MS-E1999 Special Topics in the Finite Element Method

1. Prove the following result in one space dimension.

   For $v \in H^1(K)$ it holds
   $$\|v\|^2_{0,\partial K} \lesssim \Big(h_K^{-1}\|v\|^2_{0,K} + h_K \|\nabla v\|^2_{0,K}\Big).$$

   Solution: Assume, without loss of generality, that $I = (0, h)$, and for $v \in H^1(I)$ write
   $$v(x) - v(0) = \int_0^x v'(x)\, dx\,.$$

   It follows, using Cauchy-Schwartz inequality, that
   $$v(0)^2 = \Big(v(x) - \int_0^x v'(x)\, dx\Big)^2 \lesssim v(x)^2 + (\int_0^x v'(x)\, dx)^2 \le v(x)^2 + \int_0^h 1^2 dx \int_0^h v'(x)^2\, dx$$
   $$= v(x)^2 + h\|v'\|^2_{0,I}$$

   Integrating the previous inequality in x over I, yields
   $$hv(0)^2 \lesssim \|v\|^2_{0,I} + h^2\|v'\|^2_{0,I} \qquad \Leftrightarrow \qquad v(0)^2 \lesssim h^{-1}\|v\|^2_{0,I} + h\|v'\|^2_{0,I}\,.$$

   Similarly, writing
   $$v(h) - v(x) = \int_x^h v'(x)\, dx\,,$$
   one obtains
   $$v(h)^2 \lesssim h^{-1}\|v\|^2_{0,I} + h\|v'\|^2_{0,I},$$
   and thus
   $$v(0)^2 + v(h)^2 \lesssim h^{-1}\|v\|^2_{0,I} + h\|v'\|^2_{0,I}\,.$$

2. Consider the problem with a (positive) diffusion coefficient: find $u \in H^1(\Omega)$ such that
   $$\begin{aligned} -\mathrm{div}\,(k\nabla u) &= f \text{ in } \Omega,\\ u &= 0 \text{ on } \Gamma_D,\\ k\frac{\partial u}{\partial n} &= g \text{ on } \Gamma_N,\ \Gamma_D \cup \Gamma_N = \partial\Omega. \end{aligned}$$

   How is the error estimator now defined?

   Solution: The variational and the FE formulations read now as follows
   $$(k\nabla u, \nabla v) = (f, v) + \langle g, v\rangle_{\Gamma_N} \quad \forall v \in H^1_D(\Omega), \tag{0.1}$$
   and
   $$(k\nabla u_h, \nabla v) = (f, v) + \langle g, v\rangle_{\Gamma_N} \quad \forall v \in V_h. \tag{0.2}$$

The energy norm is defined as

$$|||v||| := \|k^{1/2}\nabla v\|_0 \,.$$

As before, we then define $e = u - u_h$, let $I_h e \in V_h$ be the Clément interpolant of e and calculate

$$\begin{aligned}
|||e|||^2 &= (k\nabla e, \nabla e) = (k\nabla u, \nabla(e - I_h e)) - (k\nabla u_h, \nabla(e - I_h e)) \\
&= \sum_{K\in\mathcal{C}_h} (f, e - I_h e)_K - \sum_{K\in\mathcal{C}_h} (k\nabla u_h, \nabla(e - I_h e))_K + \langle g, e - I_h e\rangle_{\Gamma_N} \\
&= \sum_{K\in\mathcal{C}_h} (\mathrm{div}(k\nabla u_h) + f, e - I_h e)_K \\
&\quad + \langle g, e - I_h e\rangle_{\Gamma_N} - \sum_{K\in\mathcal{C}_h} \langle k\nabla u_h \cdot \boldsymbol{n}_K, e - I_h e\rangle_{\partial K}.
\end{aligned}$$

Since the energy norm has been redefined, we need to write the Clément interpolation estimate in the form

$$\sum_{K\in\mathcal{C}_h} \frac{k}{h_K^2} ||e - I_h e||_{0,K}^2 + \sum_{E\in\Omega\cup\Gamma_N} \frac{k}{h_E} ||e - I_h e||_{0,E}^2 \lesssim |||e|||^2 \,.$$

Consequently, the local error estimators are defined as

$$\begin{aligned}
\eta_K &= \frac{h_K}{k}\|\mathrm{div}(k\nabla u_h) + f\|_{0,K}, \qquad K \in \mathcal{C}_h, \\
\eta_{E,\Omega} &= \frac{h_E^{1/2}}{k}\|[\![k\frac{\partial u_h}{\partial n_E}]\!]\|_{0,E}, \qquad E \in \Omega, \\
\eta_{E,N} &= \frac{h_E^{1/2}}{k}\|k\frac{\partial u_h}{\partial n} - g\|_{0,E}, \qquad E \subset \Gamma_N.
\end{aligned}$$

The global error indicator is, as before,

$$\eta^2 = \sum_{K\in\mathcal{C}_h} \eta_K^2 + \sum_{E\subset\Omega} (\eta_{E,\Omega})^2 + \sum_{E\subset\Gamma_N} (\eta_{E,N})^2.$$

The rest of proof goes as in the lecture notes for the Poisson problem. For example, we estimate

$$\begin{aligned}
&\sum_{K\in\mathcal{C}_h} (\mathrm{div}(k\nabla u_h) + f, e - I_h e)_K \\
&\quad \le \sum_{K\in\mathcal{C}_h} \|\mathrm{div}(k\nabla u_h) + f\|_{0,K}\|e - I_h e\|_{0,K} \\
&\quad \le \sum_{K\in\mathcal{C}_h} \frac{h_K}{k^{1/2}}\|\mathrm{div}(k\nabla u_h) + f\|_{0,K}\frac{k^{1/2}}{h_K}\|e - I_h e\|_{0,K} \\
&\quad \le \Big(\sum_{K\in\mathcal{C}_h} \frac{h_K^2}{k}\|\mathrm{div}(k\nabla u_h) + f\|_{0,K}^2\Big)^{1/2}\Big(\sum_{K\in\mathcal{C}_h} \frac{k}{h_K^2}\|e - I_h e\|_{0,K}^2\Big)^{1/2} \\
&\quad \lesssim \eta|||e||| \,.
\end{aligned}$$

3. Assume that $\Gamma_N = \emptyset$, and that the regularity estimate

$$\|u\|_2 \lesssim \|f\|_0$$

holds in the Poisson problem. Use Nitsche's trick and the Lagrange interpolation operator to show that

$$\|u - u_h\|_0 \lesssim \Big(\sum_{K\in\mathcal{C}_h} h_K^4 \|\Delta u_h + f\|_{0,K}^2 + \sum_{E\subset\Omega} h_E^3 \|[\![\frac{\partial u_h}{\partial n_E}]\!]\|_{0,E}^2 \Big)^{1/2}.$$

<u>Solution:</u> Let $u \in H_D^1(\Omega)$ and $u_h \in V_H^1$ be respectively the exact and the FEM solution to the Poisson problem. Define $e = u - u_h$ as the error and φ as the solution to the dual problem

$$\begin{aligned} -\Delta\varphi &= e \quad \text{in } \Omega, \\ \varphi &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Assume, moreover, that the elliptic regularity estimate

$$\|\varphi\|_2 \lesssim \|e\|_0$$

holds and let $I_h : H_D^1(\Omega) \to V_h^1$ be the Lagrange interpolation operator for which we have the estimate

$$\Big(\sum_{K\in\mathcal{C}_h} \{ h_K^{-4} ||v - I_h v||_{0,K}^2 + h_E^{-3} ||v - I_h \varphi||_{0,\partial K}^2 \} \Big)^{1/2} \lesssim \|v\|_2. \tag{0.3}$$

It follows that

$$\|e\|_0^2 = -(e, \Delta\varphi) = (\nabla e, \nabla\varphi) = (\nabla e, \nabla(\varphi - I_h\varphi)) = (\nabla u, \nabla(\varphi - I_h\varphi)) - (\nabla u_h, \nabla(\varphi - I_h\varphi)),$$

where we have used the Galerkin orthogonality $(\nabla e, \nabla I_h\varphi) = 0$. Therefore

$$\begin{aligned} \|e\|_0^2 &= (f, \varphi - I_h\varphi) - (\nabla u_h, \nabla(\varphi - I_h\varphi)) \\ &= \sum_{K\in\mathcal{C}_h} (f, \varphi - I_h\varphi)_K - \sum_{K\in\mathcal{C}_h} (\nabla u_h, \nabla(\varphi - I_h\varphi))_K \\ &= \sum_{K\in\mathcal{C}_h} (\Delta u_h + f, \varphi - I_h\varphi)_K - \sum_{K\in\mathcal{C}_h} \langle \nabla u_h \cdot \boldsymbol{n}_K, \varphi - I_h\varphi \rangle_{\partial K} \\ &\le \sum_{K\in\mathcal{C}_h} h_K^2 \|\Delta u_h + f\|_{0,K}\, h_K^{-2} \|\varphi - I_h\varphi\|_{0,K} + \sum_{E\subset\Omega} h_E^{3/2} \|[\![\frac{\partial u_h}{\partial n_E}]\!]\|_{0,E}\, h_E^{-3/2} \|\varphi - I_h\varphi\|_{0,E} \\ &\le \Big(\sum_{K\in\mathcal{C}_h} h_K^4 \|\Delta u_h + f\|_{0,K}^2 \Big)^{1/2} \Big(\sum_{K\in\mathcal{C}_h} h_K^{-4} \|\varphi - I_h\varphi\|_{0,K}^2 \Big)^{1/2} \\ &\quad + \Big(\sum_{E\subset\Omega} h_E^3 \|[\![\frac{\partial u_h}{\partial n_E}]\!]\|_{0,E}^2 \Big)^{1/2} \Big(\sum_{E\subset\Omega} h_E^{-3} \|\varphi - I_h\varphi\|_{0,E} \Big)^{1/2}. \end{aligned}$$

Defining the global error estimator

$$\eta^2 = \sum_{K\in\mathcal{C}_h} h_K^4 \|\Delta u_h + f\|_{0,K}^2 + \sum_{E\subset\Omega} h_E^3 \|[\![\frac{\partial u_h}{\partial n_E}]\!]\|_{0,E}^2$$

and using the Lagrange interpolation estimate for φ, we obtain

$$\begin{aligned}\|e\|_0^2 &\lesssim \Big(\sum_{K\in\mathcal{C}_h} h_K^4 \|\Delta u_h + f\|_{0,K}^2 + \sum_{E\subset\Omega} h_E^3 \|[\![\frac{\partial u_h}{\partial n_E}]\!]\|_{0,E}^2 \Big)^{1/2} \|\varphi\|_2 \\ &\lesssim \eta \|\Delta\varphi\|_0 = \eta\|e\|_0,\end{aligned}$$

from which the assertion follows after division by $\|e\|_0$.

4. Show that the strain vanishes if and only if the displacement is a infinitesimal rigid body motion, i.e.

$$\boldsymbol{\varepsilon}(\boldsymbol{v}) = \mathbf{0} \;\Leftrightarrow\; \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{a} + \boldsymbol{b}\times\boldsymbol{x} \quad \text{for some } \boldsymbol{a},\, \boldsymbol{b}.$$

<u>Solution:</u> Let $\boldsymbol{v} = (v_1, v_2, v_3), \boldsymbol{a} = (a_1, a_2, a_3), \boldsymbol{b} = (b_1, b_2, b_3), \boldsymbol{x} = (x, y, z)$ and write the rigid body motion componentwise as

$$\begin{aligned} v_1 &= a_1 - b_3 y + b_2 z, \\ v_2 &= a_2 - b_1 z + b_3 x, \\ v_3 &= a_3 - b_2 x + b_1 y. \end{aligned} \tag{0.4}$$

It is now easy to see by a direct computation that all components of the strain tensor

$$\boldsymbol{\varepsilon}(\boldsymbol{v}) = \begin{bmatrix} \partial_x v_1 & \frac{1}{2}(\partial_y v_1 + \partial_x v_2) & \frac{1}{2}(\partial_z v_1 + \partial_x v_3) \\ \frac{1}{2}(\partial_x v_2 + \partial_y v_1) & \partial_y v_2 & \frac{1}{2}(\partial_z v_2 + \partial_y v_3) \\ \frac{1}{2}(\partial_x v_3 + \partial_z v_1) & \frac{1}{2}(\partial_y v_3 + \partial_z v_2) & \partial_z v_3 \end{bmatrix},$$

where $\partial_x = \partial/\partial x$ vanish.

On the other hand, assuming that $\boldsymbol{\varepsilon}(\boldsymbol{v}) = \mathbf{0}$, we first obtain from the diagonal components of the strain tensor

$$v_1 = v_1(y,z), \qquad v_2 = v_2(x,z), \qquad v_3 = v_3(x,y)\,.$$

It also follows from the off-diagonal components that

$$\begin{aligned} &\partial_{xx} v_2 = 0, \quad \partial_{xx} v_3 = 0, \quad \partial_{yy} v_1 = 0, \quad \partial_{yy} v_3 = 0, \quad \partial_{zz} v_1 = 0, \quad \partial_{zz} v_2 = 0, \\ &\partial_{yz} v_1 = -\partial_{xz} v_2 = -\partial_{zx} v_2 = \partial_{yx} v_3 = \partial_{xy} v_3 = -\partial_{zy} v_1 = -\partial_{yz} v_1 \quad \Rightarrow \quad \partial_{yz} v_1 = 0\,. \end{aligned}$$

Similarly, we see that $\partial_{xz} v_2 = \partial_{xy} v_3 = 0$ and thus v_1, v_2 and v_3 are affine functions.

Integrating (formally) the off-diagonal components, we obtain

$$\begin{aligned} &v_1(y,z) = a_1 - (\partial_x v_2) y - (\partial_x v_3) z, \quad v_2(x,z) = a_2 - (\partial_y v_1) x - (\partial_y v_3) z, \\ &v_3(x,y) = a_3 - (\partial_z v_1) x - (\partial_z v_2) y, \end{aligned}$$

where $a_j, j = 1,2,3$, are constants. Thus, defining $b_1 = \partial_z v_2, b_2 = \partial_x v_3$ and $b_3 = \partial_y v_1$, we see that v_1, v_2 and v_3 are of the form (0.4).

5. Prove the Korn inequality in the case when $\Gamma_D = \partial\Omega$, i.e.

$$\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_0 \gtrsim \|\nabla \boldsymbol{v}\|_0 \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega).$$

(<u>Hint:</u> Assume that $\mathbf{v}$ is a smooth function and integrate by parts a couple of times.)

Solution: Given that $\mathbf{v}$ vanishes on $\partial\Omega$ and $\mathbf{v} = (v_1, v_2, v_3)$ is taken as a smooth function, we obtain after integrating twice by parts

$$\int_\Omega \nabla\mathbf{v} : (\nabla\mathbf{v})^T\, dx = \int_\Omega \frac{\partial v_j}{\partial x_i}\frac{\partial v_i}{\partial x_j}\, dx = -\int_\Omega \frac{\partial^2 v_j}{\partial x_j \partial x_i} v_i\, dx = \int_\Omega \frac{\partial v_j}{\partial x_j}\frac{\partial v_i}{\partial x_i}\, dx = \int_\Omega (\operatorname{div}\mathbf{v})^2\, dx \ge 0\,.$$

Therefore

$$\|\boldsymbol{\varepsilon}(\mathbf{v})\|_0^2 = \int_\Omega \boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varepsilon}(\mathbf{v})\, dx = \frac{1}{4}\int_\Omega (\nabla\mathbf{v} + (\nabla\mathbf{v})^T) : (\nabla\mathbf{v} + (\nabla\mathbf{v})^T)\, dx$$

$$\ge \frac{1}{2}\int_\Omega \nabla\mathbf{v} : \nabla\mathbf{v}\, dx = \|\nabla\mathbf{v}\|_0^2\,.$$

6. Prove that the following velocity-pressure FE space pair for the two-dimensional Stokes system yields a unique solution

$$\boldsymbol{V}_h = \{\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \,|\, \boldsymbol{v}|_K \in [P_2(K) \oplus B(K)]^2\,, K \in \mathcal{C}_h \,\}.$$

$$P_h = \{\, q \in L_0^2(\Omega) \,|\, q|_K \in P_1(K)\,, K \in \mathcal{C}_h \,\}.$$

Solution: For the discrete solution to be unique the condition

$$(q, \operatorname{div}\mathbf{v}) = 0 \ \forall \mathbf{v} \in \boldsymbol{V}_h$$

must imply that q is constant in Ω. Note that we can choose $\boldsymbol{v} \in \boldsymbol{V}_h$ conveniently. First, we take $\boldsymbol{v}|_K = b_K \nabla q|_K$ where $b_K \in P_3(K) \cap H_0^1(K)$ is a bubble function. Thus $\boldsymbol{v} = \mathbf{0}$ in $\Omega \setminus K$ and it follows that

$$0 = (q, \operatorname{div}\boldsymbol{v}) = -(\nabla q, \boldsymbol{v})_K = \int_K b_K |\nabla q|^2\, dx,$$

from where we conclude that q is a constant function at each element K since $b_K(x) > 0\ \forall x \in K$ and $K \in \mathcal{C}_h$ was arbitrary.

On the other hand, letting K and K' be two arbitrary adjacent elements, with E as their common edge, and choosing $\boldsymbol{v}$ so that it vanishes in $\Omega \setminus (K \cup K')$, we obtain, knowing that q is constant elementwise,

$$0 = (q, \operatorname{div}\boldsymbol{v}) = (q, \operatorname{div}\boldsymbol{v})_{K\cup K'} = q|_K \int_K \operatorname{div}\boldsymbol{v}\, dx + q|_K' \int_{K'} \operatorname{div}\boldsymbol{v}\, dx = (q|_K - q|_{K'}) \int_E \boldsymbol{v}\cdot\boldsymbol{n}_E\, ds.$$

where q_K and $q_{K'}$ are the constant values of q in K and K'. Assuming, furthermore, that $\boldsymbol{v}$ is such that $\int_E \boldsymbol{v}\cdot\boldsymbol{n}_E\, ds \ne 0$, we conclude that $q|_K = q|_{K'}$, that is, q is constant in Ω which implies uniqueness for the Stokes problem.

7. Let us recall the following notation. $A \lesssim B$ means: there exists a positive constant C, independent of the mesh size h (or the local mesh since h_K, such that $A \le CB$. With $A \approx B$ we mean $A \lesssim B$ and $B \lesssim A$.

Let K be a triangle or tetrahedron, $P_k(K)$ the polynomials of degree k on K, and $b_K \in P_{d+1}(K)$ be the bubble function on K ($d =$ the space dimension.)

Prove by scaling arguments that

$$\|v\|_{0,K} \approx \|b_K^{1/2} v\|_{0,K} \approx \|b_K v\|_{0,K} \quad \forall v \in P_k(K),$$

and

$$\|\nabla v\|_{0,K} \lesssim h_K^{-1} \|v\|_{0,K} \quad \forall v \in P_k(K).$$

<u>Solution:</u> Recall that in a finite-dimensional space all norms are equivalent so that the results involving the bubble function b_K are trivially valid in the reference triangle $\hat{K}$ since $b_K > 0 \, \forall x \in K$ so that $\|b_K^{1/2} v\|_{0,K}$ and $\|b_K v\|_{0,K}$ define norms. In an arbitrary triangle $K \in \mathcal{C}_h$ we define the affine mapping

$$x = F_K(\hat{x}) = B_K(\hat{x}) + b_K, \qquad B_K \in \mathbb{R}^{2\times 2}, \quad b_K \in \mathbb{R}^2,$$

and set $\hat{v} = v \circ F_K$ and $\hat{b}_{\hat{K}} = b_K \circ F_K$. It follows that

$$\begin{aligned}\|v\|_{0,K}^2 = \int_K v(x)^2\, dx = |\text{det} B_K| \int_{\hat{K}} v(F_K(\hat{x}))^2\, d\hat{x} = |\text{det} B_K| \, \|\hat{v}\|_{0,\hat{K}}^2 \\ \lesssim |\text{det} B_K| \, \|\hat{b}_{\hat{K}} \hat{v}\|_{0,\hat{K}}^2 = |\text{det} B_K| \, |\text{det} B_K|^{-1} \, \|b_K v\|_{0,K}^2 = \|b_K v\|_{0,K}^2 \, .\end{aligned}$$

The other inequalities are now obvious. In fact, given that $0 < b_K \leq 1 \; \forall x \in K$ it holds trivially

$$\|b_K v\|_{0,K} \leq \|b_K^{1/2} v\|_{0,K} \leq \|v\|_{0,K} \, .$$

In order to establish the inverse inequality, recall that

$$\hat{\nabla} \hat{v}(\hat{x}) = B_K^T \nabla v(x), \qquad \|B_K\| \leq \frac{h_K}{\rho_{\hat{K}}}, \quad \|B_K^{-1}\| \leq \frac{h_{\hat{K}}}{\rho_K}$$

where ρ_K is the diameter of the largest sphere inscribed in K and $\|\cdot\|$ is the usual matrix (operator) norm. Now, we can compute

$$\|\nabla v\|_{0,K}^2 = \int_K \|\nabla v(x)\|^2\, dx = \int_{\hat{K}} \|B_K^{-T} \hat{\nabla} \hat{v}(\hat{x}))\|^2 |\text{det} B_K| \, d\hat{x} \leq |\text{det} B_K| \, \|B_K^{-T}\|^2 \, \|\hat{\nabla} \hat{v}\|_{0,\hat{K}}^2 \, .$$

Next, write $\hat{v} \in P_k(\hat{K})$ as $\hat{v}(\hat{x}) = \sum_{j=0}^N c_j \hat{\varphi}_j$ so that

$$\|\hat{v}\|_{0,\hat{K}}^2 = c^T A c \, , \qquad \|\hat{\nabla} \hat{v}\|_{0,\hat{K}}^2 = c^T B c$$

where $c = (c_0, c_1, \ldots, c_N)$, $N = \frac{(k+1)(k+2)}{2}$ and $A, B \in \mathbb{R}^{N \times N}$ are symmetric matrices, with A positive definite and B positive semi-definite. It follows that

$$\frac{\|\hat{\nabla} \hat{v}\|_{0,\hat{K}}^2}{\|\hat{v}\|_{0,\hat{K}}^2} = \frac{c^T B c}{c^T A c} = \frac{y^T L^{-1} B L^{-T} y}{y^T y} \, ,$$

where we have written $A = LL^T$ (Cholesky decomposition) and defined $y = L^T c$. From the Rayleigh quotient it follows that

$$\frac{\|\hat{\nabla}\hat{v}\|^2_{0,\hat{K}}}{\|\hat{v}\|^2_{0,\hat{K}}} \leq \lambda_{\max}(L^{-1}BL^{-T}),$$

where $\lambda_{\max}(L^{-1}BL^{-T})$ is the largest eigenvalue of the symmetric and positive definite matrix $L^{-1}BL^{-T}$. Given that $\lambda_{\max}(L^{-1}BL^{-T})$ is independent of h, we conclude that

$$\begin{aligned}\|\nabla v\|^2_{0,K} &\leq |\mathrm{det}B_K|\,\|B_K^{-T}\|^2\,\|\hat{\nabla}\hat{v}\|^2_{0,\hat{K}} \lesssim |\mathrm{det}B_K|\,\|B_K^{-T}\|^2\;\|\hat{v}\|^2_{0,\hat{K}}\\ &= |\mathrm{det}B_K|\,\|B_K^{-T}\|^2\,|\mathrm{det}B_K|^{-1}\,\|v\|^2_{0,K} \lesssim \rho_K^{-2}\|v\|^2_{0,K} \lesssim h_K^{-2}\|v\|^2_{0,K},\end{aligned}$$

where we have used the shape-regularity of the triangulation, that is $h_K \leq C\rho_K\ \forall K \in \mathcal{C}_h$.

8. Read the section in the lecture notes where it is shown that the discrete linear system for the Stokes problem is of the form:

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix}\begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}, \tag{0.5}$$

where $U \in I\!R^N$, $P \in I\!R^M$. Note that $A \in \mathbb{R}^{N\times N}$ is a symmetric and positive definite matrix.

Show that this can be interpreted as the discrete optimisation problem: find U which minimises the objective function

$$\frac{1}{2}V^T AV - F^T V \tag{0.6}$$

subject to the linear constraint

$$B^T V = G. \tag{0.7}$$

Show that the problem has a unique solution if, and only if, $N(B) = \{0\}$, or equivalently $R(B^T) = I\!R^M$, with N and R denoting the nullspace and range, respectively.

<u>Solution:</u> Let us define the quadratic objective function $\mathcal{F} : \mathbb{R}^N \to \mathbb{R}$ by

$$\mathcal{F}(V) = \frac{1}{2}V^T AV - F^T V$$

and consider the equality-constrained minimization problem

$$\min_{V\in\mathbb{R}^N} \mathcal{F}(V) \quad \text{subject to} \quad B^T V - G = 0,$$

where $B \in \mathbb{R}^{N\times M}, F \in \mathbb{R}^N, G \in \mathbb{R}^M$. Defining, furthermore, the Lagrangian function

$$\mathcal{L}(V, Q) = Q(V) + P^T(B^T V - G)$$

the first-order necessary conditions for optimality at (U, P) (Karush-Kuhn-Tucker conditions) read as follows

$$\left\{\begin{array}{l} \nabla_V\mathcal{L}(U, P) = 0 \\ B^T U - G = 0 \end{array}\right. \qquad \Leftrightarrow \qquad \begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix}\begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix},$$

which, choosing, $G = 0$, is of the form (0.5). The KKT matrix

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix}$$

is non-singular if and only if B has full column rank. In fact, if

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} V \\ Q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

then

$$0 = \begin{pmatrix} V \\ Q \end{pmatrix}^T \begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} V \\ Q \end{pmatrix} = V^T AV + V^T BQ + Q^T B^T V = V^T AV,$$

given that $B^T V = 0$. Consequently $V = 0$ since A is positive definite and therefore $BQ = 0$. We thus conclude that $Q = 0$ and the solution is unique if and only if B has full column rank.

9. In the lectures we proved the stability of the lowest order Crouzeix-Raviart element, i.e. the FE pair

$$\begin{aligned} \boldsymbol{V}_h &= \{\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \,|\, \boldsymbol{v}|_K \in [P_2(K)]^2 \; K \in \mathcal{C}_h \,\}, \\ P_h &= \{\, q \in L_0^2(\Omega) \,|\, q|_K \in P_0(K) \; K \in \mathcal{C}_h \,\}. \end{aligned}$$

A common (mis)belief is that the same method works in 3D, i.e, $\Omega \subset I\!R^3$ and

$$\begin{aligned} \boldsymbol{V}_h &= \{\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \,|\, \boldsymbol{v}|_K \in [P_2(K)]^3 \; K \in \mathcal{C}_h \,\}, \\ P_h &= \{\, q \in L_0^2(\Omega) \,|\, q|_K \in P_0(K) \; K \in \mathcal{C}_h \,\}. \end{aligned} \tag{0.8}$$

Question: does the 2D stability proof (or even the uniqueness proof) carry over to 3D?

Solution: There are 10 degrees of freedom in $[P_2(K)]^3$ and, to ensure continuity from element to element, each face of the tetrahedron has to have 6 degrees of freedom, all of them are situated along the edges of the face. Thus there are no degrees of freedom in the interior of the faces (or the tetrahedron). Consequently, we cannot construct bubble functions on the faces to prove stability (or uniqueness) as in the 2D case.

10. Let $\mathcal{C}_h$ be a partitioning into quadrilaterals and consider the Stokes pair

$$\begin{aligned} \boldsymbol{V}_h &= \{\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \,|\, \boldsymbol{v}|_K \in [Q_2(K)]^2 \; K \in \mathcal{C}_h \,\}, \\ P_h &= \{\, q \in L_0^2(\Omega) \,|\, q|_K \in P_1(K) \; K \in \mathcal{C}_h \,\}. \end{aligned}$$

Verify the stability. How is it with the method in 3D?

Solution: Note that there are nine degrees of freedom in the quadrilateral element $Q_2(K)$, namely the vertices and the midpoints of the edges (to ensure continuity) and the center of K. Now decompose $q \in P_h$ in each K as $q = \bar{q} + \hat{q}$ where $\bar{q}$ is piecewise constant and $\hat{q}$ its orthogonal component, i.e. $(\hat{q}, \bar{q}) = 0$. For $\bar{q} \in P_h$ we choose $\bar{\boldsymbol{v}} \in \boldsymbol{V}_h$ in such a way that $\bar{\boldsymbol{v}}|_E = h_E b_E [\![\bar{q}]\!]_E$ where b_E is the edge bubble which vanishes at the endpoints and is equal to one in the midpoint of E and $[\![\bar{q}]\!]_E$ is the jump of $\bar{q}$ over the edge $E \subset \Omega$. It follows that $\bar{\boldsymbol{v}}|_K = \sum_{E \subset \partial K} h_E b_E [\![\bar{q}]\!]_E$ so that

$$(\operatorname{div} \bar{\boldsymbol{v}}, \bar{q}) = \sum_{K \in \mathcal{C}_h} (\operatorname{div} \bar{\boldsymbol{v}}, \bar{q})_K = \sum_{E \subset \Omega} \langle \bar{\boldsymbol{v}} \cdot \mathbf{n}_E, [\![\bar{q}]\!]_E \rangle_E = \sum_{E \subset \Omega} h_E \| b_E^{1/2} [\![\bar{q}]\!] \|_{0,E}^2 \approx \|\bar{q}\|_h^2 \,.$$

Moreover,

$$\sum_{K\in\mathcal{C}_h} \|\bar{\boldsymbol{v}}\|_{1,K}^2 \lesssim \sum_{K\in\mathcal{C}_h} h_K^{-2}\|\bar{\boldsymbol{v}}\|_{0,K}^2 = \sum_{K\in\mathcal{C}_h} \| \sum_{E\subset\partial K} b_E [\![\bar{q}]\!]_E \|_{0,K}^2 \lesssim \sum_{K\in\mathcal{C}_h} \sum_{E\subset\partial K} [\![\bar{q}]\!]_E^2 \|b_E\|_{0,K}^2$$
$$\approx \sum_{K\in\mathcal{C}_h} h_K^2 \sum_{E\subset\partial K} [\![\bar{q}]\!]_E^2 \approx \sum_{K\in\mathcal{C}_h} \sum_{E\subset\partial K} h_E \|[\![\bar{q}]\!]\|_{0,E}^2 \approx \sum_{E\subset\Omega} h_E \, \|[\![\bar{q}]\!]\|_{0,E}^2 = \|\bar{q}\|_h^2 \,,$$

where we have used the inverse inequality and the facts that

$$h_E \sim h_K \,, \qquad \|b_E\|_{0,K}^2 \sim h_K^2 \,, \qquad \|[\![\bar{q}]\!]\|_{0,E}^2 \sim h_E [\![\bar{q}]\!]_E^2 \,,$$
$$\|\bar{q}\|_h^2 = \sum_{K\in\mathcal{C}_h} h_K^2 \|\nabla\bar{q}\|_{0,K}^2 + \sum_{E\subset\Omega} h_E \|[\![\bar{q}]\!]\|_{0,E}^2 = \sum_{E\subset\Omega} h_E \|[\![\bar{q}]\!]\|_{0,E}^2 .$$

It thus follows that

$$\sup_{\boldsymbol{v}\boldsymbol{V}_h} \frac{(\mathrm{div}\,\boldsymbol{v},\bar{q})}{\|\boldsymbol{v}\|_1} \gtrsim \|\bar{q}\|_h \qquad \forall \bar{q}\in P_h^0 := \{\, q\in L_0^2(\Omega) \,|\, q|_K \in P_0(K)\; K\in\mathcal{C}_h \,\},$$

which, by Theorem 4.4, ensures stability also in the continuous norms, that is

$$\sup_{\boldsymbol{v}\boldsymbol{V}_h} \frac{(\mathrm{div}\,\boldsymbol{v},\bar{q})}{\|\boldsymbol{v}\|_1} \gtrsim \|\bar{q}\|_0 \qquad \forall \bar{q}\in P_h^0 \,. \tag{0.9}$$

Next, assuming that

$$\hat{q}\in P_h^{\perp} := \{\, q\in L_0^2(\Omega) \,|\, q|_K \in P_1(K)\; K\in\mathcal{C}_h \,, \quad (q,\bar{q}) = 0 \;\forall \bar{q}\in P_h^0 \}$$

is given, we choose $\hat{\boldsymbol{v}} \in \boldsymbol{V}_h$ such that $\hat{\boldsymbol{v}}|_K = -h_K^2 b_K \nabla\hat{q} \;\forall K\in\mathcal{C}_h$, where $b_K\in\boldsymbol{V}_h$ is a non-negative bubble function in K. Given that $\hat{\boldsymbol{v}}$ vanishes on ∂K, we obtain

$$(\mathrm{div}\,\hat{\boldsymbol{v}},\hat{q}) = -(\hat{\boldsymbol{v}},\nabla\hat{q}) = \sum_{K\in\mathcal{C}_h} \int_K h_K^2 b_K |\nabla\hat{q}|^2\,dx = \sum_{K\in\mathcal{C}_h} h_K^2 \|b_K^{1/2}\nabla\hat{q}\|_{0,K}^2$$
$$\gtrsim \sum_{K\in\mathcal{C}_h} h_K^2 \|\nabla\hat{q}\|_{0,K}^2 \gtrsim \sum_{K\in\mathcal{C}_h} \|\hat{q}\|_{0,K}^2 = \|\hat{q}\|_0^2,$$

where we have used the Poincaré's inequality

$$\|\hat{q}\|_{0,K} = \|q-\bar{q}\|_{0,K} \lesssim h_K \|\nabla q\|_{0,K} = h_K \|\nabla\hat{q}\|_{0,K} \,.$$

Moreover, recalling the inverse inequality it is easy to see that

$$\|\nabla\hat{\boldsymbol{v}}\|_{0,K} \lesssim h_K^{-1}\|\hat{\boldsymbol{v}}\|_{0,K} = h_K \|b_K\nabla\hat{q}\|_{0,K} \lesssim h_K\|\nabla\hat{q}\|_{0,K} \le \|\hat{q}\|_{0,K},$$

which implies, after summing over the elements, that

$$\|\hat{\boldsymbol{v}}\|_1 \lesssim \|\hat{q}\|_0 \,.$$

The stability condition

$$\sup_{\boldsymbol{v}\boldsymbol{V}_h} \frac{(\mathrm{div}\,\boldsymbol{v},\hat{q})}{\|\boldsymbol{v}\|_1} \gtrsim \|\hat{q}\|_0 \qquad \forall \hat{q}\in P_h^{\perp} \tag{0.10}$$

is thus established directly in the continuous norms.

Finally, for any $q \in P_h$, we write $q = \hat{q} + \bar{q}$, let $\delta > 0$, and choose $\boldsymbol{v} = \hat{\boldsymbol{v}} + \delta\bar{\boldsymbol{v}}$, where $\hat{\boldsymbol{v}}$ and $\bar{\boldsymbol{v}}$ are functions for which the stability conditions (0.9) and (0.10) hold. We may assume that $\|\bar{\boldsymbol{v}}\|_1 = \|\bar{q}\|_0$ and $\|\hat{\boldsymbol{v}}\|_1 = \|\hat{q}\|_0$.

We can now estimate (note that $(\operatorname{div}\hat{\boldsymbol{v}}, \bar{q}) = 0$ since $\hat{\boldsymbol{v}}$ vanishes at ∂K and $\nabla\bar{q} = 0$ in K),

$$
\begin{aligned}
(\operatorname{div}(\hat{\boldsymbol{v}} + \delta\bar{\boldsymbol{v}}), \hat{q} + \bar{q}) &= (\operatorname{div}\hat{\boldsymbol{v}}, \hat{q}) + \delta(\operatorname{div}\bar{\boldsymbol{v}}, \bar{q}) + \delta(\operatorname{div}\bar{\boldsymbol{v}}, \hat{q}) \\
&\geq C_1\|\hat{q}\|_0^2 + \delta\, C_2\,\|\bar{q}\|_0^2 - \delta\,\|\bar{\boldsymbol{v}}\|_1\|\hat{q}\|_0 \\
&\geq C_1\|\hat{q}\|_0^2 + \delta\, C_2\,\|\bar{q}\|_0^2 - \frac{\delta\, C_2}{2}\|\bar{\boldsymbol{v}}\|_1 - \frac{\delta}{2\, C_2}\|\hat{q}\|_0 \\
&\gtrsim \|\bar{q}\|_0^2 + \|\hat{q}\|_0 = \|q\|_0^2\,,
\end{aligned}
$$

where we have chosen $\delta < 2C_1C_2$. Moreover, $\|\boldsymbol{v}\|_1 = \|\hat{\boldsymbol{v}} + \delta\bar{\boldsymbol{v}}\|_1 \lesssim \|q\|_0$ and thus

$$
\sup_{\boldsymbol{v}\boldsymbol{V}_h} \frac{(\operatorname{div}\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1} \gtrsim \|q\|_0 \qquad \forall q \in P_h\,.
$$

The stability of the $Q_2(K) - P_1(K)$ element in $3D$ (hexahedral elements) can be established similarly since there are 27 degrees of freedom and, to ensure continuity, 20 are located along the edges (8 in corner points, 12 in the midpoints of edges) 6 at the centers of the faces) and one in the center of the hexahedron. Thus one can construct similar bubble functions as above to show stability.

11. The lowest order quadrilateral Taylor-Hood method (continuous pressures) consists of the following spaces

$$
\begin{aligned}
\boldsymbol{V}_h &= \{\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \,|\, \boldsymbol{v}|_K \in [Q_2(K)]^2\; K \in \mathcal{C}_h \,\}, \\
P_h &= \{\, q \in L_0^2(\Omega) \cap C(\Omega) \,|\, q|_K \in Q_1(K)\; K \in \mathcal{C}_h \,\}.
\end{aligned}
$$

with the mesh $\mathcal{C}_h$ consists of quadrilaterals. For the case of rectangles, prove the uniqueness of the solution. Hint: use a patch of two elements and the fact that Simpson's rule is exact for cubic polynomials.

<u>Solution:</u> Consider the patch of two adjacent rectangular elements, say $K_1 = [0, h_1] \times [0, h_2]$ and $K_2 = [h_1, h_3] \times [0, h_2]$ and define the corresponding shape functions in $Q_1(K_1)$ and $Q_1(K_2)$ (numbered counterclockwise from the lower left-hand corner of K_j) by

$$
\begin{aligned}
&\eta_{1,1}(x_1, x_2) = \Big(1 - \frac{x_1}{h_1}\Big)\Big(1 - \frac{x_2}{h_2}\Big), && \eta_{2,1}(x_1, x_2) = \frac{x_1}{h_1}\Big(1 - \frac{x_2}{h_2}\Big), && (x_1, x_2) \in K_1, \\
&\eta_{3,1}(x_1, x_2) = \frac{x_1}{h_1}\frac{x_2}{h_2}, && \eta_{4,1}(x_1, x_2) = \Big(1 - \frac{x_1}{h_1}\Big)\frac{x_2}{h_2}, && (x_1, x_2) \in K_1, \\
&\eta_{1,2}(x_1, x_2) = \frac{h_3}{h_3 - h_1}\Big(1 - \frac{x_1}{h_3}\Big)\Big(1 - \frac{x_2}{h_2}\Big), && && (x_1, x_2) \in K_2 \\
&\eta_{2,2}(x_1, x_2) = \frac{h_1}{h_1 - h_3}\Big(1 - \frac{x_1}{h_1}\Big)\Big(1 - \frac{x_2}{h_2}\Big), && && (x_1, x_2) \in K_2, \\
&\eta_{3,2}(x_1, x_2) = \frac{h_1}{h_1 - h_3}\Big(1 - \frac{x_1}{h_1}\Big)\frac{x_2}{h_2}, && \eta_{4,2}(x_1, x_2) = \frac{h_3}{h_3 - h_1}\Big(1 - \frac{x_1}{h_3}\Big)\frac{x_2}{h_2}, && (x_1, x_2) \in K_2,
\end{aligned}
$$

Thus $q \in P_h$ is expressed in K_1 and K_2 as

$$q|_{K_1} = q_1\eta_{1,1} + q_2\eta_{2,1} + q_3\eta_{3,1} + q_4\eta_{4,1} \qquad q|_{K_2} = q_2\eta_{1,2} + q_5\eta_{2,2} + q_6\eta_{3,2} + q_3\eta_{4,2}\,.$$

The nine degrees of freedom in

$$Q_2(K) = \operatorname{span}\{1, x_1, x_2, x_1x_2, x_1^2, x_2^2, x_1x_2^2, x_1^2x_2, x_1^2x_2^2\}$$

are the vertices, the midpoints of the edges and the center of K. Noting that the integrand in

$$\begin{aligned}(\operatorname{div}\boldsymbol{v}, q) &= \int_{K_1} \operatorname{div}\boldsymbol{v}\, q\, dx = -\int_{K_1} \boldsymbol{v}\cdot\nabla q\, dx = -\int_{K_1} \Big(\boldsymbol{v}_1\frac{\partial q}{\partial x_1} + \boldsymbol{v}_2\frac{\partial q}{\partial x_2}\Big)\, dx \\ &= -\int_0^{h_2} \Big(\int_0^{h_1} \Big(\boldsymbol{v}_1\frac{\partial q}{\partial x_1} + \boldsymbol{v}_2\frac{\partial q}{\partial x_2}\Big)\, dx_1\Big) dx_2\end{aligned}$$

is (at most) a third-order polynomial in x_1 or in x_2, we may compute the integrals in x_1 and in x_2 exactly using Simpson's rule. Choose $\boldsymbol{v} \in \boldsymbol{V}_h$ in such a way that both $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ vanish in the nodes at $\partial K_1 \cup \partial K_2$ as well as in the center node of K_2 and that first $\boldsymbol{v}_1(\mathbf{x}^9) = 1, \boldsymbol{v}_2(\mathbf{x}^9) = 0$, where $\mathbf{x}^9 = (\frac{h_1}{2}, \frac{h_2}{2})$ is the center node of K_1, and then $\boldsymbol{v}_1(\mathbf{x}^9) = 0, \boldsymbol{v}_2(\mathbf{x}^9) = 1$. It follows that

$$\begin{aligned} 0 &= -\frac{1}{2}\Big(-q_1 + q_2 + q_3 - q_4\Big)\,, \\ 0 &= -\frac{1}{2}\Big(-q_1 - q_2 + q_3 + q_4\Big)\,. \end{aligned}$$

Similarly, choosing $\boldsymbol{v} \in \boldsymbol{V}_h$ in such a way that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ vanish in the nodes at $\partial K_1 \cup \partial K_2$ as well as in the center node of K_1 and $\boldsymbol{v}_1(\mathbf{x}^{15}) = 1, \boldsymbol{v}_2(\mathbf{x}^{15}) = 0$, respectively $\boldsymbol{v}_1(\mathbf{x}^{15}) = 0, \boldsymbol{v}_2(\mathbf{x}^{15}) = 1$, where $\mathbf{x}^{15} = (\frac{(h_1+h_3)}{2}, \frac{h_2}{2})$ is the center node of K_2, we obtain

$$\begin{aligned} 0 &= -\frac{1}{2}\Big(-q_2 + q_5 + q_6 - q_3\Big)\,, \\ 0 &= -\frac{1}{2}\Big(-q_2 - q_5 + q_6 + q_3\Big)\,. \end{aligned}$$

These equations imply that

$$q_1 = q_3 = q_5 = c_1\,, \qquad q_2 = q_4 = q_6 = c_2$$

where c_1 and c_2 are arbitrary constants. Finally choosing $\boldsymbol{v} \in \boldsymbol{V}_h$ in such a way that that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ vanish at $\partial(K_1 \cup \partial K_2)$ and $\boldsymbol{v}_2(\mathbf{x}^9) = 0, \boldsymbol{v}_2(\mathbf{x}^{15}) = 0, \boldsymbol{v}_2(\mathbf{x}^4) = 1$ and $\boldsymbol{v}_1(\mathbf{x}^9) = \boldsymbol{v}_1(\mathbf{x}^{15}) = \boldsymbol{v}_1(\mathbf{x}^4) = 0$, where $\mathbf{x}^4 = (h_1, \frac{h_2}{2})$ is the midpoint of the common edge $\partial K_1 \cap \partial K_2$, we obtain

$$0 = -\frac{1}{2}\Big(-q_2 + q_3\Big)\,.$$

Thus $c_1 = c_2$ which means that q is constant in $K_1 \cup K_2$ and consequently everywhere in Ω since K_1 and K_2 were arbitrary rectangles. For the proof in the general case (quadrilateral elements), see Stenberg, Analysis of Mixed Finite Element Methods for the Stokes Problem: A Unified Approach, *Math. Comp.* **42**, 9-23 (1984).