such that

$$(A(W),V)_H = \mathcal{B}(W,V) \quad \forall W,V \in H. \tag{4.50}$$

From the continuity condition it follows that A is bounded:

$$\|A\| \leq C. \tag{4.51}$$

2). The mapping A is bounded from below and $R(A)$ is closed: We have

$$\|A(W)\|_H = \sup_{V\in H,\, \|V\|_H=1} (A(W),V)_H = \sup_{V\in H,\, \|V\|_H=1} \mathcal{B}(W,V)_H \geq \alpha\|W\|_H. \tag{4.52}$$

Let $A(W_n)$ be a Cauchy-sequence. From above we have

$$\|A(W_n) - A(W_m)\|_H \geq \alpha\|W_n - W_m\|_H, \tag{4.53}$$

and hence W_n is also Cauchy, converging to (say) $W \in H$. Since A is bounded we have $A(W_n) \to A(W)$ which shows that $R(A)$ is closed.

3). $R(A) = H$. If not, there exists $V_0 \neq 0$ such that

$$(A(W), V_0)_H = 0 \quad \forall W \in H. \tag{4.54}$$

This is equivalent with

$$\mathcal{B}(W, V_0) = 0 \quad \forall W \in H \tag{4.55}$$

which contradicts the third condition assumed of B.

4). Next we apply the Riesz representation theorem to the right hand side as well: there exits $G \in H$ such that

$$F(V) = (G, V)_H. \tag{4.56}$$

The variational problem is now equivalent to

$$(A(U), V)_H = (G, V)_H \tag{4.57}$$

i.e.

$$A(U) = G \tag{4.58}$$

with solution

$$U = A^{-1}(G). \tag{4.59}$$

5). From the inf-sup condition we now finally have

$$\alpha \|U\|_H \leq \sup_{V \in H, V \neq 0} \frac{\mathcal{B}(U, V)}{\|V\|_H} = \sup_{V \in H, V \neq 0} \frac{F(V)}{\|V\|_H} = \|F\|'_H, \tag{4.60}$$

which also shows the uniqueness. □

4.3.2 Finite Element Discretization

Choose a subspace $H_h \subset H$ and solve $U_h \in H_h$ from

$$\mathcal{B}(U_h, V) = F(V) \quad \forall V \in H_h. \tag{4.61}$$

We then have the analog to Cea's lemma.

Theorem 4.4. *Suppose that the following discrete "inf-sup" condition is valid: there is a constant $\gamma > 0$ such that*

$$\sup_{V \in H_h, V \neq 0} \frac{\mathcal{B}(W, V)}{\|V\|_H} \geq \gamma \|W\|_H \quad \forall W \in H_h. \tag{4.62}$$

Then it holds

$$\|U - U_h\|_H \leq (1 + \frac{C}{\gamma}) \inf_{Z \in H_h} \|U - Z\|_H, \tag{4.63}$$

where C is the constant in the continuity condition.

Proof. 1). Testing with $v \in H_h$ in the the continuous problem and subtracting from the discrete formulation gives

$$\mathcal{B}(U - U_h, V) = 0 \quad \forall V \in H_h. \tag{4.64}$$

2). Let $Z \in H_h$ be arbitrary. Choosing $W = U_h - Z$ in the inf-sup then gives

$$\begin{aligned} \gamma \|U_h - Z\|_H &\leq \sup_{V \in H_h, V \neq 0} \frac{\mathcal{B}(U_h - Z, V)}{\|V\|_H} = \sup_{V \in H_h, V \neq 0} \frac{\mathcal{B}(U - Z, V)}{\|V\|_H} \\ &\leq C \|U - Z\|_H, \end{aligned} \tag{4.65}$$

i.e.

$$\|U_h - Z\|_H \leq \frac{C}{\gamma} \|U - Z\|_H. \tag{4.66}$$

The claim now follows from the triangle inequality (and "taking the inf"). □

Remark 4.1. *Alternative ways of posing the stability condition are*

- *There exist a positive constant* γ *such that*

$$\inf_{W \in H_h} \sup_{V \in H_h} \frac{\mathcal{B}(W, V)}{\|V\|_H \|W\|_H} \geq \gamma. \tag{4.67}$$

- *There exist a positive constant* γ *such that for every* $W \in H_h$ *there is* $V \in H_h$ *such that*

$$\mathcal{B}(W, V) \geq \gamma \|W\|_H^2, \quad \textit{and} \quad \|V\|_H = \|W\|_H. \tag{4.68}$$

- *There exist a positive constant* C *such that for every* $W \in H_h$ *there is* $V \in H_h$ *such that*

$$\mathcal{B}(W, V) = \|W\|_H^2, \quad \textit{and} \quad \|V\|_H \leq C \|W\|_H. \tag{4.69}$$

4.4 Application to the Stokes problem

4.4.1 The Babuska-Brezzi splitting for Stokes problem

For the Stokes equations we define $H := H_0^1(\Omega)^N \times L_0^2(\Omega)$, with the norm

$$\|(\boldsymbol{v}, q)\|_H^2 := \|\boldsymbol{v}\|_1^2 + \|q\|_0^2, \tag{4.70}$$

and the bilinear form

$$\mathcal{B}((\boldsymbol{v}, q), (\boldsymbol{z}, r)) := (\nabla \boldsymbol{v}, \nabla \boldsymbol{z}) - (\operatorname{div} \boldsymbol{z}, q) - (\operatorname{div} \boldsymbol{v}, r). \tag{4.71}$$

The dual space is now $H' = H^{-1}(\Omega)^n \times L_0^2(\Omega)$.

The inf-sup is now:

$$\sup_{(\boldsymbol{z}, r) \in H} \frac{\mathcal{B}((\boldsymbol{v}, q), (\boldsymbol{z}, r))}{\|(\boldsymbol{z}, r)\|_H} \geq \alpha \|(\boldsymbol{v}, q)\|_H \quad \forall (\boldsymbol{v}, q) \in H. \tag{4.72}$$

The Babuska-Brezzi theory for mixed (or saddle point) problems says that this is a consequence of the two conditions.

The Babuska-Brezzi conditions.

- *The ellipticity: There is a constant C_1 such that*

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{v}) \geq C_1 \|\boldsymbol{v}\|_1^2 \quad \forall \boldsymbol{v} \in H_0^1(\Omega)^n. \tag{4.73}$$

- *The LBB (Ladyshenskaya-Babuska-Brezzi) condition: there is a positive constant C_2 such that*

$$\sup_{\boldsymbol{v} \in H_0^1(\Omega)^n} \frac{(\operatorname{div} \boldsymbol{v}, q)}{\|v\|_1} \geq C_2 \|q\|_0 \quad \forall q \in L_0^2(\Omega). \tag{4.74}$$

The first estimate above is simply the Poincaré inequality.

In order to keep track of the influence of the LBB constant it will be convenient to work with the norm $\|\nabla \boldsymbol{v}\|_0$ for $\boldsymbol{v} \in H_0^1(\Omega)^n$. The LBB we then write as: there is a positive constant β such that

$$\sup_{\boldsymbol{w} \in H_0^1(\Omega)^n} \frac{(\operatorname{div} \boldsymbol{w}, q)}{\|\nabla \boldsymbol{w}\|_0} \geq \beta \|q\|_0 \quad \forall q \in L_0^2(\Omega). \tag{4.75}$$

Then we perform the analysis with the "triple-bar" norm:

$$|||(\boldsymbol{v}, q)|||_H^2 := \|\nabla \boldsymbol{v}\|_0^2 + \beta^2 \|q\|_0^2. \tag{4.76}$$

We also use the standard trick, the "arithmetic-geometric-mean inequality" (AGM):

$$|ab| \leq \frac{\varepsilon}{2} a^2 + \frac{1}{2\varepsilon} b^2, \quad a, b \in I\!R, \quad \varepsilon > 0. \tag{4.77}$$

or more precisely

$$-|ab| \geq -\frac{\varepsilon}{2} a^2 - \frac{1}{2\varepsilon} b^2, \quad a, b \in I\!R, \quad \varepsilon > 0. \tag{4.78}$$

Let's now build the inf-sup for $\mathcal{B}$ from the ellipticity and the LBB.

Proof. 1). Let $(\boldsymbol{v}, q) \in H$ be arbitrary. We first have

$$\mathcal{B}((\boldsymbol{v}, q), (\boldsymbol{v}, -q)) = (\nabla \boldsymbol{v}, \nabla \boldsymbol{v}) - (\operatorname{div} \boldsymbol{v}, q) + (\operatorname{div} \boldsymbol{v}, q) = \|\nabla \boldsymbol{v}\|_0^2. \tag{4.79}$$

2). The LBB condition reformulated is: there is $\boldsymbol{w} \in H_0^1(\Omega)^n$ such that

$$(\operatorname{div} \boldsymbol{w}, q) \geq \beta \|q\|_0^2 \quad \text{and} \quad \|\nabla \boldsymbol{w}\|_0 = \|q\|_0. \tag{4.80}$$

Using this, Schwartz and the AGM (with $\varepsilon = \beta$) gives

$$\begin{aligned}\mathcal{B}((\boldsymbol{v}, q), (-\boldsymbol{w}, 0)) &= -(\nabla \boldsymbol{v}, \nabla \boldsymbol{w}) + (\operatorname{div} \boldsymbol{w}, q) \\ &\geq -(\nabla \boldsymbol{v}, \nabla \boldsymbol{w}) + \beta \|q\|_0^2 \\ &\geq -\|\nabla \boldsymbol{v}\|_0 \|\nabla \boldsymbol{w}\|_0 + \beta \|q\|_0^2 \\ &\geq -\frac{1}{2\beta}\|\nabla \boldsymbol{v}\|_0^2 - \frac{\beta}{2}\|\nabla \boldsymbol{w}\|_0^2 + \beta \|q\|_0^2 \\ &= -\frac{1}{2\beta}\|\nabla \boldsymbol{v}\|_0^2 + \frac{\beta}{2}\|q\|_0^2.\end{aligned}$$

3). For $\delta > 0$ we then have

$$\begin{aligned}&\mathcal{B}((\boldsymbol{v}, q), (\boldsymbol{v} - \delta \boldsymbol{w}, -q) \\ &\quad= \quad \mathcal{B}((\boldsymbol{v}, q), (\boldsymbol{v}, -q)) + \delta\, \mathcal{B}((\boldsymbol{v}, q), (-\boldsymbol{w}, 0) \\ &\quad\geq \quad \Big(1 - \frac{\delta}{2\beta}\Big)\|\nabla \boldsymbol{v}\|_0^2 + \frac{\delta\beta}{2}\|q\|_0^2 \\ &\quad= \quad \frac{1}{2}\|\nabla \boldsymbol{v}\|_0^2 + \frac{\beta^2}{2}\|q\|_0^2 = \frac{1}{2}|||(\boldsymbol{v}, q)|||_H^2.\end{aligned}$$

when choosing $\delta = \beta$.

4). For $\boldsymbol{z} = \boldsymbol{v} - \delta \boldsymbol{w} = \boldsymbol{v} - \beta \boldsymbol{w}$ and $r = -q$ we thus have

$$\mathcal{B}((\boldsymbol{v}, q), (\boldsymbol{z}, r)) \geq \frac{1}{2}|||(\boldsymbol{v}, q)|||_H^2 \tag{4.81}$$

and (AGM again)

$$\begin{aligned}|||(\boldsymbol{z}, r)|||_H &\leq \Big(\|\nabla(\boldsymbol{v} - \beta \boldsymbol{w})\|_0 + \beta \|q\|_0\Big) \\ &\leq \Big(\|\nabla \boldsymbol{v}\|_0 + \beta \|\nabla \boldsymbol{w}\|_0 + \beta \|q\|_0\Big) \\ &= \Big(\|\nabla \boldsymbol{v}\|_0 + 2\beta \|q\|_0\Big) \leq 2\Big(\|\nabla \boldsymbol{v}\|_0 + \beta \|q\|_0\Big) \\ &\leq 2\sqrt{2}|||(\boldsymbol{v}, q)|||_H.\end{aligned}$$

5). Combining gives (if I calculated the constants right)

$$\sup_{(\boldsymbol{z}, r) \in H} \frac{\mathcal{B}((\boldsymbol{v}, q), (\boldsymbol{z}, r))}{|||(\boldsymbol{z}, r)|||_H} \geq \frac{\sqrt{2}}{8}|||(\boldsymbol{v}, q)|||_H \quad \forall (\boldsymbol{v}, q) \in H. \tag{4.82}$$

Since $\|\cdot\|_H$ and $|||\cdot|||_H$ are equivalent the claim is proved. □

4.5 The Babuska-Brezzi condition for the FEM

We discretize Stokes: find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times P_h =: H_h \subset H$ such that

$$\mathcal{B}((\boldsymbol{u}_h, p_h), (\boldsymbol{v}, q)) = (\boldsymbol{f}, \boldsymbol{v}) \qquad \forall (\boldsymbol{v}, q) \in H_h. \tag{4.83}$$

The discrete inf-sup follows from the ellipticity condition and the discrete LBB condition. From Cea's lemma we get the following result.

Theorem 4.5. *Suppose that the discrete spaces satisfy the condition: there is a constant $\beta > 0$, independent of h, such that*

$$\sup_{\boldsymbol{w}\in \boldsymbol{V}_h} \frac{(\operatorname{div} \boldsymbol{w}, q)}{\|\nabla \boldsymbol{w}\|_0} \geq \beta \|q\|_0 \quad \forall q \in P_h. \tag{4.84}$$

Then there is a constant $C > 0$, independent of h, such that

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + \|p - p_h\|_0 \leq C\{ \inf_{\boldsymbol{v}\in \boldsymbol{v}_h} \|\boldsymbol{u} - \boldsymbol{v}\|_1 + \inf_{q\in P_h} \|p - q\|_0\}. \tag{4.85}$$

The purpose of using the triple-bar norm above was to trace the influence of the LBB constant. When we redo it for the discrete problem we get the estimates:

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 \leq C\beta^{-1}\{ \inf_{\boldsymbol{v}\in \boldsymbol{V}_h} \|\boldsymbol{u} - \boldsymbol{v}\|_1 + \inf_{q\in P_h} \|p - q\|_0\}. \tag{4.86}$$

and

$$\|p - p_h\|_0 \leq C\beta^{-2}\{ \inf_{\boldsymbol{v}\in \boldsymbol{V}_h} \|\boldsymbol{u} - \boldsymbol{v}\|_1 + \inf_{q\in P_h} \|p - q\|_0\}. \tag{4.87}$$

These hold also in cases when $\beta > 0$ depends on the mesh length h (e.g. the Q_1–P_0) and then shows that the accuracy is degenerated for unstable methods.

4.6 Verifying the stability condition

It has turned out that the proof of the Stokes inf-sup condition is easiest by first proving the corresponding condition with a mesh dependent norm for the pressure. In the the FE subspace P_h we define

$$\|q\|_h^2 = \sum_{K\in\mathcal{C}_h} h_K^2 \|\nabla q\|_{0,K}^2 + \sum_{E\subset\Omega} h_E \|[\![q]\!]\|_{0,E}^2, \quad q \in P_h. \tag{4.88}$$

The modified stability condition is

$$\sup_{\boldsymbol{v}\in \boldsymbol{V}_h} \frac{(\operatorname{div} \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1} \gtrsim \|q\|_h \quad q \in P_h. \tag{4.89}$$

The reason why this is advantageous is that it can be proved locally.

The following result is called the Pitkäranta–Verfürt trick [11, 17].

Theorem 4.6. *Suppose that the FE subspaces satisfy the stability condition* (4.89)*. Then the condition* (4.84) *is valid.*

Proof. Let $q \in P_h$ be arbitrary. By the continuous stability condition (4.74) there exists $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$ such that

$$(\operatorname{div} \boldsymbol{v}, q) \geq C_1 \|q\|_0 \|\boldsymbol{v}\|_1. \tag{4.90}$$

Let $I_h \boldsymbol{v} \in \boldsymbol{V}_h$ be the Clément interpolant to $\boldsymbol{v}$. We first write

$$(\operatorname{div} I_h \boldsymbol{v}, q) = (\operatorname{div} \boldsymbol{v}, q) - (\operatorname{div} (\boldsymbol{v} - I_h \boldsymbol{v}), q). \tag{4.91}$$

By element by element integration by parts and using the Cauchy and Schwarz inequalities we get

$$\begin{aligned} &|(\operatorname{div} (\boldsymbol{v} - I_h \boldsymbol{v}), q)| = |\sum_{K \in \mathcal{C}_h} (\operatorname{div} (\boldsymbol{v} - I_h \boldsymbol{v}), q)_K| \\ &= |-\sum_{K \in \mathcal{C}_h} (\boldsymbol{v} - I_h \boldsymbol{v}, \nabla q)_K + \sum_{E \subset \Omega} \langle (\boldsymbol{v} - I_h \boldsymbol{v}) \cdot \boldsymbol{n}_E, [\![q]\!] \rangle_E| \\ &\lesssim \Big(\sum_{K \in \mathcal{C}_h} h_K^{-2} \|\boldsymbol{v} - I_h \boldsymbol{v}\|_{0,K}^2 + \sum_{E \subset \Omega} h_E^{-1} \|(\boldsymbol{v} - I_h \boldsymbol{v}) \cdot \boldsymbol{n}_E\|_{0,E}^2 \Big)^{1/2} \|q\|_h. \end{aligned} \tag{4.92}$$

Hence, Theorem 2.1 yields

$$|(\operatorname{div} (\boldsymbol{v} - I_h \boldsymbol{v}), q)| \leq C_2 \|\boldsymbol{v}\|_1 \|q\|_h. \tag{4.93}$$

Combining (4.90), (4.91), and (4.93) yields

$$(\operatorname{div} I_h \boldsymbol{v}, q) \geq (C_1 \|q\|_0 - C_2 \|q\|_h) \|\boldsymbol{v}\|_1. \tag{4.94}$$

If now $C_1 \|q\|_0 - C_2 \|q\|_h < 0$, the assertion is proved.

In the other case, we use the fact that $C_3 \|I_h \boldsymbol{v}\|_1 \leq \|\boldsymbol{v}\|_1$ to obtain

$$\frac{(\operatorname{div} I_h \boldsymbol{v}, q)}{\|I_h \boldsymbol{v}\|_1} \geq C_3 (C_1 \|q\|_0 - C_2 \|q\|_h), \tag{4.95}$$

that is

$$\sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{(\operatorname{div} \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1} \geq C_1' \|q\|_0 - C_2' \|q\|_h. \tag{4.96}$$

The stability in the mesh dependent norm is

$$\sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{(\operatorname{div} \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1} \geq C_3' \|q\|_h \quad q \in P_h. \tag{4.97}$$

Taking a convex compination of these estimates, with $0 < t < 1$, gives

$$\begin{aligned} &\sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{(\operatorname{div} \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1} = t \sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{(\operatorname{div} \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1} + (1-t) \sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{(\operatorname{div} \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1} \\ &\geq (t(C_3' + C_2') - C_2') \|q\|_h + (1-t) C_1' \|q\|_0. \end{aligned} \tag{4.98}$$

Choosing $1 > t > \frac{C_2'}{C_2' + C_3'}$, proves the claim.

□

Next, let us prove the stability for the methods considered before.

The MINI element.

$$\begin{aligned} \boldsymbol{V}_h &= \{\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \,|\, \boldsymbol{v}|_K \in [P_1(K) \oplus B(K)]^d \; K \in \mathcal{C}_h \,\}, \\ P_h &= \{\, q \in L_0^2(\Omega) \cap C(\Omega) \,|\, q|_K \in P_1(K) \; K \in \mathcal{C}_h \,\}. \end{aligned} \tag{4.99}$$

Let $q \in P_h$ be given. $\boldsymbol{v} \in \boldsymbol{V}_h$ we define as $\boldsymbol{v}|_K = -h_K^2 b_K \nabla q|_K$ for each $K \in \mathcal{C}_h$. This gives

$$\begin{aligned} (\operatorname{div} \boldsymbol{v}, q) = -(\boldsymbol{v}, \nabla q) &= \sum_{K \in \mathcal{C}_h} h_K^2 \| b_K^{1/2} \nabla q \|_{0,K}^2 \\ &\gtrsim \sum_{K \in \mathcal{C}_h} h_K^2 \| \nabla q \|_{0,K}^2 \gtrsim \| q \|_h^2. \end{aligned} \tag{4.100}$$

On the element K it holds

$$\| \nabla \boldsymbol{v} \|_{0,K} \lesssim h_K^{-1} \| \boldsymbol{v} \|_{0,K} \lesssim h_K \| b_K \nabla q \|_{0,K} \lesssim h_K \| \nabla q \|_{0,K}. \tag{4.101}$$

Hence, we have

$$\| \boldsymbol{v} \|_1 \lesssim \| \nabla \boldsymbol{v} \|_0 \lesssim \| q \|_h, \tag{4.102}$$

and the stability is proved.

The lowest order Crouzeix–Raviart element

$$\begin{aligned} \boldsymbol{V}_h &= \{\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \,|\, \boldsymbol{v}|_K \in [P_2(K)]^2 \; K \in \mathcal{C}_h \,\}, \\ P_h &= \{\, q \in L_0^2(\Omega) \,|\, q|_K \in P_0(K) \; K \in \mathcal{C}_h \,\}. \end{aligned} \tag{4.103}$$

For $q \in P_h$ we define $\boldsymbol{v}$ by $\boldsymbol{v}|_E = h_K b_E [\![q]\!]|_E$. This gives

$$(\operatorname{div} \boldsymbol{v}, q) = \sum_{E \subset \Omega} \langle \boldsymbol{v} \cdot \boldsymbol{n}_E, [\![q]\!] \rangle_E = \sum_{E \subset \Omega} h_E \int_E b_E |[\![q]\!]|^2 \, ds \approx \| q \|_h^2, \tag{4.104}$$

and by the inverse inequality and scaling gives

$$\| \boldsymbol{v} \|_1 \lesssim \| q \|_h. \tag{4.105}$$

The second lowest order Crouzeix–Raviart element

$$\begin{aligned} \boldsymbol{V}_h &= \{\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \,|\, \boldsymbol{v}|_K \in [P_2(K) \oplus B(K)]^2 \; K \in \mathcal{C}_h \,\}. \\ P_h &= \{\, q \in L_0^2(\Omega) \,|\, q|_K \in P_1(K) \; K \in \mathcal{C}_h \,\}. \end{aligned} \tag{4.106}$$

We now "glue together" the estimates from the previous examples. To this end $q \in P_h$ we split in the piecewise constant component and its orthogonal component $q = \bar{q} + \hat{q}$, with

$$\bar{q}|_K = \frac{\int_K q \, dx}{\int_K \, dx}. \tag{4.107}$$

From the lowest order CR, we know that there exists $\bar{\boldsymbol{v}} \in \boldsymbol{V}_h$ such that

$$(\operatorname{div} \bar{\boldsymbol{v}}, \bar{q}) = \|\bar{q}\|_0^2 \quad \text{and} \quad C\|\bar{\boldsymbol{v}}\|_1 \le \|\bar{q}\|_0. \tag{4.108}$$

By using the bubbles we have $\hat{\boldsymbol{v}} \in \boldsymbol{V}_h$ such that (we can use the same constant here)

$$(\operatorname{div} \hat{\boldsymbol{v}}, \bar{q}) = \|\hat{q}\|_0^2 \quad \text{and} \quad C\|\hat{\boldsymbol{v}}\|_1 \le \|\hat{q}\|_0. \tag{4.109}$$

It also holds that

$$(\operatorname{div} \hat{\boldsymbol{v}}, \bar{q}) = 0. \tag{4.110}$$

Let $\delta > 0$, and estimate using the AGM (with γ)and (4.108)

$$\begin{aligned}
&(\operatorname{div}(\hat{\boldsymbol{v}} + \delta\bar{\boldsymbol{v}}), q) = (\operatorname{div}(\hat{\boldsymbol{v}} + \delta\bar{\boldsymbol{v}}), \bar{q} + \hat{q})\\
&= (\operatorname{div} \hat{\boldsymbol{v}}, \hat{q}) + \delta(\operatorname{div} \bar{\boldsymbol{v}}, \bar{q}) + \delta(\operatorname{div} \bar{\boldsymbol{v}}, \hat{q})\\
&= \|\hat{q}\|_0^2 + \delta\|\bar{q}\|_0^2 + \delta(\operatorname{div} \bar{\boldsymbol{v}}, \hat{q})\\
&\ge \|\hat{q}\|_0^2 + \delta\|\bar{q}\|_0^2 - \delta\|\bar{\boldsymbol{v}}\|_1\|\hat{q}\|_0\\
&\ge \|\hat{q}\|_0^2 + \delta\|\bar{q}\|_0^2 - \frac{\delta\gamma}{2}\|\bar{\boldsymbol{v}}\|_1^2 - \frac{\delta}{2\gamma}\|\hat{q}\|_0^2\\
&\ge (1 - \frac{\delta}{2\gamma})\|\hat{q}\|_0^2 + \delta(1 - \frac{\gamma}{2C^2})\|\bar{q}\|_0^2.
\end{aligned} \tag{4.111}$$

Hence, choosing first γ such that $1 - \frac{\gamma}{2C^2} > 0$ and then δ so that $1 - \frac{\delta}{2\gamma} > 0$, gives

$$(\operatorname{div}(\hat{\boldsymbol{v}} + \delta\bar{\boldsymbol{v}}), q) \gtrsim (\|\bar{q}\|_h^2 + \|\hat{q}\|_0^2) \gtrsim \|q\|_0^2. \tag{4.112}$$

Since

$$\|\hat{\boldsymbol{v}} + \delta\bar{\boldsymbol{v}}\|_1 \lesssim \|q\|_0, \tag{4.113}$$

the assertion is proved.

The above examples show the technique to prove the stability for those elements for which it can be done element by element. For the methods where this cannot be done it looks more complicated. For the Taylor–Hood method it seems pretty difficult to explicitly construct the stability. It can, however, be proved [15] that if one can prove the local uniqueness on a patch of elements, and this independently of the geometrical shape of the element, then the stability is valid.

Let us end this section by giving a list of known stable combinations.

Triangular and tetrahedral Taylor–Hood

The finite element partitioning $\mathcal{C}_h$ consist of triangles ($d = 2$) or tetrahedrons ($d = 3$) and $k \ge 1$.

$$\begin{aligned}
\boldsymbol{V}_h &= \{\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \,|\, \boldsymbol{v}|_K \in [P_{k+1}(K)]^d \; K \in \mathcal{C}_h \,\},\\
P_h &= \{\, q \in L_0^2(\Omega) \cap C(\Omega) \,|\, q|_K \in P_k(K) \; K \in \mathcal{C}_h \,\}.
\end{aligned} \tag{4.114}$$

The proofs are given in $[3, 5, 12, 14, 17]$ for the 2D problems and $[4, 13]$ in 3D.

Quadrilateral and hexahedral Taylor–Hood

For discontinuous pressure we have seen that the lowest order Crouzeix-Raviart elements need both internal and edge/face bubbles. The lowes order methods in this family are the following.

The finite element partitioning $\mathcal{C}_h$ consist of quadrilateral ($d = 2$) or hexahedral ($d = 3$) and $k \geq 1$.

$$\begin{aligned} \boldsymbol{V}_h &= \{ \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \,|\, \boldsymbol{v}|_K \in [Q_{k+1}(K)]^d \; K \in \mathcal{C}_h \}, \\ P_h &= \{ q \in L_0^2(\Omega) \cap C(\Omega) \,|\, q|_K \in Q_k(K) \; K \in \mathcal{C}_h \}. \end{aligned} \tag{4.115}$$

The proofs are given in [3, 12, 14] for the 2D problems and [13] for $k = 1$ in 3D.

The lowest order method with linear velocities is the MINI element.

Exercise 4.2. *Construct the MINI element for quadrilaterals.*

The Fortin element [8]

$\mathcal{C}_h$ consists of triangles/tetrahedrons. Denote an edge and face bubble function by b_E and the edge/face bubble subspace as

$$\boldsymbol{S}(K) = \text{span}\{b_E \boldsymbol{n}_E, \; E \subset \partial K \}. \tag{4.116}$$

$$\begin{aligned} \boldsymbol{V}_h &= \{ \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \,|\, \boldsymbol{v}|_K \in [P_1(K)]^d \oplus \boldsymbol{S}(K) \; K \in \mathcal{C}_h \}, \\ P_h &= \{ q \in L_0^2(\Omega) \,|\, q|_K \in P_0(K) \; K \in \mathcal{C}_h \}. \end{aligned} \tag{4.117}$$

$Q_{k+1} \; - \; P_k$ **-discontinuous**

The finite element partitioning $\mathcal{C}_h$ consist of quadrilateral ($d = 2$) or hexahedral ($d = 3$) and $k \geq 1$.

$$\begin{aligned} \boldsymbol{V}_h &= \{ \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \,|\, \boldsymbol{v}|_K \in [Q_{k+1}(K)]^d \; K \in \mathcal{C}_h \}, \\ P_h &= \{ q \in L_0^2(\Omega) \,|\, q|_K \in P_k(K) \; K \in \mathcal{C}_h \}. \end{aligned} \tag{4.118}$$

The stability is easily verified [8, 12, 16].

4.6.1 A posteriori estimate

In the preceding section we assumed homogeneous Dirichlet boundary conditions as these are the "worst case" when studuing stability. In ths section we will assume more general boundary conditions and also a non-vanishing loading in the second equation, i.e. the problem: find $(\boldsymbol{u}, p)$ such that

$$-\boldsymbol{A}\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{4.119}$$

$$\text{div}\, \boldsymbol{u} = g \quad \text{in } \Omega, \tag{4.120}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_D, \tag{4.121}$$

$$(\boldsymbol{\varepsilon}(\boldsymbol{u}) - p\boldsymbol{I})\boldsymbol{n} = \boldsymbol{t} \quad \text{on } \Gamma_N. \tag{4.122}$$

With
$$\mathcal{F}(\boldsymbol{v},q) = (\boldsymbol{f},\boldsymbol{v}) + \langle \boldsymbol{t},\boldsymbol{v}\rangle_{\Gamma_N} - (g,q), \tag{4.123}$$
the variational formulation is: Find $(\boldsymbol{u},p) \in \boldsymbol{H}^1_D(\Omega)\times L^2(\Omega)$ such that
$$\mathcal{B}(\boldsymbol{u},p;\boldsymbol{v},q) = \mathcal{F}(\boldsymbol{v},q) \quad \forall (\boldsymbol{v},q)\in \boldsymbol{H}^1_D(\Omega)\times L^2(\Omega). \tag{4.124}$$
The finite element method is then. find $(\boldsymbol{u}_h,p_h)\in \boldsymbol{V}_h\times P_h$ such that
$$\mathcal{B}_h(\boldsymbol{u}_h,p_h;\boldsymbol{v},q) = \mathcal{F}(\boldsymbol{v},q) \quad \forall (\boldsymbol{v},q)\in \boldsymbol{V}_h\times P_h. \tag{4.125}$$
The local error indicators are
$$\eta_K^2 = h_K^2\|\boldsymbol{A}\boldsymbol{u}_h - \nabla p_h + \boldsymbol{f}\|^2_{0,K} + \|\operatorname{div}\boldsymbol{u}_h - g\|^2_{0,K} \tag{4.126}$$
and
$$\eta_E^2 = \begin{cases} h_E\|[\![(\boldsymbol{\varepsilon}(\boldsymbol{u}_h) - p_h\boldsymbol{I})\boldsymbol{n}]\!]\|^2_{0,E}, & \text{when } E\subset\Omega,\\ h_E\|(\boldsymbol{\varepsilon}(\boldsymbol{u}_h) - p_h\boldsymbol{I})\boldsymbol{n} - \boldsymbol{t}\|^2_{0,E}, & \text{when } E\subset\Gamma_N. \end{cases} \tag{4.127}$$
By $\mathcal{E}_h$ we denote the collection of edges/faces in Ω and on Γ_N. The global error estimator is then defined as
$$\eta^2 = \sum_{K\in\mathcal{C}_h}\eta_K^2 + \sum_{E\in\mathcal{E}_h}\eta_E^2. \tag{4.128}$$
The upper bound is given by the following theorem.

Theorem 4.7. *It holds*
$$(\|\boldsymbol{u}-\boldsymbol{u}_h\|_1 + \|p-p_h\|_0) \lesssim \eta. \tag{4.129}$$

Proof. By the stability of the continuous problem there exists $(\boldsymbol{v},q)\in \boldsymbol{H}^1_D(\Omega)\times L^2(\Omega)$ with
$$\|\boldsymbol{v}\|_1 + \|q\|_0 = 1 \tag{4.130}$$
and
$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_1 + \|p-p_h\|_0 \lesssim \mathcal{B}(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h;\boldsymbol{v},q). \tag{4.131}$$
Let $I_h\boldsymbol{v}\in\boldsymbol{V}_h$ be the Clément interpolant (Lemma 2.1) of $\boldsymbol{v}$ for which we recall the estimate
$$\Big(\sum_{K\in\mathcal{C}_h} h_K^{-2}\|\boldsymbol{v}-I_h\boldsymbol{v}\|^2_{0,K} + \sum_{E\in\mathcal{E}_h} h_E^{-1}\|\boldsymbol{v}-I_h\boldsymbol{v}\|^2_{0,E}\Big)^{1/2} \lesssim \|\boldsymbol{v}\|_1 \lesssim 1. \tag{4.132}$$
Choosing the pair $(\boldsymbol{v},q) = (I_h\boldsymbol{v},0)$ in the finite element formulation (4.125), we get
$$\mathcal{B}(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h, I_h\boldsymbol{v},0) = 0. \tag{4.133}$$
Subtracting this from the right hand side in (4.176), we obtain
$$\mathcal{B}(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h;\boldsymbol{v},q) = \mathcal{B}(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h;\boldsymbol{v}-I_h\boldsymbol{v},q). \tag{4.134}$$

Next, we have

$$
\begin{aligned}
&\mathcal{B}(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h; \boldsymbol{v}-I_h\boldsymbol{v}, q) = \mathcal{B}(\boldsymbol{u}, p; \boldsymbol{v}-I_h\boldsymbol{v}, q) - \mathcal{B}(\boldsymbol{u}_h, p_h; \boldsymbol{v}-I_h\boldsymbol{v}, q)\\
&\quad = (\boldsymbol{f}, \boldsymbol{v}-I_h\boldsymbol{v}) + \langle \boldsymbol{t}, \boldsymbol{v}-I_h\boldsymbol{v}\rangle_{\Gamma_N} - (g, q) - \mathcal{B}(\boldsymbol{u}_h, p_h; \boldsymbol{v}-I_h\boldsymbol{v}, q)\\
&\quad = (\boldsymbol{f}, \boldsymbol{v}-I_h\boldsymbol{v}) + \langle \boldsymbol{t}, \boldsymbol{v}-I_h\boldsymbol{v}\rangle_{\Gamma_N} - (\boldsymbol{\varepsilon}(\boldsymbol{u}_h), \boldsymbol{\varepsilon}(\boldsymbol{v}-I_h\boldsymbol{v})) + (\operatorname{div}(\boldsymbol{v}-I_h\boldsymbol{v}), p_h)\\
&\qquad + (\operatorname{div}\boldsymbol{u}_h - g, q).
\end{aligned}
\tag{4.135}
$$

The terms are estimated as follows.

$$
(\operatorname{div}\boldsymbol{u}_h - g, q) \le \|\operatorname{div}\boldsymbol{u} - g\|_0 \|q\|_0 = \|\operatorname{div}\boldsymbol{u} - g\|_0 \lesssim \eta. \tag{4.136}
$$

Integrating by parts gives

$$
\begin{aligned}
&-(\boldsymbol{\varepsilon}(\boldsymbol{u}_h), \boldsymbol{\varepsilon}(\boldsymbol{v}-I_h\boldsymbol{v})) - (\operatorname{div}(\boldsymbol{v}-I_h\boldsymbol{v}), p_h)\\
&\quad = \sum_{K\in\mathcal{C}_h} (\boldsymbol{A}\boldsymbol{u}_h - \nabla p_h, \boldsymbol{v}-I_h\boldsymbol{v})_K\\
&\quad - \sum_{E\subset\Omega} \langle [\![(\boldsymbol{\varepsilon}(\boldsymbol{u}_h) - p_h\boldsymbol{I})\boldsymbol{n}]\!], \boldsymbol{v}-I_h\boldsymbol{v}\rangle_E\\
&\quad - \sum_{E\subset\Gamma_N} \langle (\boldsymbol{\varepsilon}(\boldsymbol{u}_h) - p_h\boldsymbol{I})\boldsymbol{n}, \boldsymbol{v}-I_h\boldsymbol{v}\rangle_E.
\end{aligned}
\tag{4.137}
$$

Hence, we get

$$
\begin{aligned}
&(\boldsymbol{f}, \boldsymbol{v}-I_h\boldsymbol{v}) + \langle \boldsymbol{t}, \boldsymbol{v}-I_h\boldsymbol{v}\rangle_{\Gamma_N} - (\boldsymbol{\varepsilon}(\boldsymbol{u}_h), \boldsymbol{\varepsilon}(\boldsymbol{v}-I_h\boldsymbol{v})) - (\operatorname{div}(\boldsymbol{v}-I_h\boldsymbol{v}), p_h)\\
&= \sum_{K\in\mathcal{C}_h} (\boldsymbol{A}\boldsymbol{u}_h - \nabla p_h + \boldsymbol{f}, \boldsymbol{v}-I_h\boldsymbol{v})_K - \sum_{E\subset\Omega} \langle [\![(\boldsymbol{\varepsilon}(\boldsymbol{u}_h) - p_h\boldsymbol{I})\boldsymbol{n}]\!], \boldsymbol{v}-I_h\boldsymbol{v}\rangle_E\\
&\quad - \sum_{E\subset\Gamma_N} \langle (\boldsymbol{\varepsilon}(\boldsymbol{u}_h) - p_h\boldsymbol{I})\boldsymbol{n} - \boldsymbol{t}, \boldsymbol{v}-I_h\boldsymbol{v}\rangle_E\\
&\le \sum_{K\in\mathcal{C}_h} \|\boldsymbol{A}\boldsymbol{u}_h - \nabla p_h + \boldsymbol{f}\|_{0,K} \|\boldsymbol{v}-I_h\boldsymbol{v}\|_{0,K} + \sum_{E\subset\Omega} \|[\![(\boldsymbol{\varepsilon}(\boldsymbol{u}_h) - p_h\boldsymbol{I})\boldsymbol{n}]\!]\|_{0,K} \|\boldsymbol{v}-I_h\boldsymbol{v}\|_{0,E}\\
&\quad \sum_{E\subset\Gamma_N} \|(\boldsymbol{\varepsilon}(\boldsymbol{u}_h) - p_h\boldsymbol{I})\boldsymbol{n} - \boldsymbol{t}\|_{0,K} \|\boldsymbol{v}-I_h\boldsymbol{v}\|_{0,E}\\
&\le \eta\,\Big(\sum_{K\in\mathcal{C}_h} h_K^{-2} \|\boldsymbol{v}-I_h\boldsymbol{v}\|_{0,K}^2 + \sum_{E\in\mathcal{E}_h} h_E^{-1} \|\boldsymbol{v}-I_h\boldsymbol{v}\|_{0,E}^2\Big)^{1/2} \lesssim \eta\,\|\boldsymbol{v}\|_1 \lesssim \eta.
\end{aligned}
\tag{4.138}
$$

The assertion now follows by combining the above estimates. □

We define $\operatorname{osc}_K(\boldsymbol{f})$ by

$$
\operatorname{osc}_K(\boldsymbol{f}) = h_K \|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,K}, \tag{4.139}
$$

where $\boldsymbol{f}_h \in \boldsymbol{V}_h$ is the interpolant of $\boldsymbol{f}$. Similarly, we define

$$
\operatorname{osc}_E(\boldsymbol{t}) = h_E^{1/2} \|\boldsymbol{t} - \boldsymbol{t}_h\|_{0,E}, \tag{4.140}
$$

with $\boldsymbol{t}_h \in \boldsymbol{V}_h|_{\Gamma_N}$ being the interpolant . The global oscillation terms are defined through

$$\operatorname{osc}(\boldsymbol{f})^2 = \sum_{K \in \mathcal{C}_h} \operatorname{osc}_K(\boldsymbol{f})^2 \quad \text{and} \quad \operatorname{osc}(\boldsymbol{t})^2 = \sum_{E \subset \Gamma_N} \operatorname{osc}_E(\boldsymbol{t})^2. \tag{4.141}$$

The efficiency is obtained exactly in the same way as for the model Poisson problem.

Theorem 4.8. *For all $(\boldsymbol{v}, q) \in \boldsymbol{V}_h \times P_h$ it holds:*

$$h_K \|\boldsymbol{A}\boldsymbol{v} - \nabla q + \boldsymbol{f}\|_{0,K} \lesssim \|\boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{v})\|_{0,K} + \|p - q\|_{0,K} + \operatorname{osc}_K(\boldsymbol{f}) \quad \forall K \in \mathcal{C}_h. \tag{4.142}$$

For E in the interior of Ω

$$h_E^{1/2} \|[\![(\boldsymbol{\varepsilon}(\boldsymbol{v}) - q\boldsymbol{I})\boldsymbol{n}]\!]\|_{0,E} \lesssim \|\boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{v})\|_{0,\omega_E} + \|p - q\|_{0,\omega_E} + \sum_{K \subset \omega_E} \operatorname{osc}_K(\boldsymbol{f}) \tag{4.143}$$

and for $E \subset \Gamma_N$

$$\begin{aligned} h_E^{1/2} \|(\boldsymbol{\varepsilon}(\boldsymbol{v}) - q\boldsymbol{I})\boldsymbol{n} - \boldsymbol{t}\|_{0,E} \lesssim & \|\boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{v})\|_{0,\omega_E} + \|p - q\|_{0,\omega_E} \\ & + \sum_{K \subset \omega_E} \operatorname{osc}_K(\boldsymbol{f}) + \operatorname{osc}_E(\boldsymbol{t}). \end{aligned} \tag{4.144}$$

Exercise 4.3. *Prove this lemma.*

From the local bounds, the global bound follows

$$\eta \lesssim (\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + \|p - p_h\|_0) + \operatorname{osc}(\boldsymbol{f}) + \operatorname{osc}(\boldsymbol{t}). \tag{4.145}$$

4.7 Stabilised methods

In this section we will study stabilisation by which it is possible to design methods for which a stability condition for the subspaces can be avoided. To this end, we first recall the Stokes problem

$$\begin{aligned} -\boldsymbol{A}\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } \Omega, & (4.146)\\ \operatorname{div} \boldsymbol{u} &= 0 \quad \text{in } \Omega, & (4.147)\\ \boldsymbol{u} &= \boldsymbol{0} \quad \text{on } \Gamma_D, & (4.148)\\ (\boldsymbol{\varepsilon}(\boldsymbol{u}) - p\boldsymbol{I})\boldsymbol{n} &= \boldsymbol{0} \quad \text{on } \Gamma_N, & (4.149) \end{aligned}$$

where $\boldsymbol{A}\boldsymbol{u} = \mathbf{div}\, \boldsymbol{\varepsilon}(\boldsymbol{u})$, define the (big) bilinear and linear forms

$$\mathcal{B}(\boldsymbol{w}, r; \boldsymbol{v}, q) = (\boldsymbol{\varepsilon}(\boldsymbol{w}), \boldsymbol{\varepsilon}(\boldsymbol{v})) - (\operatorname{div} \boldsymbol{v}, r) - (\operatorname{div} \boldsymbol{w}, q), \quad \mathcal{F}(\boldsymbol{v}, q) = (\boldsymbol{f}, \boldsymbol{v}),$$

and recall the variational formulation for the continuous problem: Find $(\boldsymbol{u}, p) \in \boldsymbol{H}_D^1(\Omega) \times L^2(\Omega)$ such that

$$\mathcal{B}(\boldsymbol{u}, p; \boldsymbol{v}, q) = \mathcal{F}(\boldsymbol{v}, q) \quad \forall (\boldsymbol{v}, q) \in \boldsymbol{H}_D^1(\Omega) \times L^2(\Omega).$$

Next, we define the following terms

$$\mathcal{S}_h(\boldsymbol{w}, r; \boldsymbol{v}, q) = \sum_{K \in \mathcal{C}_h} h_K^2(-\boldsymbol{A}\boldsymbol{w} + \nabla r, -\boldsymbol{A}\boldsymbol{v} + \nabla q)_K, \tag{4.150}$$

$$\mathcal{L}_h(\boldsymbol{v}, q) = \sum_{K \in \mathcal{C}_h} h_K^2(\boldsymbol{f}, -\boldsymbol{A}\boldsymbol{v} + \nabla q)_K, \tag{4.151}$$

for a FE function pairs $(\boldsymbol{w}, r), (\boldsymbol{v}, q) \in \boldsymbol{V}_h \times P_h$, and then the forms

$$\mathcal{B}_h(\boldsymbol{w}, r; \boldsymbol{v}, q) = \mathcal{B}(\boldsymbol{w}, r; \boldsymbol{v}, q) - \alpha \mathcal{S}_h(\boldsymbol{w}, r; \boldsymbol{v}, q) \tag{4.152}$$

and

$$\mathcal{F}_h(\boldsymbol{v}, q) = \mathcal{F}(\boldsymbol{v}, q) - \alpha \mathcal{L}_h(\boldsymbol{v}, q), \tag{4.153}$$

where α is a positive constant less than a fixed constant C_I to be specified below. The stabilised finite element method reads as follows: Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times P_h$ such that

$$\mathcal{B}_h(\boldsymbol{u}_h, p_h; \boldsymbol{v}, q) = \mathcal{F}_h(\boldsymbol{v}, q) \quad \forall (\boldsymbol{v}, q) \in \boldsymbol{V}_h \times P_h. \tag{4.154}$$

Before explaining the added value of this modification, let us show that it at least does not do any harm. The formulation is consistent:

Theorem 4.9. *Suppose that $\boldsymbol{f} \in \boldsymbol{L}_2(\Omega)$. Then the finite element method* (4.154) *is consistent, in the sense that the exact solution $(\boldsymbol{u}, p) \in \boldsymbol{H}_D^1(\Omega) \times L^2(\Omega)$ to* (4.124) *satisfies the discrete variational form*

$$\mathcal{B}_h(\boldsymbol{u}, p; \boldsymbol{v}, q) = \mathcal{F}_h(\boldsymbol{v}, q) \quad \forall (\boldsymbol{v}, q) \in \boldsymbol{V}_h \times P_h. \tag{4.155}$$

Proof. The differential equation (4.146) has to be interpreted in the sense of distributions. However, with the assumption $\boldsymbol{f} \in \boldsymbol{L}_2(\Omega)$ the sum $-\boldsymbol{A}\boldsymbol{u} + \nabla p$ is in $\boldsymbol{L}_2(\Omega)$ and hence both $\mathcal{S}_h(\boldsymbol{u}, p; \boldsymbol{v}, q)$ and $\mathcal{L}_h(\boldsymbol{v}, q)$ are well defined and equal. Hence, it holds

$$\mathcal{S}_h(\boldsymbol{u}, p; \boldsymbol{v}, q) = \mathcal{L}_h(\boldsymbol{v}, q) \quad \forall (\boldsymbol{v}, q) \in \boldsymbol{V}_h \times P_h, \tag{4.156}$$

from which the claim follow. □

Let us turn to the stability of the method. The constant C_I is given from the following inequality (which follows from the standard inverse inequality)

$$C_I \sum_{K \in \mathcal{C}_h} h_K^2 \|\boldsymbol{A}\boldsymbol{v}\|_{0,K}^2 \leq \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_0^2. \tag{4.157}$$

Next, we turn to the stability. Assume that $0 < \alpha < C_I$. Then it holds

$$\begin{aligned} \mathcal{B}_h(\boldsymbol{w}, r; \boldsymbol{w}, -r) &= \|\boldsymbol{\varepsilon}(\boldsymbol{w})\|_0^2 - \alpha \sum_{K \in \mathcal{C}_h} h_K^2 \|\boldsymbol{A}\boldsymbol{w}\|_{0,K}^2 + \alpha \sum_{K \in \mathcal{C}_h} h_K^2 \|\nabla r\|_{0,K}^2 \\ &\geq (1 - \frac{\alpha}{C_I}) \|\boldsymbol{\varepsilon}(\boldsymbol{w})\|_0^2 + \alpha \sum_{K \in \mathcal{C}_h} h_K^2 \|\nabla r\|_{0,K}^2 \\ &\gtrsim (\|\boldsymbol{\varepsilon}(\boldsymbol{w})\|_0^2 + \sum_{K \in \mathcal{C}_h} h_K^2 \|\nabla r\|_{0,K}^2). \end{aligned} \tag{4.158}$$

The conclusion is that we have the stability (in the mesh dependent norm) for continuous pressures. For discontinuous pressures we have stability for all components except the piecewise constant component. The stability for this must hence be obtained from the original bilinear form. More precisely, for a discontinuous pressure we denote the piecewise constants by

$$P_h^0 = \{\ q \in L^2(\Omega) \mid q|_K \in P_0(K)\ \forall K \in \mathcal{C}_h\ \}, \tag{4.159}$$

and we have to assume that the following discrete stability inequality is valid.

$$\sup_{\boldsymbol{v}\in \boldsymbol{V}_h} \frac{(\operatorname{div} \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1} \gtrsim \|q\|_0 \quad \forall q \in P_h^0. \tag{4.160}$$

Theorem 4.10. *Suppose that one of the following conditions is valid:*

(i) $P_h \subset \mathcal{C}^0(\Omega)$,

(ii) *the stability inequality* (4.160) *is valid.*

For $0 < \alpha < C_I$ *it then holds*

$$\sup_{(\boldsymbol{v},q)\in \boldsymbol{V}_h\times P_h} \frac{\mathcal{B}_h(\boldsymbol{w}, r; \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1 + \|q\|_0} \gtrsim (\|\boldsymbol{w}\|_1 + \|r\|_0) \quad \forall (\boldsymbol{w}, r) \in \boldsymbol{V}_h \times P_h. \tag{4.161}$$

For the proof of the stability we need the following result.

Lemma 4.1. *Under the assumptions of Theorem 4.10, it holds that*

$$\sup_{\boldsymbol{v}\in \boldsymbol{V}_h} \frac{(\operatorname{div} \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1} \geq \Big(C_1 \|q\|_0 - C_2 (\sum_{K\in\mathcal{C}_h} h_K^2 \|\nabla q\|_{0,K}^2)^{1/2} \Big) \quad \forall q \in P_h. \tag{4.162}$$

Proof. For continuous pressures

$$\|q\|_h^2 = \sum_{K\in\mathcal{C}_h} h_K^2 \|\nabla q\|_{0,K}^2 + \sum_{E\subset\Omega} h_E \|[\![q]\!]\|_{0,E}^2 = \sum_{K\in\mathcal{C}_h} h_K^2 \|\nabla q\|_{0,K}^2,$$

and thus the estimate follows from Theorem 4.6, see equation (4.96). For a discontinuous pressure $q \in P_h$ we write

$$q = \Pi_h q + (I - \Pi_h) q, \tag{4.163}$$

where $\Pi_h : L^2(\Omega) \to P_h^0$ is the L^2-projection. Observe that

$$\|q\|_0^2 = \|\Pi_h q\|_0^2 + \|(I - \Pi_h) q\|_0^2, \qquad \Rightarrow \qquad \|\Pi_h q\|_0 \geq \|q\|_0 - \|(I - \Pi_h) q\|_0,$$

and recall the interpolation estimate

$$\|(I - \Pi_h) q\|_0 \leq C_3 (\sum_{K\in\mathcal{C}_h} h_K^2 \|\nabla q\|_{0,K}^2)^{1/2}. \tag{4.164}$$

Now, estimate (4.160) implies that there exists a constant $C_4 > 0$ and $\boldsymbol{v} \in \boldsymbol{V}_h$, with $\|\boldsymbol{v}\|_1 = 1$, such that

$$(\operatorname{div} \boldsymbol{v}, \Pi_h q) \geq C_4 \, \|\Pi_h q\|_0 \qquad \forall q \in P_h \,.$$

Therefore,

$$\begin{aligned}(\operatorname{div} \boldsymbol{v}, q) &= (\operatorname{div} \boldsymbol{v}, \Pi_h q) + (\operatorname{div} \boldsymbol{v}, (I - \Pi_h) q) \geq C_4 \, \|\Pi_h q\|_0 - \|v\|_1 \|(I - \Pi_h) q\|_0 \\ &\geq C_4 \|q\|_0 - C_5 \big(\sum_{K \in \mathcal{C}_h} h_K^2 \|\nabla q\|_{0,K}^2 \big)^{1/2} ,\end{aligned}$$

and the assertion is proved. □

Proof of Theorem 4.10. For any given $(\boldsymbol{v}, q) \in \boldsymbol{V}_h \times P_h$, let $\hat{\boldsymbol{v}} \in \boldsymbol{V}_h$ be a function for which the supremum is attained in the stability estimate (4.162) and assume that $\|\hat{\boldsymbol{v}}\|_1 = \|q\|_0$. We thus obtain, using the inverse inequality (4.157) and the norm equivalence $\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_0 \approx \|\boldsymbol{v}\|_1$, that

$$\begin{aligned}\mathcal{B}_h(\boldsymbol{v}, q; -\hat{\boldsymbol{v}}, 0) &= -(\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\hat{\boldsymbol{v}})) + (\operatorname{div} \hat{\boldsymbol{v}}, q) - \alpha \sum_{K \in \mathcal{C}_h} h_K^2 (-\boldsymbol{A}\boldsymbol{v} + \nabla q, \boldsymbol{A}\hat{\boldsymbol{v}})_K \\ &\geq -\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_0 \|\boldsymbol{\varepsilon}(\hat{\boldsymbol{v}})\|_0 + \|\hat{\boldsymbol{v}}\|_1 \Big(C_1 \|q\|_0 - C_2 \big(\sum_{K \in \mathcal{C}_h} h_K^2 \|\nabla q\|_{0,K}^2 \big)^{1/2} \Big) \\ &\quad + \alpha \sum_{K \in \mathcal{C}_h} h_K^2 (\boldsymbol{A}\boldsymbol{v}, \boldsymbol{A}\hat{\boldsymbol{v}})_K - \alpha \sum_{K \in \mathcal{C}_h} h_K^2 (\nabla q, \boldsymbol{A}\hat{\boldsymbol{v}})_K \\ &\geq -\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_0 \|\boldsymbol{\varepsilon}(\hat{\boldsymbol{v}})\|_0 + C_1 \|q\|_0^2 - C_2 \big(\sum_{K \in \mathcal{C}_h} h_K^2 \|\nabla q\|_{0,K}^2 \big)^{1/2} \|q\|_0 \\ &\quad - \alpha \, C_I^{-1} \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_0 \|\boldsymbol{\varepsilon}(\hat{\boldsymbol{v}})\|_0 - \alpha \, C_I^{1/2} \big(\sum_{K \in \mathcal{C}_h} h_K^2 \|\nabla q\|_{0,K}^2 \big)^{1/2} \|\boldsymbol{\varepsilon}(\hat{\boldsymbol{v}})\|_0 \\ &\geq -C_3 \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_0 \|q\|_0 + C_1 \|q\|_0^2 - C_4 \big(\sum_{K \in \mathcal{C}_h} h_K^2 \|\nabla q\|_{0,K}^2 \big)^{1/2} \|q\|_0 \\ &\geq C_4 \|q\|_0^2 - C_5 \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_0^2 - C_6 \sum_{K \in \mathcal{C}_h} h_K^2 \|\nabla q\|_{0,K}^2 \,.\end{aligned}$$

On the other hand, we already proved that for $0 < \alpha < C_I$, it holds that

$$\mathcal{B}_h(\boldsymbol{v}, q; \boldsymbol{v}, -q) \geq C_7 \Big(\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_0^2 + \sum_{K \in \mathcal{C}_h} h_K^2 \|\nabla q\|_{0,K}^2 \Big).$$

Therefore,

$$\begin{aligned}\mathcal{B}_h(\boldsymbol{v}, q; \boldsymbol{v} - \delta \hat{v}, -q) &= \mathcal{B}_h(\boldsymbol{v}, q; \boldsymbol{v}, -q) + \delta \mathcal{B}_h(\boldsymbol{v}, q; -\hat{\boldsymbol{v}}, 0) \\ &\geq (C_7 - \delta C_5) \, \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_0^2 + \delta \, C_4 \|q\|_0^2 + (C_7 - \delta C_6) \sum_{K \in \mathcal{C}_h} h_K^2 \|\nabla q\|_{0,K}^2 \,.\end{aligned}$$

Thus, choosing $0 < \delta < \min\{C_7/C_5, C_7/C_6\}$, and defining $\boldsymbol{w} = \boldsymbol{v} - \delta \, \hat{v}, r = -q$, we conclude that

$$\mathcal{B}_h(\boldsymbol{v}, q; \boldsymbol{w}, r) \gtrsim \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_0^2 + \|q\|_0^2 \,.$$

Moreover,

$$\|\boldsymbol{\varepsilon}(\boldsymbol{w})\|_0+\|r\|_0 \lesssim \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_0+\|q\|_0,$$

which proves the assertion. □

From the stability and the consistency the a priori estimate follows.

Theorem 4.11. *It holds*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_1+\|p-p_h\|_0 \lesssim \inf_{\boldsymbol{v}\in\boldsymbol{V}_h}\|\boldsymbol{u}-\boldsymbol{v}\|_1+\inf_{q\in P_h}\|p-q\|_0+\operatorname{osc}(\boldsymbol{f}). \tag{4.165}$$

Proof. Let $(\boldsymbol{v},q)\in\boldsymbol{V}_h\times P_h$ be arbitrary. By the stability estimate (4.161) there exists $(\boldsymbol{w},r)\in\boldsymbol{V}_h\times P_h$ with

$$\|\boldsymbol{w}\|_1+\|r\|_0=1 \tag{4.166}$$

and

$$\|\boldsymbol{u}_h-\boldsymbol{v}\|_1+\|p_h-q\|_0 \lesssim \mathcal{B}_h(\boldsymbol{u}_h-\boldsymbol{v},p_h-q;\boldsymbol{w},r). \tag{4.167}$$

Using (4.154), (4.153), (4.124) and (4.152) yields

$$\begin{aligned}
\mathcal{B}_h(\boldsymbol{u}_h-\boldsymbol{v},p_h-q;\boldsymbol{w},r) &= \mathcal{B}_h(\boldsymbol{u}_h,p_h,\boldsymbol{w},r)-\mathcal{B}_h(\boldsymbol{v},q;\boldsymbol{w},r)\\
&= \mathcal{F}_h(\boldsymbol{w},r)-\mathcal{B}_h(\boldsymbol{v},q;\boldsymbol{w},r)\\
&= \mathcal{F}(\boldsymbol{w},r)-\alpha\mathcal{L}_h(\boldsymbol{w},r)-\mathcal{B}_h(\boldsymbol{v},q;\boldsymbol{w},r)\\
&= \mathcal{B}(\boldsymbol{u},p;\boldsymbol{w},r)-\alpha\mathcal{L}_h(\boldsymbol{w},r)-\mathcal{B}_h(\boldsymbol{v},q;\boldsymbol{w},r)\\
&= \mathcal{B}(\boldsymbol{u},p;\boldsymbol{w},r)-\alpha\mathcal{L}_h(\boldsymbol{w},r)-\mathcal{B}(\boldsymbol{v},q;\boldsymbol{w},r)+\alpha\mathcal{S}_h(\boldsymbol{v},q;\boldsymbol{w},r)\\
&= \mathcal{B}(\boldsymbol{u}-\boldsymbol{v},p-q;\boldsymbol{w},r)+\alpha(\mathcal{S}_h(\boldsymbol{v},q;\boldsymbol{w},r)-\mathcal{L}_h(\boldsymbol{w},r)).
\end{aligned} \tag{4.168}$$

From the boundedness of the bilinear form $\mathcal{B}$ and the normalization (4.166), we have

$$\mathcal{B}(\boldsymbol{u}-\boldsymbol{v},p-q;\boldsymbol{w},r)\lesssim(\|\boldsymbol{u}-\boldsymbol{v}\|_1+\|p-q\|_0). \tag{4.169}$$

From the definitions (4.150) and (4.151) we have

$$\mathcal{S}_h(\boldsymbol{v},q;\boldsymbol{w},r)-\mathcal{L}_h(\boldsymbol{w},r)=\sum_{K\in\mathcal{C}_h}h_K^2(-\boldsymbol{A}\boldsymbol{v}+\nabla q-\boldsymbol{f},-\boldsymbol{A}\boldsymbol{w}+\nabla r)_K. \tag{4.170}$$

The Cauchy–Schwarz inequality then yields

$$\begin{aligned}
&|\mathcal{S}_h(\boldsymbol{v},q;\boldsymbol{w},r)-\mathcal{L}_h(\boldsymbol{w},r)|\\
&\quad\le\Big(\sum_{K\in\mathcal{C}_h}h_K^2\|-\boldsymbol{A}\boldsymbol{v}+\nabla q-\boldsymbol{f}\|_{0,K}^2\Big)^{1/2}\Big(\sum_{K\in\mathcal{C}_h}h_K^2\|-\boldsymbol{A}\boldsymbol{w}+\nabla r\|_{0,K}^2\Big)^{1/2}.
\end{aligned}$$

By local inverse inequalities we have

$$\begin{aligned}
\Big(\sum_{K\in\mathcal{C}_h}h_K^2\|-\boldsymbol{A}\boldsymbol{w}+\nabla r\|_{0,K}^2\Big)^{1/2} &\le \Big(2\sum_{K\in\mathcal{C}_h}h_K^2(\|\boldsymbol{A}\boldsymbol{w}\|_{0,K}^2+\|\nabla r\|_{0,K}^2)\Big)^{1/2}\\
&\lesssim (\|\boldsymbol{w}\|_1+\|r\|_0).
\end{aligned}$$

Hence, (4.142) gives

$$\begin{aligned} |\mathcal{S}_h(\boldsymbol{v},q;\boldsymbol{w},r) - \mathcal{L}_h(\boldsymbol{w},r)| &\lesssim \Big(\sum_{K\in\mathcal{C}_h} h_K^2\|-\boldsymbol{A}\boldsymbol{v}+\nabla q-\boldsymbol{f}\|_{0,K}^2\Big)^{1/2} \\ &\lesssim (\|\boldsymbol{u}-\boldsymbol{v}\|_1+\|p-q\|_0+\operatorname{osc}(\boldsymbol{f})). \end{aligned}$$

The assertion now follows from (4.167), (4.168), (4.169) and (4.171). □

We finish the section by the a posteriori analysis.

For the a posteriori estimates we define the local estimators

$$\eta_K^2 = h_K^2\|\boldsymbol{A}\boldsymbol{u}_h - \nabla p_h + \boldsymbol{f}\|_{0,K}^2+\|\operatorname{div}\boldsymbol{u}_h - \boldsymbol{g}\|_{0,K}^2 \tag{4.171}$$

and

$$\eta_E^2 = \begin{cases} h_E\|[\![(\boldsymbol{\varepsilon}(\boldsymbol{u}_h)-p_h\boldsymbol{I})\boldsymbol{n}]\!]\|_{0,E}^2, & \text{when } E\subset\Omega, \\ h_E\|(\boldsymbol{\varepsilon}(\boldsymbol{u}_h)-p_h\boldsymbol{I})\boldsymbol{n}-\boldsymbol{t}\|_{0,E}^2, & \text{when } E\subset\Gamma_N. \end{cases} \tag{4.172}$$

By $\mathcal{E}_h$ we denote the collection of edges/faces in Ω and on Γ_N. The global error estimator is then defined as

$$\eta^2 = \sum_{K\in\mathcal{C}_h}\eta_K^2 + \sum_{E\in\mathcal{E}_h}\eta_E^2. \tag{4.173}$$

Taking $(\boldsymbol{v},q) = (\boldsymbol{u}_h,p_h)$ in Theorem4.8 yields a local lower bound for the error. Now we will prove the following upper bound.

Theorem 4.12. *It holds*

$$(\|\boldsymbol{u}-\boldsymbol{u}_h\|_1+\|p-p_h\|_0) \lesssim \eta. \tag{4.174}$$

Proof. From the stability of the continuous problem, it follows that, for any $(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)\in \boldsymbol{H}_D^1(\Omega)\times L^2(\Omega)$, there exists $(\boldsymbol{v},q)\in\boldsymbol{H}_D^1(\Omega)\times L^2(\Omega)$, such that

$$\|\boldsymbol{v}\|_1+\|q\|_0= 1 \tag{4.175}$$

and

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_1+\|p-p_h\|_0\lesssim \mathcal{B}(\boldsymbol{u}-\boldsymbol{u}_h,p-p_h;\boldsymbol{v},q). \tag{4.176}$$

Let $I_h\boldsymbol{v}\in\boldsymbol{V}_h$ be the Clément interpolant of $\boldsymbol{v}$ for which we have the estimates

$$\Big(\sum_{K\in\mathcal{C}_h} h_K^{-2}\|\boldsymbol{v}-I_h\boldsymbol{v}\|_{0,K}^2+\sum_{E\in\mathcal{E}_h} h_E^{-1}\|\boldsymbol{v}-I_h\boldsymbol{v}\|_{0,E}^2\Big)^{1/2}\lesssim\|\boldsymbol{v}\|_1\lesssim 1,$$

$$\|I_h\boldsymbol{v}\|_1\lesssim\|\boldsymbol{v}\|_1\,.$$

Choosing the pair $(\boldsymbol{v},q)=(I_h\boldsymbol{v},0)$ in the finite element formulation (4.125) and the consistency equation (4.155), we get

$$\mathcal{B}_h(\boldsymbol{u}-\boldsymbol{u}_h,p-p_h,I_h\boldsymbol{v},0)=0. \tag{4.177}$$

Subtracting this from the right hand side in (4.176), and using the definition of $\mathcal{B}_h$, we obtain

$$\begin{aligned}\mathcal{B}(\boldsymbol{u}-\boldsymbol{u}_h,p-p_h;\boldsymbol{v},q) &= \mathcal{B}(\boldsymbol{u}-\boldsymbol{u}_h,p-p_h;\boldsymbol{v},q)-\mathcal{B}_h(\boldsymbol{u}-\boldsymbol{u}_h,p-p_h,I_h\boldsymbol{v},0)\\ &= \mathcal{B}(\boldsymbol{u}-\boldsymbol{u}_h,p-p_h;\boldsymbol{v}-I_h\boldsymbol{v},q)-\ \alpha\mathcal{S}_h(\boldsymbol{u}-\boldsymbol{u}_h,p-p_h;I_h\boldsymbol{v},0).\end{aligned}$$

The first term above is estimated exactly as in Theorem 4.7 using element by element integration by parts and the interpolation estimate (2.1). This results in

$$\mathcal{B}(\boldsymbol{u}-\boldsymbol{u}_h,p-p_h;\boldsymbol{v}-I_h\boldsymbol{v},q)\lesssim\eta. \tag{4.178}$$

Recalling definition (4.150), equation (4.146), and using an inverse inequality together with estimate (2.1), we then get

$$\begin{aligned}|\mathcal{S}_h(\boldsymbol{u}-\boldsymbol{u}_h,p{-}p_h;I_h\boldsymbol{v},0)| &= |\sum_{K\in\mathcal{C}_h}h_K^2(-\boldsymbol{A}\boldsymbol{u}+\nabla p+\boldsymbol{A}\boldsymbol{u}_h-\nabla p_h,-\boldsymbol{A}(I_h\boldsymbol{v}))_K|\\ &\le (\sum_{K\in\mathcal{C}_h}h_K^2\|\boldsymbol{f}+\boldsymbol{A}\boldsymbol{u}_h-\nabla p_h\|_{0,K})^{1/2}(\sum_{K\in\mathcal{C}_h}h_K^2\|\boldsymbol{A}(I_h\boldsymbol{v})\|_{0,K})^{1/2}\\ &\lesssim \eta\|I_h\boldsymbol{v}\|_1\lesssim\eta\|\boldsymbol{v}\|_1\lesssim\eta.\end{aligned}$$

The assertion now follows by combining the above estimates. □

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