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MS-E1997: Abstract Algebra II

Problem Set II

Problem 1:

(a) Consider the action of S_3 on itself by conjugation: $gx := gxg^1$. Describe the orbits of (12) and (123) under this action. How many orbits are there in total in S_3? Find the stabilizer of each of the elements above.

(b) Consider the action of S_4 on itself by conjugation: $gx := gxg^1$. Describe the orbits of (12), (123), and $(12)(34)$ under this action. How many orbits are there in total in S_4? Find the stabilizer of each of the elements above.

Problem 2:

(a) Determine all Sylow subgroups of the symmetric group S_3 and of $\mathbb{Z}/4900\mathbb{Z}$.

(b) Let G be a group of order 12. Determine the possible values of s_2 and s_3 the 2- and 3-Sylow subgroups of G. Show that from $s_3 = 4$ there follows $s_2 = 1$, and that if $s_2 = 1 = s_3$ the group G is abelian.

(c) Let G be a finite group and H a normal subgroup of G. Show that the p-Sylow subgroups of H are exactly the intersections of p-Sylow subgroups of G with H.

Work: **(a)** $|S_3| = 3! = 6$ and hence we expect p-Sylow subgroups for $p = 2$ and $p = 3$. Knowing $s_3 \mid 6$ and $s_3 \equiv 1 \pmod 3$ the only value for s_3 is 1 and the corresponding subgroup is given by $\{\mathrm{id}, (123), (321)\}$. The remaining elements of S_3 are $(12), (13), (23)$ and these give rise to the 2-Sylow subgroups $\{\mathrm{id}, (12)\}$, $\{\mathrm{id}, (13)\}$ and $\{\mathrm{id}, (23)\}$. The group $\mathbb{Z}_{4900}$ is abelian, and for this reason $s_2 = s_5 = s_7 = 1$. From the main theorem characterizing finitely generated abelian groups we further know that the Sylow subgoups are isomorphic to $\mathbb{Z}_4$, $\mathbb{Z}_{25}$, and $\mathbb{Z}_{49}$, respectively.

(b) $s_2 \equiv 1 \pmod 2$ and $s_2 \mid 12$ yields $s_2 \in \{1, 3\}$ and similarly we get $s_3 \in \{1, 4\}$. If $s_3 = 4$ then these 4 different 3-Sylow subgroups cover 9 elements of G. The remaining 3 elements together with the identity fit in exactly one 2-Sylow subgroup of order 4, and this proves the claim. If now $s_2 = 1 = s_3$ then the corresponding Sylow subgroups U_2 and U_3 are normal in G and intersect only trivially. This yields already $G = U_2U_3$ where the elements of U_2 and U_3 commute with each other by 1.17 (e). Both subgroups U_2 and U_3 are abelian, because this is trivially true for $U_3 \cong \mathbb{Z}_3$ and because we can easily check that a U_2 as a 4-element group must be abelian. From all we get that G is abelian.

(c) If Q is a p-Sylow subgroup of H then by Sylow' theorems we have a p-Sylow subgroup P of G with $Q \leq P$. Then we have $Q := Q \cap H \leq P \cap H$ where the latter is a p-subgroup of H. As Q is a maximal p-subgroup of H we have $Q = P \cap H$.

Let on the other hand P be a p-Sylow subgroup of G and consider $Q := P \cap H$. Then Q is a p-subgroup of H which is contained in a p-Sylow subgroup Q' of H by Sylow's first theorem. Applying Sylow's second theorem we find $g \in G$ such that $gQ'g^{-1} \leq P$, and as H is normal even $gQ'g^{-1} \leq P \cap H = Q$. Hence $|Q'| \leq |Q|$ showing that Q is already a Sylow subgroup of H.

Problem 3:

(a) For prime numbers p, q with $p \geq q$ and arbitrary $r \in \mathbb{N}$ every group of order $p^r q$ is solvable.

(b) Every group of order 100 is solvable.

Work: **(a)** If $p = q$ then the claim is true because every p-group is solvable. Assume therefore $p > q$, then we have p-Sylow subgroups of order p^r and q-Sylow subgroups of order q in G. We try to prove that $s_p = 1$, meaning that we have a unique and normal p-Sylow subgroup U_p in G. Then G/U_p is solvable as a q-group and U_p is solvable as a p-group which yields G to be solvable according to proposition 1.67. The possible values of s_p are $\{1, q, p, \ldots, p^r, pq, \ldots, p^r q\}$. The assumption that $s_p \equiv 1 \pmod p$ reduces these possibilities to $\{1, q\}$ and as $2 \leq q < p$, we find that $s_p = 1$, as claimed.

(b) We expect 2-Sylow subgroups of order 4 and 5-Sylow subgroups of order 25. For s_5 we have the possibilities $\{1, 2, 4, 5, 10, 20, 25, 50, 100\}$ from which all but the first are removed by the condition $s_5 \equiv 1 \pmod 5$. Hence, we have a unique normal 5-Sylow subgroup U_5 and again G/U_5 is solvable because of being a 2-group and U_5 is solvable because of being a 5-group. This shows the result again by application of proposition 1.68.

Problem 4: Let G be a group. For $a, b \in G$ define the commutator $[a, b] := aba^{-1}b^{-1}$ of a and b. For arbitrary subgroups U, V of G define $[U, V] := \langle [u, v] \mid u \in U, v \in V \rangle$. Now show the following:

(a) If U, V are normal subgroups of G, then so is $[U, V]$.

(b) $[G, G]$ is the smallest normal subgroup of G for which the factor group is abelian.

(c) Setting $G^{(0)} := G$ und $G^{(i)} := [G^{(i-1)}, G^{(i-1)}]$ for all $i \in \mathbb{N}$, we find that G is solvable if and only if there exists $n \in \mathbb{N}$ such that $G^{(n)} = \{e\}$.

(d) For $n \geq 5$ let U be a subgroup of S_n and N a normal subgroup of U for which U/N is abelian. Show that if U contains all 3-cycles of S_n, then also N will contain these.
Hint: If $u, v, w, x, y \in \{1, \ldots, n\}$ are distinct elements, then there holds the equation

$$(u, v, w) = (u, v, x)(w, y, u)(x, v, u)(u, y, w).$$

(e) Show that this implies that the symmetric group S_n is not solvable for $n \geq 5$.

Work: **(a)** It is easily verified for an arbitrary subset M of a group G that $\langle M \rangle$ is normal in G if and only if $gMg^{-1} \subseteq \langle M \rangle$. In the current context M can just be chosen to be the set of commutators $\{[u, v] \mid u \in U,\, v \in V\}$. For each of these commutators we find $g[u, v]g^{-1} = [gug^{-1}, gvg^{-1}]$ and as U and V are normal, we now even have $gMg^{-1} \subseteq M$. This proves the claim.

(b) We have just seen that $[G, G]$ is normal in G. The quotient group $G/[G, G]$ is abelian, since $gh = [g, h]hg$ for all $g, h \in G$. If now H is a normal subgroup of G with G/H being abelian, then we have $xyH = yxH$ for all $x, y \in G$ which can be equivalently written as $[x, y]H = H$, for all $x, y \in G$. This however shows that H contains a generator of $[G, G]$ and hence $[G, G]$ itself.

(c) If this commutator series terminates at $\{e\}$ then clearly we have a solvable group by definition. If on the other hand G is solvable, then we have a normal series $G = H_0 \geq H_1 \geq \ldots \geq H_n = \{1\}$, and first observe $G = G^{(0)} \subseteq H_0$. We show that $G^{(i)} \subseteq H_i$ for all $i = 0, \ldots, n$, by which then commutator series will then terminate in $\{1\}$. This follows by induction considering that $G^{(i+1)} = [G^{(i)}, G^{(i)}] \subseteq [H_i, H_i] \subseteq H_{i+1}$ because H_i/H_{i+1} being abelian implies that $[H_i, H_i] \subseteq H_{i+1}$.

(d) According to the hint every 3-cycle of S_n is a commutator provided $n \geq 5$ (otherwise we have not enough space, as one could say). If now U contains all 3-cycles of S_n then also $[U, U]$ will contain them. If then N is normal in U with U/N abelian, then $[U, U] \leq N$ as we have seen before, and hence N contains all 3-cycles. This shows the claim.

(e) If S_n were solvable then we would have a normal series $S_n = H_0 \geq H_1 \geq \ldots \geq H_k = \{\mathrm{id}\}$ for some $k \in \mathbb{N}$. Clearly S_n contains all 3-cycles of S_n, and using our claim in **(d)** we inductively see that if H_i contains all 3-cycles then H_{i+1} must contain those for all $i = 0, \ldots, k-1$. This however will finally contradict the fact that $H_k = \{\mathrm{id}\}$, and thus shows that S_n cannot be solvable for all $n \geq \mathbb{N}$.

You are encouraged to collaborate in preparing solutions, however, please submit individual write-ups.